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ON MARCINKIEWICZ-ZYGMUND LAWS OF LARGE NUMBERS IN BANACH SPACES AND RELATED RATES OF CONVERGENCE

BY

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Abstract. The paper studies asymptotic almost sure and tail behavior of sums $(X_1+\ldots+X_n)/n^{1/p}$, $1 \leqslant p < 2$, for independent, centered random vectors X_n, $n = 1, 2, \ldots$, taking values in Banach space E. The obtained results are in the spirit of Mazurkiewicz--Zygmund, Hsu-Robbins-Erdös-Spitzer, and Brunk theorems for real random variables and show the essential role played by the geometry of E in the infinite-dimensional case.

1. Introduction and preliminaries. Let $(E, \|\cdot\|)$ be a real separable Banach space. In the present paper we study strongly measurable random vectors X on a probability space $(\Omega, \mathscr{F}, P)$ with values in E. If $\mathrm{E}\|X\| < \infty$, then $\mathrm{E}X$ stands for the Bochner integral, and throughout the paper $(X_i)_{i=1,2,\ldots}$ will be independent random vectors in E, with $S_0 = 0$, $S_n = X_1+\ldots+X_n$, $n = 1, 2, \ldots$, and (r_i) will stand for a *Rademacher sequence*, i.e., a sequence of real independent random variables with $P(r_i = \pm 1) = 1/2$.

We recall a couple of definitions (for more information cf., e.g., [14]).

Definition 1.1. Let $1 \leqslant p \leqslant 2$. A Banach space E is said to be of *Rademacher type p* (*R-type p*) if there exists C such that for every $n \in N$ and for all $x_1, \ldots, x_n \in E$

$$\mathrm{E}\Big\|\sum_{i=1}^{n} r_i x_i\Big\| \leqslant C\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}.$$

Definition 1.2. Let $1 \leqslant p \leqslant 2$. l_p is said to be *finitely representable* in E if for every $\varepsilon > 0$ and every $n \in N$ there exist $x_1, \ldots, x_n \in E$ such that for all $\alpha_1, \ldots, \alpha_n \in \boldsymbol{R}$

$$\Big(\sum_{i=1}^{n} |\alpha_i|^p\Big)^{1/p} \leqslant \Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \leqslant (1+\varepsilon)\Big(\sum_{i=1}^{n} |\alpha_i|^p\Big)^{1/p}.$$

Example 1.1. l_p is of R-type $\min(p, 2)$ for any $p \geqslant 1$. l_p is finitely representable in l_q for any $q \leqslant p$, but l_p is not finitely representable in l_q if $q > p$. On the other hand, by Dvoretzky's theorem, l_2 is finitely representable in E for any infinite dimensional E.

Definition 1.3. A sequence (X_i) of random vectors in E is said to have *uniformly bounded tail probabilities by tail probabilities of a real random variable* X_0 if there exists $C > 0$ such that for every $t > 0$ and every $i \in N$

$$P(\|X_i\| > t) \leqslant CP(|X_0| > t).$$

The main results of the paper deal with the almost sure convergence of sums $S_n/n^{1/p}$ and with the rate of convergence to zero of tail probabilities $P(\|S_n/n^{1/p}\| > \varepsilon)$ under restrictions on individual random vectors X_i and on geometric structure of E. For real-valued independent identically distributed (X_i) $(E = \boldsymbol{R})$ the problem of rates of convergence was studied in a series of papers by Erdös [3], Spitzer [12], Baum and Katz [1], and in the case of a general Banach space E certain interesting results have been obtained by Jain [4].

As far as the strong and weak laws of large numbers of Marcinkiewicz--Zygmund type (i.e., for $S_n/n^{1/p}$ and i.i.d. (X_i)) are concerned the following is known:

In the case $p = 1$, R. Fortet and M. Mourier proved in 1953 that, without any restrictions on E, if (X_i) are i.i.d., $\mathrm{E}\|X_1\| < \infty$ and $\mathrm{E}X_1 = 0$, then $S_n/n \to 0$ a.s. On the other hand, Maurey and Pisier [10] have shown that $(r_1 x_1 + \ldots + r_n x_n)/n^{1/p} \to 0$ a.s. for any bounded sequence $(x_n) \subset E$ if and only if l_p is not finitely representable in E $(1 \leqslant p < 2)$. In 1977, Marcus and Woyczyński [8], [9] proved that $S_n/n^{1/p} \to 0$ in probability for any i.i.d. (X_i) satisfying the condition

$$\lim_{n\to\infty} n^p P(\|X_1\| > n) = 0$$

if and only if l_p is not finitely representable in E.

In this paper we show, in particular, that for independent (X_i) with uniformly bounded tail probabilities the implication "if $\mathrm{E}\|X_i\|^p < \infty$ and $\mathrm{E}X_i = 0$, then $S_n/n^{1/p} \to 0$ a.s." also depends in an essential way on l_p not being finitely representable in E. We also prove that a Banach space analogue of Brunk's strong law of large numbers (cf. [2], [11]) depends on the R-type of E. Brunk's type strong law is particularly useful in cases where one has information about existence of moments of X_i's of orders greater than 2. Such information may not be utilized in the framework of Kolmogorov--Chung's strong law.

