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ON CONVERGENCE OF L_1-BOUNDED MARTINGALES INDEXED BY DIRECTED SETS

BY

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Abstract. Let $(\mathscr{F}_t)$ be an increasing family of σ-algebras indexed by a directed set J. In this paper* it is shown that every L_1-bounded real-valued martingale converges essentially if and only if a weak type of maximal inequality holds for all martingales. A new covering condition C stated in terms of multivalued stopping times is introduced and characterized in terms of maximal inequalities. C is shown to be strictly weaker than the Vitali condition V, than SV (see [15]), and also sigma-SV. Under C, L_1-bounded martingales taking values in a Banach space with the Radon-Nikodým property converge essentially.

It was shown by Dieudonné [5] that Doob's martingale convergence theorem in general fails when the index set is not totally ordered. In 1956, Krickeberg introduced the *Vitali condition* V (also denoted by V_0 and V_∞) on the σ-algebras, and proved that V was sufficient for essential convergence of L_1-bounded martingales ([9], or [17], p. 99). In a recent note [15], we showed that V was not *necessary*, replacing it by the condition SV, the logical union of conditions SV(m), $m = 0, 1, 2, \ldots$ Informally, V may be stated as follows:

Every adapted 2-valued process can be stopped by a (genuine) stopping time τ as close as desired to the essential lim sup.

The condition SV(m) allows stopping by *multivalued* stopping times, with excess bounded in L_∞ by the integer m. A new condition C introduced in the sequel ensures the approximation of ess lim sup up to ε, the excess of the stopping time being bounded in L_∞ by a number depending on ε (precise definitions are given in Section 1). We show in Section 4 that C

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is strictly weaker than SV, and also than sigma-SV (the space is a countable union of properly measurable sets each of which satisfies SV). In Section 3, C is shown to be sufficient for essential convergence of L_1-bounded martingales taking values in a Banach space with the Radon-Nikodým property.

As is our wont, at the beginning of the paper, in Section 1, we characterize our new condition by appropriate maximal inequalities. In Section 2 we connect maximal inequalities with convergence of real-valued martingales. The main result, Theorem 2.2, asserts that every L_1-bounded martingale converges essentially if and only if every martingale satisfies a simple maximal inequality. Section 3 proves convergence in the Banach valued case and Section 4 compares different covering conditions.

Martingale theory in part traces its origins to the point-derivation theory. Thus Krickeberg's condition V was an adaptation of R. de Possel's abstraction of the classical Vitali property; similarly, SV is the stochastic version of the Besicovitch property (cf. [7] and [8]). We attempt to repay the debt, offering in [16] a point-derivation version of condition C, sufficient to obtain Lebesgue's theorem.

In an independent work, Astbury [2] introduced a remarkable new sufficient condition A for convergence of real-valued L_1-bounded martingales. In [16] we show that, in the presence of a countable cofinal subset, A is equivalent to C; also other equivalent conditions are given.

1. Condition C and maximal inequalities. Let J be a *directed set filtering to the right*, i.e., a set of indices partially ordered by $\leqslant$, such that for each pair t_1, t_2 of elements of J there exists an element t_3 of J such that $t_1 \leqslant t_3$ and $t_2 \leqslant t_3$. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Sets and random variables are considered equal if they are equal almost surely. All considered sets and functions are measurable. Let $X = (X_t)$ be a family of random variables taking values in $\bar{R}$. The *stochastic upper limit* of X,

$$\tilde{X} = \mathrm{s}\limsup X_t,$$

is the essential infimum of the set of random variables Y such that $\lim P(Y < X_t) = 0$. The *stochastic lower limit* of X is

$$\mathrm{s}\liminf X_t = -\mathrm{s}\limsup(-X_t).$$

The directed family is said to *converge stochastically* (or *in probability*) if

$$\mathrm{s}\limsup X_t = \mathrm{s}\liminf X_t.$$

The *essential upper limit* of X, $X^* = \limsup X_t$, is defined by

$$X^* = \operatorname*{ess\,inf}_s (\operatorname*{ess\,sup}_{t \geqslant s} X_t).$$

The *essential lower limit* of X, $X_* = \liminf X_t$, is $-\limsup(-X_t)$.

The directed family X is said to *converge essentially* if $X^* = X_*$; this common value is called the *essential limit* of X, $\lim X_t$. If $A = (A_t)$ is a directed family of measurable sets, the *stochastic upper limit* of A, $\tilde{A} = \operatorname{s\,lim\,sup} A_t$, is the set defined by

$$1_{\tilde{A}} = \operatorname{s\,lim\,sup} 1_{A_t};$$

the *essential upper limit* of A, $A^* = \limsup A_t$, is the set defined by

$$1_{A^*} = \limsup 1_{A_t}.$$

A *stochastic basis* $(\mathscr{F}_t)$ is an increasing family of sub-σ-algebras of $\mathscr{F}$ (i.e., for every $s \leqslant t$, $\mathscr{F}_s \subset \mathscr{F}_t$). A *stochastic process* X is a family of random variables $X_t: \Omega \to \boldsymbol{R}$ such that X_t is $\mathscr{F}_t$-measurable for every t. The process is called *integrable* (*positive*) if X_t is integrable (positive) for every t. A family of sets A is *adapted* if $A_t \in \mathscr{F}_t$ for every $t \in J$.

Denote by $\mathscr{J}$ the set of finite subsets of J. An (*incomplete*) *multivalued stopping time* is a map τ from Ω (from a subset of Ω called $D(\tau)$) to $\mathscr{J}$ such that $R(\tau) = \bigcup \tau(\omega)$ is finite and such that, for every $t \in J$,

$$\{\tau = t\} \overset{\text{def}}{=} \{\omega \in \Omega : t \in \tau(\omega)\} \in \mathscr{F}_t$$

(cf. [13]). Denote by M (IM) the set of (incomplete) multivalued stopping times. A *simple stopping time* is an element τ of M such that, for every ω, $\tau(\omega)$ is a singleton; the set of simple stopping times is denoted by T. The *excess function* of $\tau \in IM$ is

$$e_\tau = \sum 1_{\{\tau = t\}} - 1_{D(\tau)}.$$

Let σ and τ be in IM; we say that $\sigma \leqslant \tau$ if, for every s and every t, $\{\sigma = s\} \cap \{\tau = t\} \neq \emptyset$ implies $s \leqslant t$. For the order $\leqslant$, M is a directed set filtering to the right. Let $\tau \in IM$; if X is a positive stochastic process, we set

$$X(\tau) = \sup (1_{\{\tau = t\}} X_t);$$

if A is an adapted family of sets, we set

$$A(\tau) = \bigcup_t (\{\tau = t\} \cap A_t).$$

Clearly, $1_{A(\tau)} = 1_A(\tau)$ for every $\tau \in IM$. The stochastic basis $(\mathscr{F}_t)$ satisfies the *covering condition* C if for every $\varepsilon > 0$ there exists a constant $M_\varepsilon > 0$ such that for every adapted family of sets A there exists $\tau \in IM$ with

$$e_\tau \leqslant M_\varepsilon \quad \text{and} \quad P[A^* \backslash A(\tau)] \leqslant \varepsilon.$$

The following theorem gives several equivalent formulations of the covering condition C in terms of maximal inequalities:

THEOREM 1.1. *Let* $(\mathscr{F}_t)$ *be a stochastic basis. The following conditions are equivalent*:

(1) $(\mathscr{F}_t)$ *satisfies the condition* C.

(2) *For every* $\varepsilon > 0$ *there exists a number* $M_\varepsilon > 0$ *such that for every adapted family of sets* A *there exists* $\tau \in IM$ *with*

$$e_\tau \leqslant M_\varepsilon \quad \text{and} \quad P[A^* \Delta A(\tau)] \leqslant \varepsilon.$$

(3) *For every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that for every adapted family of sets* A *there exists* $\tau \in IM$ *with*

$$e_\tau \leqslant M_\varepsilon \quad \text{and} \quad P(A^*) - P[A(\tau)] \leqslant \varepsilon.$$

(4) *There exists a constant* $\alpha > 0$ *such that for every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that for every adapted family of sets* A *there exists* $\tau \in IM$ *with*

$$e_\tau \leqslant M_\varepsilon \quad \text{and} \quad P[A^* \cap A(\tau)] \geqslant \alpha P(A^*) - \varepsilon.$$

(5) *For every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that for every adapted family of sets* A *satisfying* $P(A^*) > \varepsilon$ *there exists* $\tau \in IM$ *with*

$$1 \leqslant \|e_\tau\|_\infty \leqslant M_\varepsilon P[A(\tau)].$$

(6) $\lim\limits_{n \to \infty} \{\sup\limits_{A} P[A^* \backslash \mathrm{s}\limsup\limits_{e_\tau \leqslant n} A(\tau)]\} = 0.$

(7) *For every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that, for every* $\lambda > 0$ *and every positive stochastic process* X,

$$P[X^* \geqslant \lambda] \leqslant \varepsilon + \frac{1}{\lambda} \limsup_{e_\tau \leqslant M_\varepsilon} \mathrm{E}[X(\tau)].$$

(8) *There exists* $K > 0$ *such that for every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that, for every adapted family of sets* A,

$$P(A^*) \leqslant \varepsilon + K \limsup_{e_\tau \leqslant M_\varepsilon} P[A(\tau)].$$

Proof. Obviously, (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3) and (7) $\Rightarrow$ (8). It is easy to see that, given any index s, one may require the stopping times given in (1)-(5) to be larger than s, and in M. Indeed, given $\tau \in IM$, there exists $\tau' \in M$ with $\tau'|_{D(\tau)} = \tau$ and $e_{\tau'} = e_\tau$.

(3) $\Rightarrow$ (7). Fix $\varepsilon > 0$; let X be a positive stochastic process and let $\lambda > 0$. Fix $s \in J$ and δ, $0 < \delta < \lambda$; set $A_t = \{X_t > \lambda - \delta\}$ if $t \geqslant s$, $A_t = \emptyset$ otherwise, and let $\tau \in M$ be such that $e_\tau \leqslant M_\varepsilon$ and $P(A^*) - P[A(\tau)] < \varepsilon$. Then

$$P[X^* > \lambda] \leqslant P[A^*] \leqslant P[A(\tau)] + \varepsilon \leqslant \varepsilon + P\big[\bigcup (A_t \cap \{\tau = t\})\big]$$
$$\leqslant \varepsilon + \frac{1}{\lambda - \delta} \mathrm{E}[\sup (1_{\{\tau = t\}} X_t)] \leqslant \varepsilon + \frac{1}{\lambda - \delta} \sup_{\tau \geqslant s, e_\tau \leqslant M_\varepsilon} \mathrm{E}[X(\tau)].$$

The maximal inequality follows on letting $s \to \infty$ and $\delta \to 0$.

(8) $\Rightarrow$ (4). We may and do assume that $K > 1$. Fix $\varepsilon > 0$ and let A be an adapted family of sets such that $P(A^*) > 0$. Choose $s \in J$ such that

$$P[\operatorname*{ess\,sup}_{t \geq s} A_t \backslash A^*] \leqslant \varepsilon,$$

and choose $\tau \in M$ such that

$$\tau \geqslant s, \quad e_\tau \leqslant M_\varepsilon \quad \text{and} \quad |P[A(\tau)] - \limsup_{e_\sigma \leqslant M_\varepsilon} P[A(\sigma)]| \leqslant \varepsilon.$$

Then

$$A(\tau) \subset \operatorname*{ess\,sup}_{t \geqslant s} A_t,$$

and, applying (8), we obtain

$$\begin{aligned} P[A^* \cap A(\tau)] &\geqslant P[\operatorname*{ess\,sup}_{t \geqslant s} A_t \cap A(\tau)] - P[\operatorname*{ess\,sup}_{t \geqslant s} A_t \backslash A^*] \\ &\geqslant P[A(\tau)] - \varepsilon \geqslant \limsup_{e_\sigma \leqslant M_\varepsilon} P[A(\sigma)] - 2\varepsilon \\ &\geqslant \frac{1}{K}[P(A^*) - \varepsilon] - 2\varepsilon \geqslant \frac{P(A^*)}{K} - 3\varepsilon. \end{aligned}$$