As far as the rates of convergence are concerned a number of simple remarks are in order here. Directly from definitions and from Chebyshev's inequality one can obtain the following "trivial" rate:

PROPOSITION 1.1. *Let $1 \leqslant p \leqslant 2$ and let E be of R-type p. If (X_i) are i.i.d. with* $\mathrm{E}\,\|X_1\|^p < \infty$ *and* $\mathrm{E}X_1 = 0$, *then*

$$P(\|S_n/n\| \geqslant \varepsilon) = O(n^{1-p}) \quad \text{for every } \varepsilon > 0.$$

Also some exponential rates can be immediately obtained without any restrictions on the geometric structure of E.

PROPOSITION 1.2. *If (X_i) are i.i.d. with* $\mathrm{E}X_1 = 0$ *and with the property that for every $\varepsilon > 0$ there exist C_ε and β_ε such that for every $\beta \leqslant \beta_\varepsilon$*

$$\mathrm{E}\exp[\beta\,\|X_1\|] \leqslant C_\varepsilon \exp[\beta\varepsilon],$$

then for every $\varepsilon > 0$ there exists $\alpha < 1$ such that

$$P(\|S_n/n\| > \varepsilon) = O(\alpha^n).$$

Proof. By Chebyshev's inequality and for $\delta < \varepsilon$ we get

$$\begin{aligned} P(\|S_n/n\| > \varepsilon) &\leqslant \exp[-\beta_\delta n\varepsilon]\,\mathrm{E}\exp[\beta_\delta\,\|S_n\|] \\ &\leqslant \exp[-\beta_\delta n\varepsilon]\,(\mathrm{E}\exp[\beta_\delta\,\|X_1\|])^n \leqslant C_\delta\big(\exp[(\delta-\varepsilon)\beta_\delta]\big)^n. \end{aligned}$$

It is also interesting to notice that a sufficiently rapid rate of convergence to zero of tail probabilities $P(\|S_n/a_n\| > \varepsilon)$ implies similar rates of convergence in the strong law, i.e., for the suprema.

PROPOSITION 1.3. *Let E be a Banach space and let (X_i) be independent symmetric random vectors in E. Let (a_i), (b_i), $(c_i) \subset \boldsymbol{R}$ be such that*

$$0 < a_i \uparrow \infty, \quad b_i, c_i \downarrow 0 \quad \text{and} \quad \sum_{i=1}^{j} 2^i b_{2^i} = O(2^j c_{2^j})$$

and let

$$\sum_{n=1}^{\infty} c_n P(\|S_n/a_n\| > \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.$$

Then

$$\sum_{n=1}^{\infty} b_n P(\sup_{k\geqslant n}\|S_k/a_k\| > \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.$$

Proof. Grouping the terms in exponential blocks (n: $2^j < n \leqslant 2^{j+1}$) we get

$$\begin{aligned} A \equiv \sum_{n=1}^{\infty} b_n P(\sup_{k\geqslant n}\|S_k/a_k\| > \varepsilon) &\leqslant \sum_{i=1}^{\infty} b_{2^i}\cdot 2^i P(\sup_{k\geqslant 2^i}\|S_k/a_k\| > \varepsilon) \\ &\leqslant \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} b_{2^i}\cdot 2^i P(\max_{2^j<k\leqslant 2^{j+1}}\|S_k/a_k\| > \varepsilon) \end{aligned}$$

and, by Lévy's inequality,

$$A \leqslant 2 \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} b_{2^i} \cdot 2^i P(\|S_{2^{j+1}}/a_{2^{j+1}}\| > \varepsilon)$$

$$= 2 \sum_{j=1}^{\infty} \Big(\sum_{i=1}^{j} b_{2^i} \cdot 2^i \Big) P(\|S_{2^{j+1}}/a_{2^{j+1}}\| > \varepsilon)$$

$$\leqslant 2C \sum_{j=1}^{\infty} c_{2^j} \cdot 2^j P(\|S_{2^{j+1}}/a_{2^{j+1}}\| > \varepsilon).$$

Now, by the symmetry assumptions, grouping the terms again as follows:

$$S_n = S_{2^{j+1}} - X_{2^{j+1}} - X_{2^{j+1}-1} - \ldots - X_{n+1}, \quad 2^{j-1} \leqslant n < 2^j,$$

we obtain

$$A \leqslant 8C \sum_{n=1}^{\infty} c_n P(\|S_n/a_n\| > 2\varepsilon).$$

Two special cases of Proposition 1.3 will be of interest later on.

COROLLARY 1.1. *Let E be a Banach space and let (X_i) be independent symmetric random vectors in E. Then*

(i) *for every $q > 1$ there exists $C > 0$ such that*

$$\sum_{n=1}^{\infty} n^{-q} P(\sup_{k \geqslant n} \|S_k/a_k\| > \varepsilon) \leqslant C \sum_{n=1}^{\infty} n^{-q} P(\|S_n/a_n\| > \varepsilon);$$

(ii) *there exists $C > 0$ such that*

$$\sum_{n=1}^{\infty} n^{-1} P(\sup_{k \geqslant n} \|S_k/a_k\| > \varepsilon) \leqslant C \sum_{n=1}^{\infty} n^{-1} (\log n) P(\|S_n/a_n\| > \varepsilon).$$

2. Rates of convergence based on the Marcinkiewicz-Zygmund inequality. In Proposition 1.1 we could have only used moments of order p, $1 \leqslant p \leqslant 2$, and in Proposition 1.2 exponential moments were needed. The following analogue of the Marcinkiewicz-Zygmund inequality (cf. also results by P. Assouad and B. Maurey and G. Pisier quoted in [14]) permits us to use the information on moments of arbitrary order.