(4) $\Rightarrow$ (5). Fix $\varepsilon > 0$ and let A be an adapted family of sets such that $P(A^*) \geqslant \varepsilon$. Applying (4), choose $\tau_1 \in IM$ such that

$$e_{\tau_1} \leqslant M_{\alpha\varepsilon/4} \quad \text{and} \quad P[A^* \cap A(\tau_1)] \geqslant \alpha P(A^*) - \frac{\alpha\varepsilon}{4}.$$

Let s_1 be an index larger than τ_1, and set $A_t^1 = A_t \backslash A(\tau_1)$ if $t \geqslant s$, and $A_t^1 = \emptyset$ otherwise; then

$$\limsup A_t^1 = A^* \backslash A(\tau_1).$$

Let $\tau_2 \in IM$ satisfy

$$\tau_2 \geqslant s_1, \quad e_{\tau_2} \leqslant M_{\alpha\varepsilon/4} \quad \text{and} \quad P[(A^* \backslash A(\tau_1)) \cap A(\tau_2)] \geqslant \alpha P[A^* \backslash A(\tau_1)] - \frac{\alpha\varepsilon}{4}.$$

Define by induction an increasing sequence of stopping times $\tau_n \in IM$ such that $e_{\tau_n} \leqslant M_{\alpha\varepsilon/4}$ and

$$P[(A^* \backslash \bigcup_{j<n} A(\tau_j)) \cap A(\tau_n)] \geqslant \alpha P[A^* \backslash \bigcup_{j<n} A(\tau_j)] - \frac{\alpha\varepsilon}{4}.$$

Then, for every n,

$$P[A^*\setminus \bigcup_{j\leqslant n} A(\tau_j)] \leqslant (1-\alpha)P[A^*\setminus \bigcup_{j<n} A(\tau_j)]+\frac{\alpha\varepsilon}{4}$$

$$\leqslant (1-\alpha)^n P(A^*)+\frac{\alpha\varepsilon}{4}[1+(1-\alpha)+\ldots+(1-\alpha)^{n-1}]$$

$$\leqslant (1-\alpha)^n P(A^*)+\frac{\varepsilon}{4}.$$

Choose N such that $(1-\alpha)^N \leqslant 1/12$; define $\tau \in IM$ by

$$\{\tau = t\} = \{\tau_j = t\} \cap A_t \cap \big(\bigcup_{k<j} A(\tau_k)\big)^c \quad \text{for every } t\in R(\tau_j),\ j=1,\ldots,N.$$

Then $e_\tau \leqslant M_{\alpha\varepsilon/4}$, and since $A(\tau) = \bigcup_{j\leqslant N} A(\tau_j)$, we have

$$P[A(\tau)\cap A^*] \geqslant 2P(A^*)\cdot 3^{-1}.$$

Let t be an index larger than τ; similarly define a multivalued stopping time $\sigma \geqslant t$ such that

$$e_\sigma \leqslant M_{\alpha\varepsilon/4} \quad \text{and} \quad P[A(\sigma)\cap A^*] \geqslant 2P(A^*)\cdot 3^{-1}.$$

The multivalued stopping time ϱ defined by $\{\varrho = s\} = \{\sigma = s\}\cup\{\tau = s\}$ satisfies

$$1 \leqslant \|e_\varrho\|_\infty \leqslant 2[M_{\alpha\varepsilon/4}+1] \leqslant 2[M_{\alpha\varepsilon/4}+1]\,P(A^*)\varepsilon^{-1}$$

$$\leqslant 3[M_{\alpha\varepsilon/4}+1]\,\varepsilon^{-1}P[A(\varrho)].$$

(5) $\Rightarrow$ (2). Fix $\varepsilon > 0$ and let A be an adapted family of sets such that $P(A^*) \geqslant \varepsilon$. Choose an index s_1 such that

$$P[\operatorname{ess\,sup}_{t\geqslant s_1} A_t\setminus A^*] \leqslant \frac{\varepsilon}{2},$$

and let $\tau_1 \in IM$ satisfy

$$\tau_1 \geqslant s_1 \quad \text{and} \quad 1 \leqslant \|e_{\tau_1}\|_\infty \leqslant M_{\varepsilon/2}P[A(\tau_1)].$$

Then $e_{\tau_1} \leqslant M_{\varepsilon/2}$ and $P[A(\tau_1)] \geqslant 1/M_{\varepsilon/2}$. If $P[A^*\setminus A(\tau_1)] \geqslant \varepsilon/2$, let s_2 be an index larger than τ_1, and apply (5) to the adapted family of sets defined by $A_t\setminus A(\tau_1)$ if $t \geqslant s_2$ and by $\emptyset$ otherwise; there exists $\tau_2 \in IM$ such that

$$\tau_2 \geqslant s_2 \quad \text{and} \quad 1 \leqslant \|e_{\tau_2}\|_\infty \leqslant M_{\varepsilon/2}P[A(\tau_2)\setminus A(\tau_1)].$$

If $\tau_1,\ldots,\tau_k$ have been defined and if

$$P[A^*\setminus \bigcup_{j\leqslant k} A(\tau_j)] \geqslant \frac{\varepsilon}{2},$$

let s_{k+1} be an index larger than τ_k, and apply (5) to the adapted family of sets defined by $A_t \setminus \bigcup_{j \leqslant k} A(\tau_j)$ if $t \geqslant s_{k+1}$ and by $\emptyset$ otherwise; there exists $\tau_{k+1} \in IM$ such that

$$\tau_{k+1} \geqslant s_{k+1} \quad \text{and} \quad 1 \leqslant \|e_{\tau_{k+1}}\|_\infty \leqslant M_{\varepsilon/2} P\big[A(\tau_{k+1}) \setminus \bigcup_{j \leqslant k} A(\tau_j)\big].$$

Let N be the first integer such that

$$P\big[A^* \setminus \bigcup_{j \leqslant N} A(\tau_j)\big] < \frac{\varepsilon}{2};$$

define $\tau \in IM$ by

$$\{\tau = t\} = \{\tau_k = t\} \setminus \bigcup_{j<k} A(\tau_j) \quad \text{for every } t \in R(\tau_k),\ k = 1, \ldots, N.$$

Clearly, $e_\tau \leqslant M_{\varepsilon/2}$ and $P[A^* \setminus A(\tau)] < \varepsilon/2$; therefore

$$P[A^* \Delta A(\tau)] \leqslant P[\operatorname*{ess\,sup}_{t \geqslant s_1} A_t \setminus A^*] + P[A^* \setminus A(\tau)] \leqslant \varepsilon.$$

If $P(A^*) < \varepsilon$, then the proof of the existence of a multivalued stopping time τ such that $e_\tau \leqslant M_{\varepsilon/2}$ and $P[A^* \Delta A(\tau)] \leqslant \varepsilon$ is trivial.

(1) $\Rightarrow$ (6). For every integer n and every adapted family of sets A, put

$$\tilde{A}_n = \operatorname{s\,lim\,sup}_{e_\tau \leqslant n} A(\tau);$$

clearly, $\tilde{A}_n$ is an increasing sequence of subsets of A^*. By the definition of s lim sup, $\tilde{A}_n$ is the smallest set C such that

$$\lim_{e_\tau \leqslant n} P[A(\tau) \setminus C] = 0.$$

Hence, for every n,

$$\lim_{e_\tau \leqslant n} P[A(\tau) \cap (A^* \setminus \tilde{A}_n)] = \lim_{e_\tau \leqslant n} P[A(\tau) \cap \tilde{A}_n^c] - \lim_{e_\tau \leqslant n} P[A(\tau) \cap A^{*c}] = 0.$$

Fix $\varepsilon > 0$, and let $s \in J$; there exists $\tau \in M$ such that $e_\tau \leqslant M_\varepsilon$, $\tau \geqslant s$ and

$$P[(A^* \setminus \tilde{A}_{M_\varepsilon}) \cap A(\tau)] \geqslant P[A^* \setminus \tilde{A}_{M_\varepsilon}] - \varepsilon.$$

Hence

$$0 = \limsup_{e_\tau \leqslant M_\varepsilon} P[(A^* \setminus \tilde{A}_{M_\varepsilon}) \cap A(\tau)] \geqslant P[A^* \setminus \tilde{A}_{M_\varepsilon}] - \varepsilon$$

holds for every adapted family of sets A. It follows that

$$\sup_A P[A^* \setminus \tilde{A}_{M_\varepsilon}] \leqslant \varepsilon.$$

(6) $\Rightarrow$ (3). The proof depends on the following lemma [14], which may be interpreted to mean that the *stochastic* version of the condition SV(N) is always true:

LEMMA 1.1. *Let A be an adapted family of sets and let N be an integer. Then for every $\varepsilon > 0$ and every index s there exists $\tau \in M$ with*

$$\tau \geqslant s, \quad e_\tau \leqslant N \quad \textit{and} \quad P[\operatorname{s\,lim\,sup}_{e_\sigma \leqslant N} A(\sigma) \backslash A(\tau)] < \varepsilon.$$

Now suppose that (6) holds, and for every adapted family of sets A put

$$\tilde{A}_n = \operatorname{s\,lim\,sup}_{e_\sigma \leqslant n} A(\sigma).$$

Fix $\varepsilon > 0$, let N be such that

$$\sup_A P[A^* \backslash \tilde{A}_N] < \varepsilon,$$

and fix a family A. Applying Lemma 1.1, we may choose $\tau \in M$ with $e_\tau \leqslant N$ and $P[\tilde{A}_N \backslash A(\tau)] \leqslant \varepsilon$, which implies

$$PA^* \leqslant P\tilde{A}_N + \varepsilon \leqslant P[A(\tau)] + 2\varepsilon.$$

Thus the proof of the theorem is completed.

The following proposition states that C implies a maximal inequality for positive submartingales:

PROPOSITION 1.1. *Let $(\mathscr{F}_t)$ be a stochastic basis satisfying the condition* C. *For every $\varepsilon > 0$ there exists a number $M_\varepsilon > 0$ such that for every positive submartingale X and $\lambda > 0$ we have*

$$P(\limsup X_t \geqslant \lambda) \leqslant \left(\frac{M_\varepsilon}{\lambda} \lim \mathrm{E} X_t\right) \vee \varepsilon.$$

Proof. Fix $\varepsilon > 0$, $\lambda > 0$ and let X be a positive submartingale. If $P(X^* \geqslant \lambda) > \varepsilon$, fix $M > 0$, let $\tau \in IM$ satisfy $e_\tau \leqslant M$, and let v be an index larger than τ; then

$$\begin{aligned} \mathrm{E}[X(\tau)] &\leqslant \mathrm{E}\left[\sum 1_{\{\tau = t\}} X_t\right] \leqslant \mathrm{E}\left[\sum 1_{\{\tau = t\}} X_v\right] \\ &\leqslant \mathrm{E}[(e_\tau + 1) X_v] \leqslant (M+1) \lim \mathrm{E} X_t. \end{aligned}$$

Hence applying Theorem 1.1 (7) we have, letting $M = M_{\varepsilon/2}$,

$$\begin{aligned} P(X^* \geqslant \lambda) &\leqslant \frac{\varepsilon}{2} + \frac{M_{\varepsilon/2}+1}{\lambda} \lim \mathrm{E} X_t \\ &\leqslant 2^{-1} P[X^* \geqslant \lambda] + \frac{M_{\varepsilon/2}+1}{\lambda} \lim \mathrm{E} X_t. \end{aligned}$$

Therefore,

$$P(X^* > \lambda) \leqslant [2(M_{\varepsilon/2}+1)\lambda^{-1} \lim \mathrm{E} X_t] \vee \varepsilon,$$

which completes the proof.

2. Maximal inequalities and essential convergence of martingales: Real case. Recall that an *ordered stopping time* is a simple stopping time τ such that the elements $t_1, t_2, \ldots, t_n$ in the range of τ are linearly ordered. Denote by T' the set of ordered stopping times. The forthcoming lemma states that the stochastic Vitali condition holds for every stochastic basis. The proof, being similar to that of (6) $\Rightarrow$ (3) in Theorem 1.1, is omitted. The lemma is completely proved in [14].