PROPOSITION 2.1. *Let $1 \leqslant p \leqslant 2$ and $q \geqslant 1$. The following properties of E are equivalent:*

(i) *E is of R-type p.*

(ii) *There exists C such that for every $n \in N$ and for any sequence (X_i) of independent random vectors in E with* $\mathrm{E}X_i = 0$

$$\mathrm{E} \Big\| \sum_{i=1}^{n} X_i \Big\|^q \leqslant C\, \mathrm{E} \Big(\sum_{i=1}^{n} \|X_i\|^p \Big)^{q/p}.$$

Proof. (i) $\Rightarrow$ (ii). Let $(\tilde{X}_i) = (X_i - X_i')$ be a symmetrization of (X_i) and let (r_i) be independent of (X_i) and (X_i'). Then

$$\mathrm{E}\Big\|\sum_{i=1}^{n} X_i\Big\|^q \leqslant \mathrm{E}\Big\|\sum_{i=1}^{n} \tilde{X}_i\Big\|^q = \mathrm{E}\Big\|\sum_{i=1}^{n} r_i \tilde{X}_i\Big\|^q$$

$$\leqslant C\,\mathrm{E}\Big(\sum_{i=1}^{n} \|\tilde{X}_i\|^p\Big)^{q/p} \leqslant C\cdot 2^q\,\mathrm{E}\Big(\sum_{i=1}^{n} \|X_i\|^p\Big)^{q/p},$$

where the first inequality follows from the condition $\mathrm{E}X_i = 0$, and because (X_i') are independent of (X_i), the equality holds by symmetry of $(\tilde{X}_i)$, the second inequality by R-type of E and Fubini's theorem, and the third one by the triangle inequality.

The implication (ii) $\Rightarrow$ (i) follows from the proof of Theorem 3.1 given in the sequel.

COROLLARY 2.1. *Let E be of R-type p and $q \geqslant p$. If (X_n) are i.i.d. random vectors in E with* $\mathrm{E}\|X_1\|^q < \infty$ *and* $\mathrm{E}X_1 = 0$, *then* $\mathrm{E}\|S_n\|^q = O(n^{q/p})$.

Proof. If $p = q$, the estimate follows directly from the definition of R-type p. If $q > p$, then by Hölder's inequality with exponents q/p and $q/(q-p)$ and by Proposition 2.1 we have

$$\mathrm{E}\Big\|\sum_{i=1}^{n} X_i\Big\|^q \leqslant C\,\mathrm{E}\Big(\sum_{i=1}^{n} \|X_i\|^p\Big)^{q/p}$$

$$\leqslant C\,\mathrm{E}\Big(\sum_{i=1}^{n} \|X_i\|^q\Big) n^{(q-p)/p} = Cn^{q/p}\,\mathrm{E}\|X_1\|^q.$$

Hence, by Chebyshev's inequality we obtain immediately

COROLLARY 2.2. *Let E be of R-type p and $q \geqslant p$. If (X_n) are i.i.d. with* $\mathrm{E}\|X_1\|^q < \infty$ *and* $\mathrm{E}X_1 = 0$, *then*

$$P(\|S_n/n\| > \varepsilon) = O(n^{q(1/p-1)}) \quad \textit{for every } \varepsilon > 0.$$

Remark 2.1. Jurek and Urbanik [5], studying stable measures on E, define E as being of *type* (s, r), $s \geqslant 0$, $r > 0$, whenever there exists C such that for all (X_i) independent and symmetric in E

$$\mathrm{E}\Big\|\sum_{i=1}^{n} X_i\Big\|^r \leqslant Cn^s \sum_{i=1}^{n} \mathrm{E}\|X_i\|^r.$$

Proposition 2.1 implies (as in the proof of Corollary 2.1) that if E is of R-type p, then

$$\mathrm{E}\Big\|\sum_{i=1}^{n} X_i\Big\|^q \leqslant Cn^{q/p-1} \sum_{i=1}^{n} \mathrm{E}\|X_i\|^q \quad \text{for every } q \geqslant p,$$

i.e. E is also of Jurek-Urbanik's type $(q/p-1, q)$ or, equivalently, E is of type $(s, p(s+1))$ for every $s \geqslant 0$. One can also show (as in Theorem 3.1

below) that if for some $s > 0$ the space E is of type $(s, p(s+1))$, then E is of R-type p.

3. Brunk's type strong law and related rates of convergence. The following result extends the Kolmogorov-Chung type strong law in E obtained by the author and J. Hoffmann-Jørgensen and G. Pisier (cf. [14], p. 390, where E is of R-type p, $1 \leqslant p \leqslant 2$, and $q = 1$). In the case $E = \boldsymbol{R}$, $p = 2$, $q \geqslant 1$, the theorem is due to Brunk [2] and Prohorov [11].

THEOREM 3.1. (a) *Let $1 \leqslant p \leqslant 2$, let E be of R-type p, and $q \geqslant 1$. If (X_n) are independent zero-mean random vectors in E such that*

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\,\|X_n\|^{pq}}{n^{pq+1-q}} < \infty, \tag{3.1}$$

then $S_n/n \to 0$ a.s. in norm.