LEMMA 2.1. *Let A be an adapted family of sets. Then for every $\varepsilon > 0$ and $s \in J$ there exists $\tau \in T'$ such that*

$$\tau \geqslant s \quad \text{and} \quad P\big(\mathrm{s}\limsup A_t \Delta A(\tau)\big) < \varepsilon.$$

The following theorem gives the stochastic maximal inequality; part (i) appears in [14].

THEOREM 2.1. *Let X be a positive stochastic process.*

(i) *For every $\lambda > 0$,*

$$P(\mathrm{s}\limsup X_t \geqslant \lambda) \leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} \mathrm{E}\, X(\tau).$$

(ii) *If, in addition, $\big(X(\tau), \tau \in T'\big)$ is uniformly integrable, then for every $\lambda > 0$, letting $A = \{\mathrm{s}\limsup X_t \geqslant \lambda\}$, we have*

$$P(A) \leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} \int_A X(\tau)\, dP.$$

Remark. For a positive uniformly integrable submartingale X, $\big(X(\tau), \tau \in T\big)$ is always uniformly integrable.

Proof. Fix a positive process X and $\lambda > 0$. For a number α with $0 < \alpha < \lambda$, set $A_t = \{X_t > \lambda - \alpha\}$; then

$$\{\mathrm{s}\limsup X_t \geqslant \lambda\} \subset \mathrm{s}\limsup A_t.$$

Given $\varepsilon > 0$, choose $s \in J$ and $\tau \in T'$ such that $\tau \geqslant s$ and, letting $\tilde{A} = \mathrm{s}\limsup A_t$, $P\big(\tilde{A} \Delta A(\tau)\big) \leqslant \varepsilon$. Then we have

$$P(\tilde{A}) \leqslant P[A(\tau)] + \varepsilon \leqslant \frac{1}{\lambda - \alpha} \mathrm{E}[1_{A(\tau)} X(\tau)] + \varepsilon.$$

The maximal inequality in (i) follows on letting $s \to \infty$, $\alpha \to 0$ and $\varepsilon \to 0$. If $\big(X(\tau), \tau \in T'\big)$ is uniformly integrable, then given $\delta > 0$ choose $\varepsilon < \delta$ such that $P(B) < 2\varepsilon$ implies

$$\sup_{\tau \in T'} \mathrm{E}[1_B X(\tau)] \leqslant \delta\lambda.$$

Let $\alpha < \lambda/2$ be such that $P(\tilde{A}\setminus\{\tilde{X} \geqslant \lambda\}) < \varepsilon$, and given an index $s \in J$ choose $\tau \in T'$ such that $\tau \geqslant s$ and $P(\tilde{A}\Delta A(\tau)) \leqslant \varepsilon$. Then

$$P(\tilde{A}) \leqslant \frac{1}{\lambda-\alpha} \mathrm{E}\,[1_{\{\tilde{X}\geqslant\lambda\}} X(\tau)] + \frac{\delta\lambda}{\lambda-\alpha} + \varepsilon \leqslant \frac{1}{\lambda-\alpha} \mathrm{E}\,[1_{\{\tilde{X}\geqslant\lambda\}} X(\tau)] + 3\delta.$$

The maximal inequality in (ii) follows on letting $s \to \infty$, $\alpha \to 0$ and $\delta \to 0$.

We now characterize the convergence of L_1-bounded martingales in terms of maximal inequalities for martingales. A non-negative, finitely additive set function on an algebra $\mathscr{A}$ is called a *charge*. A *pure charge* on $\mathscr{A}$ is a charge which does not dominate any non-trivial measure. Two charges λ and μ are said to be *nearly orthogonal* if for every $\varepsilon > 0$ and every $\delta > 0$ there exists $A \in \mathscr{A}$ such that $\lambda(A) < \varepsilon$ and $\mu(A^c) < \delta$. The following known variant of the Yosida-Hewitt theorem [4] gives the decomposition of a charge into a measure and a pure charge nearly orthogonal to this measure.

Proposition 2.1. *Let λ be a charge defined on an algebra $\mathscr{A}$. Then λ admits a unique decomposition $\lambda = \lambda_m + \lambda_c$, where λ_m is a measure and λ_c is a pure charge. Moreover, λ_c is nearly orthogonal to every measure.*

Lemma 2.2. *Let $(\mathscr{F}_t)$ be a stochastic basis such that there exists a function $M\colon \boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ which for every $\lambda > 0$ satisfies:*

(1) $\lim_{\varepsilon\to 0} M(\lambda, \varepsilon) = 0$;

(2) *for every positive martingale $\boldsymbol{X}$, $\lim \mathrm{E} X_t \leqslant \varepsilon$ implies*

$$P(\limsup X_t \geqslant \lambda) \leqslant M(\lambda, \varepsilon).$$

Let $\boldsymbol{Z}$ be a positive submartingale such that the charge π defined on $\mathscr{A} = \bigcup \mathscr{F}_t$ by

$$\pi(A) = \lim \int_A Z_t \, dP$$

is nearly orthogonal to P.

Then Z_t converges essentially to zero.

Proof. Fix $\delta > 0$, and using the near orthogonality of P and π, choose a sequence (A_k) of sets in $\mathscr{A}$ such that

$$\sum P(A_k) < \delta \quad \text{and} \quad \lim M(k^{-1}, \pi(A_k^c)) = 0;$$

here each set A_k is measurable with respect to some $\mathscr{F}_t$, say $\mathscr{F}_{t_k}$. For every k, the process $(Z_t 1_{A_k^c})_{t \geqslant t_k}$ is a positive L_1-bounded submartingale such that

$$\lim \mathrm{E}\,[1_{A_k^c} Z_t] = \pi(A_k^c).$$

This process can be extended to a submartingale $\boldsymbol{S}$ with respect to $(\mathscr{F}_t)$ by setting $S_s = 0$ if $t_k \leqslant s$ fails. Denote by T the subset of M composed of single-valued stopping times, i.e., τ such that $e_\tau = 0$. Let X be the Snell supermartingale corresponding to $\boldsymbol{S}$, i.e.,

$$X_t = \operatorname*{ess\,sup}_{t \leqslant \tau \in T} \mathrm{E}^{\mathscr{F}_t} S(\tau);$$