(b) *Conversely, if $q \geqslant 1$, $1 \leqslant p \leqslant 2$, and, for each $(x_i) \subset E$ such that $\sum \|x_i\|^{pq}/i^{pq+1-q} < \infty$,*

$$\sum_{i=1}^{n} r_i x_i/n \to 0 \quad \text{as } n \to \infty$$

a.s. in norm, then E is of R-type p.

Proof. (a) For $q = 1$ the theorem boils down to the Kolmogorov-Chung type strong law as mentioned above.

Assume $q > 1$. Then $\|S_n\|^{pq}$ is a real submartingale and, by the well-known Hajek-Rényi-Chow type inequality, we get

$$\varepsilon^{pq} P(\sup_{j\geqslant n} \|S_j/j\| > \varepsilon) = \varepsilon^{pq} \lim_{m\to\infty} P(\sup_{n\leqslant j\leqslant m} \|S_j/j\|^{pq} > \varepsilon^{pq}) \tag{3.2}$$

$$\leqslant n^{-pq}\,\mathrm{E}\,\|S_n\|^{pq} + \sum_{j=n+1}^{\infty} j^{-pq}\,\mathrm{E}(\|S_j\|^{pq} - \|S_{j-1}\|^{pq})$$

for every $\varepsilon > 0$.

By Proposition 2.1 and by Hölder's inequality,

$$\mathrm{E}\,\|S_j\|^{pq} \leqslant C\,\mathrm{E}\Big(\sum_{i=1}^{j} \|X_i\|^p\Big)^q \leqslant Cj^{q-1} \sum_{i=1}^{j} \mathrm{E}\,\|X_i\|^{pq},$$

so that by (3.1) and Kronecker's lemma we obtain

$$j^{-pq}\,\mathrm{E}\,\|S_j\|^{pq} \to 0 \quad \text{as } j \to \infty.$$

Also the series on the right-hand side of (3.2) converges because of Proposition 2.1. Hence, summing by parts,

$$\sum_{j=1}^{n} \big((j-1)^{-pq} + j^{-pq}\big)\,\mathrm{E}\,\|S_j\|^{pq} \leqslant \sum_{j=1}^{n} \big((j-1)^{-pq} + j^{-pq}\big) j^{q-1} \sum_{i=1}^{j} \mathrm{E}\,\|X_i\|^{pq}$$

$$\leqslant C \sum_{j=1}^{n} \mathrm{E}\,\|X_j\|^{pq}/j^{pq+1-q} + \sum_{i=1}^{n} \mathrm{E}\,\|X_i\|^{pq}/n^{pq+1-q}.$$

Therefore, for every $\varepsilon > 0$,

$$P(\sup_{j \geqslant n} \|S_j/j\| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

(b) Kahane's theorem (cf. [14], p. 275) states that, for any Banach space E and any p $(0 \leqslant p < \infty)$, all the $L_p(E)$-norms are equivalent on the span of $(r_i x_i)$, $(x_i) \subset E$. Hence, in view of the closed graph theorem, there exists C such that for all $(x_i) \subset E$

$$\mathrm{E}\Big\|\sum_{i=1}^{n} r_i x_i n^{-1}\Big\| \leqslant C\Big(\sum_{i=1}^{n} \frac{\|x_i\|^{pq}}{i^{pq+1-q}}\Big)^{1/pq},$$

so that

$$\mathrm{E}\Big\|\sum_{i=1}^{n} n^{-1} i^{1+(1-q)/pq} r_i x_i\Big\| \leqslant C\Big(\sum_{i=1}^{n} \|x_i\|^{pq}\Big)^{1/pq} \quad \text{for all } (x_i) \subset E.$$

Hence

$$\mathrm{E}\Big\|\sum_{i=1}^{n} r_i x_i\Big\| = \mathrm{E}\Big\|\sum_{i=n+1}^{2n} r_i x_{i-n}\Big\|$$
$$\leqslant n^{-(1-q)/pq}\, \mathrm{E}\Big\|\sum_{i=1}^{n} \frac{i^{1+(1-q)/pq}}{2n} r_i x_i + \sum_{i=n+1}^{2n} \frac{i^{1+(1-q)/pq}}{2n} r_i x_i\Big\|$$
$$\leqslant n^{-(1-q)/pq} C \cdot 2^{1/pq}\Big(\sum_{i=1}^{n} \|x_i\|^{pq}\Big)^{1/pq}.$$

Now, since for any α, β $(0 < \alpha, \beta < \infty)$ and $a_i \geqslant 0$ the inequality

$$\Big(\sum a_i^{\alpha}\Big)^{1/\alpha} \leqslant n^{1/\alpha - 1/\beta}\Big(\sum a_i^{\beta}\Big)^{1/\beta}$$

holds, we have

$$\mathrm{E}\Big\|\sum_{i=1}^{n} r_i x_i\Big\| \leqslant C \cdot 2^{1/pq} n^{-(1-q)/pq} n^{1/pq - 1/p}\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p} \leqslant C \cdot 2^{1/pq}\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}.$$

The following "rate of convergence" result for the weak law is associated with the strong law above.

THEOREM 3.2. *Let $1 \leqslant p \leqslant 2$ and $q \geqslant 1$. The following properties of a Banach space E are equivalent:*

(i) *E is of R-type p.*

(ii) *for every $\varepsilon > 0$ there exists C_ε such that for any independent zero-mean (x_i) in E*

$$\sum_{n=1}^{\infty} n^{-1} P(\|S_n/n\| > \varepsilon) \leqslant C_\varepsilon \sum_{n=1}^{\infty} \frac{\mathrm{E}\|X_n\|^{pq}}{n^{pq+1-q}}.$$

Proof. (i) $\Rightarrow$ (ii). By the Chebyshev, Marcinkiewicz-Zygmund (Proposition 2.1) and Hölder inequalities we get

$$\sum_{n=1}^{\infty} n^{-1} P(\|S_n\| > \varepsilon n) \leqslant \sum_{n=1}^{\infty} n^{-1} n^{-pq} \varepsilon^{-pq} \mathrm{E}\,\|S_n\|^{pq}$$

$$\leqslant \varepsilon^{-pq} C \sum_{n=1}^{\infty} n^{-1+(q-1)-pq} \sum_{k=1}^{n} \mathrm{E}\,\|X_k\|^{pq}$$

$$\leqslant C\varepsilon^{-pq} \sum_{k=1}^{\infty} \mathrm{E}\,\|X_k\|^{pq} \sum_{n=k}^{\infty} n^{-pq+q-2}$$

$$\leqslant C\varepsilon^{-pq} \sum_{k=1}^{\infty} \mathrm{E}\,\|X_k\|^{pq}/k^{pq+1-q}.$$

(ii) $\Rightarrow$ (i) follows directly from the proof of (b) in Theorem 3.1.

4. Marcinkiewicz-Zygmund's type strong laws and related rates of convergence.

THEOREM 4.1. *Let $1 < p < 2$. Then the following properties of a Banach space E are equivalent:*

(i) *l_p is not finitely representable in E.*

(ii) *For any sequence (X_i) of zero-mean independent random vectors in E with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^p$, the series*

$$\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}}$$

converges a.s. in norm.

(iii) *For any sequence (X_i) as in (ii), $S_n/n^{1/p} \to 0$ a.s.*

The proof of Theorem 4.1 will be based on the following

LEMMA 4.1. *Let $1 \leqslant p < 2$, let l_p be not representable in E, and let (X_n) satisfy assumptions of Theorem 4.1 (ii). Then the series*

$$\sum_{n=1}^{\infty} (X_n - \mathrm{E}Y_n)/n^{1/p},$$

where $Y_n = X_n I(\|X_n\| \leqslant n^{1/p})$, converges a.s.

Proof. Since

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(\|X_n\| > n^{1/p}) \leqslant C \sum_{n=1}^{\infty} P(|X_0| > n^{1/p}) \leqslant C_1 \mathrm{E}|X_0|^p < \infty,$$

in view of the Borel-Cantelli lemma it suffices to show that the series $\sum (Y_n - \mathrm{E}Y_n)/n^{1/p}$ converges a.s.

Let $r > p$. Then

$$\sum_{n=1}^{\infty} \mathrm{E}\,\|Y_n - \mathrm{E}Y_n\|^r n^{-r/p} \leqslant 2^{r+1} \sum_{n=1}^{\infty} \mathrm{E}\,\|Y_n\|^r n^{-r/p}$$

$$= 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \int_{\|X_n\| \leqslant n^{1/p}} \|X_n\|^r dP = 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \int_0^{n^{1/p}} t^r dP(\|X_n\| \leqslant t)$$

$$= 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \left(n^{r/p} P(\|X_n\| \leqslant n^{1/p}) - r \int_0^{n^{1/p}} P(\|X_n\| < t)\,dt\right)$$

$$\leqslant C_1 \sum_{n=1}^{\infty} \Big(1 - rn^{-r/p} \int_0^{n^{1/p}} t^{r-1} \big(1 - P(|X_0| > t)\big)\,dt\Big)$$

$$= C_1 \sum_{n=1}^{\infty} rn^{-r/p} \int_0^{n^{1/p}} t^{r-1} P(|X_0| > t)\,dt = C_1 \sum_{n=1}^{\infty} \int_0^1 P(|X_0 s^{-1/r}| > n^{1/p})\,ds$$

$$\leqslant C_2\, \mathrm{E}|X_0|^p \int_0^1 s^{-p/r} ds = C_2 \frac{r}{r-p} \mathrm{E}|X_0|^p < \infty.$$

By Maurey-Pisier's theorem (see [10] and [14], p. 371) and by assumption, there exists $r > p$ such that E is of R-type r. Therefore, the estimate above and Theorem V.7.5 in [14] give the desired a.s. convergence of $\sum (Y_n - \mathrm{E}Y_n) n^{-1/p}$.

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii). In view of Lemma 4.1 it is sufficient to prove the absolute convergence of the series $\sum \mathrm{E}Y_n n^{-1/p}$. Since $\mathrm{E}X_n = 0$ and $p > 1$, we have

$$\sum_{n=1}^{\infty} \|\mathrm{E}\,Y_n\|\, n^{-1/p} \leqslant \sum_{n=1}^{\infty} n^{-1/p} \int_{n^{1/p}}^{\infty} t\,dP(\|X_n\| \leqslant t)$$

$$= -\sum_{n=1}^{\infty} n^{-1/p} \int_{n^{1/p}}^{\infty} t\,dP(\|X_n\| > t)$$

$$= \sum_{n=1}^{\infty} (P|X_0| > n^{1/p}) + \int_1^{\infty} P(|X_0/s| > n^{1/p})\,ds \leqslant C\,\mathrm{E}|X_0|^p,$$

which gives (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows by a straightforward application of Kronecker's lemma.