we prove that X is a martingale which dominates $\boldsymbol{S}$. (The use of X_t was suggested to us by C. Stegall; see also [18].) Fix indices $s \leqslant t$; let $\sigma \in T$ be larger than s, and let u be an index such that $u \geqslant \sigma$ and $u \geqslant t$. By the submartingale property, $S(\sigma) \leqslant \mathrm{E}^{\mathscr{F}_\sigma} S_u$. Therefore,

$$\begin{aligned} X_s &= \operatorname*{ess\,sup}_{s \leqslant \sigma \in T} \mathrm{E}^{\mathscr{F}_s} S(\sigma) = \operatorname*{ess\,sup}_{t \leqslant \tau \in T} \mathrm{E}^{\mathscr{F}_s} S(\tau) \\ &= \operatorname*{ess\,sup}_{t \leqslant \tau \in T} \mathrm{E}^{\mathscr{F}_s} [\mathrm{E}^{\mathscr{F}_t} S(\tau)] \leqslant \mathrm{E}^{\mathscr{F}_s} [\operatorname*{ess\,sup}_{t \leqslant \tau \in T} \mathrm{E}^{\mathscr{F}_t} S(\tau)], \end{aligned}$$

which shows that X is a submartingale, and hence a martingale. Furthermore, since for every index t there exists a sequence $\tau_n \in T$ such that

$$X_t = \lim_n \uparrow \mathrm{E}^{\mathscr{F}_t} S(\tau_n),$$

we have

$$\mathrm{E} X_t \leqslant \sup_{t \leqslant \tau \in T} \mathrm{E} S(\tau) = \sup_{u \geqslant t} \mathrm{E} S_u = \pi(A_k^c).$$

Applying (2) to X with $\lambda = k^{-1}$ and $\varepsilon = \pi(A_k^c)$, we obtain

$$\begin{aligned} P[\limsup 1_{A_k^c} Z_t > k^{-1}] &= P[\limsup S_t > k^{-1}] \leqslant P[\limsup X_t > k^{-1}] \\ &\leqslant M(k^{-1}, \pi(A_k^c)). \end{aligned}$$

Set $A = \bigcup A_k$; then $PA < \delta$ and, for every k,

$$P[A^c \cap \{\limsup Z_t > k^{-1}\}] \leqslant P[A_k^c \cap \{\limsup Z_t > k^{-1}\}] \leqslant M(k^{-1}, \pi(A_k^c)),$$

which implies that $\limsup Z_t = 0$ a.e. on A^c. Since δ is arbitrary, Z_t converges essentially to zero.

THEOREM 2.2. *Let $(\mathscr{F}_t)$ be a stochastic basis. The following properties are equivalent*:

(i) *For every martingale X and for every $\lambda > 0$,*

$$P(\limsup |X_t| \geqslant \lambda) \leqslant \frac{1}{\lambda} \lim \mathrm{E}|X_t|.$$

(ii) *There exists a function M: $\boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that, for every $\lambda > 0$,*

(1) $\lim_{\varepsilon \to 0} M(\lambda, \varepsilon) = 0$;

(2) *for every positive martingale X, $\lim \mathrm{E} X_t \leqslant \varepsilon$ implies*

$$P(X^* \geqslant \lambda) \leqslant M(\lambda, \varepsilon).$$

(iii) *Every L_1-bounded martingale converges essentially.*

Proof. Obviously, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let X be an L_1-bounded martingale. By Krickeberg's decomposition, (X_t) is the difference of two positive L_1-bounded martingales (see, e.g., [17], p. 99); hence we may and do assume that X is positive. Let $\mathscr{A} = \bigcup \mathscr{F}_t$ and let $\mathscr{F}_\infty = \sigma(\mathscr{A})$ be the σ-algebra generated by $\mathscr{A}$. Define a charge λ on $\mathscr{A}$ by $\lambda(A) = \lim \mathrm{E}[1_A X_t]$, and let $\lambda = \lambda_m + \lambda_c$ be the decomposition of λ given by Proposition 2.1. Still denote by λ_m the (unique) extension of λ_m to $\mathscr{F}_\infty$, and let $\lambda_m = X \cdot P + \nu$ be the Lebesgue decomposition of λ_m with respect to P: $X \in L_1^+(\mathscr{F}_\infty)$, $X \cdot P\colon \mathscr{F}_\infty \to \boldsymbol{R}^+$ is defined by

$$(X \cdot P)(A) = \int_A X dP,$$

and ν is a measure singular with respect to P. Since λ_c and ν are both nearly orthogonal to P on $\mathscr{A}$, so is their sum $\pi = \lambda_c + \nu$. Set $Y_t = \mathrm{E}^{\mathscr{F}_t} X$, and write $X_t = Y_t + Z_t$; then Y is a positive uniformly integrable martingale, and Z is a positive martingale such that $\lim \mathrm{E}[1_A Z_t] = \pi(A)$ for every $A \in \mathscr{A}$. Fix $\varepsilon > 0$ and $\lambda > 0$, and choose α, $0 < \alpha < \varepsilon\lambda/2$, such that $M(\lambda/2, \alpha) \leqslant \varepsilon$. Let $Y \in \bigcup L_1(\mathscr{F}_t)$ satisfy $\|X - Y\|_1 \leqslant \alpha$; then

$$\begin{aligned}
P\{\limsup |Y_t - X| > \lambda\} &\leqslant P\{\limsup [|\mathrm{E}^{\mathscr{F}_t}(X-Y)| + |\mathrm{E}^{\mathscr{F}_t} Y - Y| + |X-Y|] > \lambda\} \\
&\leqslant P\{\limsup \mathrm{E}^{\mathscr{F}_t}|X-Y| + |X-Y| > \lambda\} \\
&\leqslant P\left\{\limsup \mathrm{E}^{\mathscr{F}_t}|X-Y| > \frac{\lambda}{2}\right\} + P\left\{|X-Y| > \frac{\lambda}{2}\right\} \\
&\leqslant M\left(\frac{\lambda}{2}, \alpha\right) + \frac{2}{\lambda}\|X-Y\|_1 \leqslant 2\varepsilon.
\end{aligned}$$

Therefore, Y_t converges essentially to X and, by Lemma 2.2, Z_t converges essentially to zero.

(iii) $\Rightarrow$ (i). Let X be an L_1-bounded martingale; since X_t converges essentially, we have

$$\limsup |X_t| = \mathrm{s}\limsup |X_t|.$$

Applying Theorem 2.1 (i) to the L_1-bounded submartingale $|X_t|$, for every $\lambda > 0$ we obtain

$$P(\limsup |X_t| \geqslant \lambda) \leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} \mathrm{E}|X(\tau)| = \frac{1}{\lambda} \lim \mathrm{E}|X_t|.$$

3. Convergence of vector-valued martingales. In this section we characterize convergence of Banach-valued martingales $\mathrm{E}^{\mathscr{F}_t} X$ in terms of maximal inequalities. We also show that if a Banach space E has the Radon-Nikodým property, and if $(\mathscr{F}_t)$ satisfies C, then every E-valued L_1-bounded martingale converges essentially. An application to derivation of Banach-valued finitely additive measures is given in [16]. For notation and definitions, see [17].

THEOREM 3.1. *Let $(\mathscr{F}_t)$ be a stochastic basis. The following properties are equivalent:*

(i) *For every Banach space $(E, |\cdot|)$, for every Bochner integrable E-valued random variable X, and for every $\lambda > 0$, letting $A = \{\limsup |E^{\mathscr{F}_t} X| \geqslant \lambda\}$, we have*

$$P(A) \leqslant \frac{1}{\lambda} \int_A |X|\, dP.$$

(ii) *There exists a function $M\colon \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that, for every $\lambda > 0$,*

(1) $\lim\limits_{\varepsilon\to 0} M(\lambda, \varepsilon) = 0$;

(2) *for every positive integrable random variable X, $EX \leqslant \varepsilon$ implies*

$$P(\limsup E^{\mathscr{F}_t} X \geqslant \lambda) \leqslant M(\lambda, \varepsilon).$$

(iii) *For every Banach space E and for every Bochner integrable E-valued random variable X, the martingale $E^{\mathscr{F}_t} X$ converges essentially.*

Proof. Obviously, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let $X \in L_1^E$; fix $\alpha > 0$, $\lambda > 0$, choose $\varepsilon < \alpha\lambda/2$ such that $M(\lambda/2, \varepsilon) < \alpha$, and choose $Y \in \bigcup L_1^E(\mathscr{F}_t)$ such that $E|X - Y| < \varepsilon$. Then, if $\mathscr{F}_\infty = \sigma(\bigcup \mathscr{F}_t)$, we have

$$P(\limsup |E^{\mathscr{F}_t} X - E^{\mathscr{F}_\infty} X| \geqslant \lambda)$$

$$\leqslant P\left(\limsup E^{\mathscr{F}_t} |X - Y| \geqslant \frac{\lambda}{2}\right) + P\left(E^{\mathscr{F}_\infty} |X - Y| \geqslant \frac{\lambda}{2}\right) \leqslant \alpha + \frac{2\varepsilon}{\lambda} \leqslant 2\alpha,$$

which proves the essential convergence of $E^{\mathscr{F}_t} X$ to $E^{\mathscr{F}_\infty} X$.

(iii) $\Rightarrow$ (i). Let $X \in L_1^E$; since $E^{\mathscr{F}_t} X$ converges essentially, we have

$$\operatorname{s}\limsup |E^{\mathscr{F}_t} X| = \limsup |E^{\mathscr{F}_t} X|.$$

Applying Theorem 2.1 (ii) and the Remark which follows it to the uniformly integrable positive submartingale $Y_t = |E^{\mathscr{F}_t} X|$, and letting $A = \{\limsup |E^{\mathscr{F}_t} X| \geqslant \lambda\}$, for every $\lambda > 0$ we obtain

$$P(\limsup |E^{\mathscr{F}_t} X| \geqslant \lambda) \leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} \int_A Y_\tau\, dP$$

$$\leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} \int_A E^{\mathscr{F}_\tau} |X|\, dP \leqslant \frac{1}{\lambda} \int_A |X|\, dP.$$

The following theorem shows that the maximal inequality of Proposition 1.1 insures essential convergence of L_1-bounded E-valued martingales:

THEOREM 3.2. *Let* $(\mathcal{F}_t)$ *be a stochastic basis such that there exists a function* $M: \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *which for every* $\lambda > 0$ *satisfies*:

(1) $\lim_{\varepsilon \to 0} M(\lambda, \varepsilon) = 0$;

(2) *for every positive martingale* X, $\lim \mathrm{E} X_t \leqslant \varepsilon$ *implies*

$$P(\limsup X_t \geqslant \lambda) \leqslant M(\lambda, \varepsilon).$$

If the Banach space E *has the Radon-Nikodým property, then every* E*-valued* L_1^E*-bounded martingale converges essentially.*

Proof. Set $\mathcal{A} = \bigcup \mathcal{F}_t$, $\mathcal{F}_\infty = \sigma(\mathcal{A})$, and let X be an L_1^E-bounded martingale. Define a finitely additive E-valued measure λ on $\mathcal{A}$ by $\lambda(A) = \lim \mathrm{E}[1_A X_t]$. We write $\mu \ll P$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies $(\mathrm{Var}\,\mu)(A) < \varepsilon$. Since $\mathrm{Var}\,\lambda = \lim \mathrm{E}|X_t| < \infty$, λ can be decomposed as $\lambda = \mu + \pi$, where μ is a σ-additive measure on $\mathcal{A}$, $\mu \ll P$, $\mathrm{Var}\,\mu < \infty$, and π is a finitely additive measure on $\mathcal{A}$ such that the positive charges $\mathrm{Var}\,\pi$ and P are nearly orthogonal (for this result due to Chatterji and Uhl, see, e.g., [4], p. 30-31). Since E has the Radon-Nikodým property, there exists $X \in L_1^E$ such that $\mu(A) = \mathrm{E}[1_A X]$ for each $A \in \mathcal{A}$. The martingale $\mathrm{E}^{\mathcal{F}_t} X$ converges by Theorem 3.1. Set $Z_t = X_t - \mathrm{E}^{\mathcal{F}_t} X$; Lemma 2.2 applied to the submartingale $|Z_t|$ shows that Z_t converges essentially to zero.