(iii) $\Rightarrow$ (i). This implication is essentially due to Maurey and Pisier [10] (cf. also [14], p. 389). We quote the proof for the sake of completeness. In view of Kronecker's lemma it suffices to construct, in any Banach space E such that l_p is finitely representable in E, a sequence $(x_n) \subset E$,

$\|x_n\| \leqslant 1$, $n = 1, 2, \ldots$, such that for a sequence $(N_k) \subset N$, $N_k \to \infty$, for all choices of $\varepsilon_n = \pm 1$ and for all $k \in N$

$$N_k^{-1/p} \Big\| \sum_{i=1}^{N_k} \varepsilon_i x_i \Big\| > \frac{1}{2}. \tag{4.1}$$

Put $N_1 = 1$ and choose any $x_1 \in E$, $\|x_1\| = 1$. Suppose $N_1, \ldots, N_k$ and $x_1, \ldots, x_{N_k}$ have been chosen so that $\|x_i\| \leqslant 1$, $i = 1, \ldots, N_k$, and for all $\varepsilon_i = \pm 1$ inequality (4.1) is satisfied. Choose $N_{k+1} \in N$ large enough for

$$N_{k+1}^{-1/p} \left[\frac{2}{3} (N_{k+1} - N_k)^{1/p} - N_k \right] > \frac{1}{2}.$$

Since l_p is finitely representable in E, we can find $x_{N_k+1}, \ldots, x_{N_{k+1}}$ such that for all $(\alpha_k) \subset \boldsymbol{R}$

$$\frac{2}{3} \Big(\sum_{i=N_k+1}^{N_{k+1}} |\alpha_i|^p \Big)^{1/p} \leqslant \Big\| \sum_{i=N_k+1}^{N_{k+1}} \alpha_i x_i \Big\| \leqslant \Big(\sum_{i=N_k+1}^{N_{k+1}} |\alpha_i|^p \Big)^{1/p}.$$

Therefore

$$N_{k+1}^{-1/p} \Big\| \sum_{i=1}^{N_{k+1}} \varepsilon_i x_i \Big\| \geqslant N_{k+1}^{-1/p} \Big[\Big\| \sum_{i=N_k+1}^{N_{k+1}} \varepsilon_i x_i \Big\| - \Big\| \sum_{i=1}^{N_k} \varepsilon_i x_i \Big\| \Big]$$

$$> N_{k+1}^{-1/p} \left[\frac{2}{3} (N_{k+1} - N_k)^{1/p} - N_k \right] > \frac{1}{2} \quad \text{for all } \varepsilon_n = \pm 1.$$

For spaces E such that l_1 is not finitely representable in E, i.e., for B-convex spaces (see [14], Chapter VII), Lemma 4.1 permits to prove the following

THEOREM 4.2. *The following properties of a Banach space E are equivalent*:

(i) *l_1 is not finitely representable in E.*

(ii) *For any sequence (X_i) of independent zero-mean random vectors in E with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L \log^+ L$, the series*

$$\sum_{n=1}^{\infty} \frac{X_n}{n}$$

converges a.s.

(iii) *For any sequence (X_i) as in* (ii), *$S_n/n \to 0$ a.s. as $n \to \infty$.*

Proof. (i) $\Rightarrow$ (ii). In view of Lemma 4.1 it suffices to prove that $\sum \|\mathrm{E} Y_n\| n^{-1}$ converges whenever $X_0 \in L \log^+ L$. Since $\mathrm{E} X_n = 0$, by integration by parts we obtain

$$\sum_{n=1}^{\infty} \|\mathrm{E}Y_n\| n^{-1} \leqslant \sum_{n=1}^{\infty} n^{-1} \int_{n}^{\infty} t dP(\|X_n\| \leqslant t)$$

$$= \sum_{n=1}^{\infty} \left[P(\|X_n\| > n) + n^{-1} \int_{n}^{\infty} P(\|X_n\| > t) dt\right]$$

$$\leqslant C_1 \left[\mathrm{E}|X_0| + \sum_{n=1}^{\infty} n^{-1} \sum_{k=n}^{\infty} P(|X_0| > k)\right]$$

$$= C_1 \left[\mathrm{E}|X_0| + \sum_{k=1}^{\infty} \sum_{n=1}^{k} n^{-1} P(|X_0| > k)\right]$$

$$= C_1 \left[\mathrm{E}|X_0| + \sum_{k=1}^{\infty} (\log k) P(|X_0| > k)\right]$$

$$\leqslant C_1 \left[\mathrm{E}|X_0| + \mathrm{E}|X_0| \log^+ |X_0|\right] < \infty.$$

(ii) $\Rightarrow$ (iii) follows directly from Kronecker's lemma, and (iii) $\Rightarrow$ (i) can be proved exactly as (iii) $\Rightarrow$ (i) in Theorem 4.1.

THEOREM 4.3. (a) *Let E be a Banach space, $1 < p < 2$, and let $\alpha \geqslant 1/p$. Then l_p is not finitely representable in E if and only if for each independent zero-mean (X_i) in E with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^p$ we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leqslant i \leqslant n} \|S_i\| > n^{\alpha} \varepsilon\right) < \infty \quad \textit{for every } \varepsilon > 0.$$

(b) *Let E be a Banach space and let $1 \leqslant p < 2$. Then l_p is not finitely representable in E if and only if for each independent zero-mean (X_i) in E with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^p \log^+ L$ we have*

$$\sum_{n=1}^{\infty} n^{-1} (\log n) P(\|S_n\| > n^{1/p} \varepsilon) < \infty \quad \textit{for every } \varepsilon > 0.$$

Proof. (a) We prove first the sufficiency of the condition of l_p not being finitely representable in E. By Theorem 4.1, $S_n/n^{1/p} \to 0$ a.s. and, as is easy to see, also

$$M_n/n^{1/p} \to 0 \text{ a.s.,} \quad \text{where } M_u = \max_{1 \leqslant i \leqslant [u]} \|S_i\|, \; u \in \boldsymbol{R}, \; [u] = \text{entier } u.$$

Hence, if we introduce Chow's delayed sums

$$S_{u,v} = \sum_{1 \leqslant j \leqslant v} X_{[u]+j}, \quad u, v \in \boldsymbol{R},$$

we get

$$M_{n,n}n^{-1/p} \leqslant (M_n+M_{2n})n^{-1/p} \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

Now, in the case $\alpha = 1/p$, since $M_{2^n,2^n}$ $(n = 1, 2, \ldots)$ are independent, from the Borel-Cantelli lemma we infer that

$$\infty > \sum_{n=1}^{\infty} P(M_{2^n,2^n} > 2^{n/p}\varepsilon) = \sum_{n=1}^{\infty} P(M_{2^n} > 2^{n/p}\varepsilon) \geqslant \int_1^{\infty} P(M_{2^t} > 2^{(t+1)/p}\varepsilon)\,dt$$

$$> (\log 2)^{-1}\int_1^{\infty} u^{-1}P(M_u > 2^{1/p}\varepsilon u^{1/p})\,du \quad \text{for every } \varepsilon > 0,$$

so that $\sum n^{-1}P(M_n > n^{1/p}\varepsilon) < \infty$ for every $\varepsilon > 0$.

In the case $\alpha > 1/p$, for $m \geqslant 1$ we have

$$(m+1)^{\alpha p/(\alpha p-1)} \geqslant m^{\alpha p/(\alpha p-1)} + \frac{\alpha p}{\alpha p-1} m^{1/(\alpha p-1)} \geqslant m^{\alpha p/(\alpha p-1)} + m^{1/(\alpha p-1)},$$

so that the random variables $M_{m^{\alpha p/(\alpha p-1)},m^{1/(\alpha p-1)}}$, $m = 1, 2, \ldots$, are independent. Moreover, by Theorem 4.1,

$$m^{-\alpha/(\alpha p-1)}M_{m^{\alpha p/(\alpha p-1)},m^{1/(\alpha p-1)}} \leqslant m^{-\alpha/(\alpha p-1)}M_{m^{\alpha p/(\alpha p-1)},m^{\alpha p/(\alpha p-1)}} \to 0 \text{ a.s.} \quad \text{as } m \to \infty.$$

Therefore, again by the Borel-Cantelli lemma we obtain

$$\infty > \sum_{m=1}^{\infty} P(M_{m^{\alpha p/(\alpha p-1)},m^{1/(\alpha p-1)}} \geqslant m^{\alpha/(\alpha p-1)}\varepsilon)$$

$$= \sum_{m=1}^{\infty} P(M_{m^{1/(\alpha p-1)}} \geqslant m^{\alpha/(\alpha p-1)}\varepsilon)$$

$$\geqslant \int_1^{\infty} P\big(M_{t^{1/(\alpha p-1)}} \geqslant (t+1)^{\alpha/(\alpha p-1)}\varepsilon\big)\,dt$$

$$\geqslant (\alpha p-1)\int_1^{\infty} u^{\alpha p-1}P(M_u \geqslant 2^{\alpha/(\alpha p-1)}u^{\alpha}\varepsilon)\,du,$$

which gives the desired rate of convergence. The necessity of the condition of l_p not being representable in E follows directly from the example developed in the proof of (iii) $\Rightarrow$ (i) in Theorem 4.1.

(b) Sufficiency. We may assume that X_n's are symmetric. The case of zero expectations can be handled by adapting in the standard way the method presented below.

Put $Y_{kn} = X_k I\,(\|X_k\| < n^{1/p})$. Then

$$\sum_{n=1}^{\infty} n^{-1}(\log n)\, P(\|S_n\| > n^{1/p}\varepsilon) \leqslant \sum_{n=1}^{\infty} n^{-1}(\log n)\, P\Big(\bigcup_{k=1}^{n} (\|X_k\| > n^{1/p}\varepsilon)\Big) + $$
$$+ \sum_{n=1}^{\infty} n^{-1}(\log n)\, P\Big(\Big\|\sum_{k=1}^{n} Y_{kn}\Big\| > n^{1/p}\varepsilon\Big).$$

The series on the right-hand side can be estimated from above by

$$C \sum_{n=1}^{\infty} (\log n)\, P(|X_0| > n^{1/p}\varepsilon) \leqslant C_1 \operatorname{E}|X_0|^p \log^+ |X_0| < \infty,$$

and the convergence of the second series can be proved as follows.

Since l_p is not finitely representable in E, by Maurey-Pisier's theorem mentioned before there exists $\delta > 0$ such that E is of R-type $(p+\delta)$. Hence, making use of Chebyshev's inequality and integrating by parts we get

$$\sum_{n=1}^{\infty} n^{-1}(\log n)\, P\Big(\Big\|\sum_{k=1}^{n} Y_{kn}\Big\| > n^{1/p}\varepsilon\Big)$$
$$\leqslant C_1 \sum_{n=1}^{\infty} n^{-1-(p+\delta)/p}(\log n) \sum_{k=1}^{n} \operatorname{E}\|Y_{kn}\|^{p+\delta}$$
$$\leqslant C_2 \sum_{n=1}^{\infty} n^{-1-(p+\delta)/p}(\log n) \sum_{k=1}^{n} \int_0^{n^{1/p}} t^{p+\delta}\, dP(\|X_k\| \leqslant t)$$
$$\leqslant C_2 \sum_{n=1}^{\infty} n^{-(p+\delta)/p}(\log n) \int_0^{n^{1/p}} t^{p+\delta-1} P(|X_0| > t)\, dt$$
$$= C_2 \int_0^1 s^{\delta/p} \sum_{n=1}^{\infty} (\log n)\, P(|X_0 s^{-1/p}| > n^{1/p})\, ds$$
$$\leqslant C_2 \int_0^1 s^{\delta/p} \operatorname{E}|X_0 s^{-1/p}|^p \log^+ |X_0 s^{-1/p}|\, ds$$
$$\leqslant C_3 \operatorname{E}|X_0|^p \log^+ |X_0| \int_0^1 s^{-1+\delta/p}\, ds < \infty.$$

This completes the proof of the sufficiency.

The necessity can be obtained exactly as in (a).

COROLLARY 4.1. *If l_p is not finitely representable in E, $1 < p < 2$, and (X_i) are i.i.d. zero-mean random vectors in E with* $\operatorname{E}\|X_1\|^p < \infty$, *then*

$$P(\|S_n/n\| > \varepsilon) = o(n^{1-p}) \quad \textit{for every } \varepsilon > 0.$$

COROLLARY 4.2. *Let E be of R-type p, $1 < p \leqslant 2$, and let (X_i) be independent zero-mean vectors in E such that*

$$P(\|X_k\| > n) = o(n^{-p}) \tag{4.2}$$

uniformly in k. Then for every $\delta > 0, \varepsilon > 0$

$$P(\|S_n/n\| > \varepsilon) = o(n^{1-p+\delta}).$$

Proof. Since E is of R-type p, $l^{p-\delta}$ is not finitely representable in E for every $\delta > 0$. From (4.2) it follows also that X_k's have tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^{p-\delta}$. Therefore, by Theorem 4.3,

$$\sum_{n=1}^{\infty} n^{p-\delta-2} P(\|S_n/n\| > \varepsilon) < \infty,$$

so that

$$n^{p-\delta-2} P(\|S_n/n\| > \varepsilon) = o(n^{-1}),$$

which gives the corollary.

From Corollary 1.1 and Theorem 4.3 we get immediately

COROLLARY 4.3. *If $1 \leqslant p < 2$ and l_p is not finitely representable in E, then for any sequence (X_i) of independent zero-mean random vectors in E with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^p$ if $1 < p < 2$ and of an $X_0 \in L\log^+ L$ if $p = 1$ we have*

$$\sum_{n=1}^{\infty} n^{p-2} P(\sup_{k \geqslant n} \|S_k/k\| > \varepsilon) < \infty \quad \textit{for every } \varepsilon > 0.$$

5. Concluding remarks.

5.1. Brunk's type strong law of large numbers in Banach spaces can be also obtained by using the methods developed by Kuelbs and Zinn [6] (J. Zinn – oral communication). These methods use however a rather powerful tool of exponential inequalities in Banach spaces.

5.2. In the i.i.d. case an alternative proof of results concerning rates of convergence is possible by applying a theorem of Jain [4] who proved that by and large, real-line "rates of convergence" results remain valid in general Banach spaces as long as S_n/n^{α} are bounded in probability. In presence of our geometric restrictions on E the latter is, of course, implied by the Marcinkiewicz-Zygmund type strong law. Other extensions along the lines of Jain's paper are also possible (e.g., Orlicz space type moment assumptions). We stuck to a simpler set up to emphasize the relation between geometric and probabilistic phenomena in E.

5.3. It also follows from Jain's paper that, for any Banach space E and any i.i.d. zero-mean (X_i), if $X_1 \in L_1(E)$, then $\sum n^{\alpha-2} P(\|S_n/n^{\alpha}\| > \varepsilon) < \infty$ for every $\varepsilon > 0$, and if, for an $\alpha \geqslant 1/p$, $\sum n^{\alpha p-2} P(\|S_n/n^{\alpha}\| > \varepsilon) < \infty$ for every $\varepsilon > 0$, then $EX_1 = 0$ and $E\|X_1\|^p < \infty$.

5.4. If E is a Hilbert space, we can prove a result somewhat stronger than Corollary 4.2. Namely, if (X_i) are i.i.d. in E with $EX_1 = 0$ and

$P(\|X_1\| \geqslant n) = o(n^{-p})$ for a $p > 1$, then $P(\|S_n/n\| > \varepsilon) = o(n^{1-p})$ for every $\varepsilon > 0$.

5.5. The validity of the Marcinkiewicz-Zygmund strong law of large numbers for i.i.d. (X_n) in E is equivalent to E being of R-type p (A. de Acosta — oral communication).

5.6. Taylor and Wei [13] studied weighted sums of independent random vectors in Banach spaces under moment conditions similar to ours, but obtained only weak laws for them (i.e., with convergence in probability).

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