From Proposition 1.3 and Theorem 3.2 we derive

THEOREM 3.3. *Let* $(\mathcal{F}_t)$ *be a stochastic basis satisfying the condition* C *and let* E *be a Banach space with the Radon-Nikodým property. Then every* L_1^E*-bounded martingale converges essentially.*

4. Conditions C and sigma-SV. In a previous paper [15] we introduced a condition SV defined as follows:

For every integer m, $(\mathcal{F}_t)$ satisfies the *condition* SV(m) if for every $\varepsilon > 0$ and for every adapted family of sets A there exists an incomplete multivalued stopping time τ such that $P(A^* \backslash A(\tau)) < \varepsilon$ and $\varepsilon_\tau \leqslant m$; SV is the logical union of the conditions SV(m).

Let E be a Banach space with the Radon-Nikodým property. We showed that the condition SV implies the essential convergence of L_1^E-bounded martingales. Therefore, the following condition sigma-SV also implies essential convergence of L_1^E-bounded martingales:

There exists a sequence of sets Ω_n in the algebra $\mathcal{A} = \bigcup \mathcal{F}_t$ such that $\Omega = \bigcup \Omega_n$ and, for every n, the restriction of $(\mathcal{F}_t)$ to Ω_n satisfies SV.

Since all the conditions SV(m), $m = 0, 1, 2, \ldots$, are different [15], sigma-SV is properly weaker than SV. The following example shows that C is strictly weaker than sigma-SV:

THEOREM 4.1. *There exists a stochastic basis which satisfies the condition* C, *but not the condition* sigma-SV.

Proof. Set $(\Omega, \mathscr{F}, P) = [0, 1)$ with Lebesgue measure; all the intervals considered are of the form $[a, b)$. Given a subinterval $[a, b)$ of Ω, an integer m, and a number $\delta > 0$, a family of sets $A_1, \ldots, A_k \subset [a, b)$ is an *(m, δ)-family of* $[a, b)$ according to the partition $\{D, I\}$ if $\{D, I\}$ is a partition of $[a, b)$ into intervals such that

(1) $\{A_i \cap D\colon i = 1, \ldots, k\}$ is a partition of D into intervals of equal length;

(2) $1_I \sum_{i \leqslant k} 1_{A_i} = m$;

(3) for any subfamily $\{B_i\}$ $(1 \leqslant i \leqslant k')$ of $\{A_i\}$, $\sum_{i \leqslant k'} 1_{B_i} \leqslant m-1$ implies

$$P(I) + P\Big(\bigcup_{i \leqslant k'} B_i\Big) \leqslant \delta P([a, b)).$$

We now show that for any interval $[a, b)$, any integer m and any $\delta > 0$, there exists an (m, δ)-family of $[a, b)$ (see also [24]). Indeed, let I be the extreme left interval of $[a, b)$ such that $P(I) < \delta(b-a)/3$, and fix $k > m$. Divide the interval $D = [a, b) \backslash I$ into k disjoint intervals of equal length $D(i)$, $1 \leqslant i \leqslant k$, and divide I into $k!\big(m!(k-m)!\big)^{-1}$ disjoint intervals of equal length $I(s_1, \ldots, s_m)$, $1 \leqslant s_1 < \ldots < s_m \leqslant k$. For every i with $1 \leqslant i \leqslant k$, set

$$A_i = D(i) \cup \{I(s_1, \ldots, s_m)\colon \exists q, s_q = i\}.$$

The construction is such that any subfamily $\{B_i\}$, $1 \leqslant i \leqslant k'$, satisfying

$$\sum_{i \leqslant k'} 1_{B_i} \leqslant m-1,$$

contains at most $m-1$ sets; hence

$$P\Big(\bigcup_{i \leqslant k'} B_i\Big) \leqslant P(I) + (m-1)(b-a)/k \leqslant 2\delta(b-a)/3 \quad \text{if } k \geqslant 3(m-1)/\delta.$$

Given a sequence of integers (n_r), which will be determined by induction, let

$$J = \{(i_1, \ldots, i_r)\colon r \geqslant 1, 1 \leqslant i_j \leqslant n_j\}$$

ordered by the relation $(i_1, \ldots, i_r) < (i'_1, \ldots, i'_k)$ iff $r < k$. We construct $(\mathscr{F}_t)$ by induction as follows:

Step 1. Fix a number $p_1(2)$ satisfying $0 < p_1(2) < 1$, and let L be the extreme left interval of Ω, of measure $p_1(2)$. Set $m_1 = 3(3-1) \cdot 2^2$ and $n_1 = 2m_1$. Define a $(2, 2^{-2})$-family of sets $A(i)$ $(1 \leqslant i \leqslant m_1)$ of the interval L according to a partition $\{D', I'\}$. Define a $(3, 2^{-2})$-family of sets $A(i)$, $m_1+1 \leqslant i \leqslant n_1$, of the remaining interval $\Omega \backslash L$, according to a partition $\{D'', I''\}$. For every i set

$$D(i) = \begin{cases} A(i) \cap D' & \text{if } 1 \leqslant i \leqslant m_1, \\ A(i) \cap D'' & \text{if } m_1+1 \leqslant i \leqslant n_1. \end{cases}$$

For every i with $1 \leqslant i \leqslant n_1$, let $\mathscr{F}(i)$ be the σ-algebra generated by $A(i)$. Let $\mathscr{G}_1$ be the σ-algebra generated by all the intervals introduced in the first step.

Step 2. Fix two numbers $P_2(j)$ with $0 < P_2(j) < 1$ for $j = 2, 3$. Let $A(i_1)$ be a fixed interval obtained in the first step; $A(i_1)$ belongs to a $(j, 2^{-2})$-family, where $j = 2$ or 3. Let $L(i_1)$ be the extreme left interval of $D(i_1)$ with $P[L(i_1)] = P_2(j)P[D(i_1)]$. Set $m_2 = 3(4-1)\cdot 2^3$ and $n_2 = 2m_2$. Define a $(j, 2^{-3})$-family of sets $A(i_1, i)$ $(1 \leqslant i \leqslant m_2)$ of $L(i_1)$ according to a partition $\{D'(i_1), I'(i_1)\}$. Define a $(j+1, 2^{-3})$-family of sets $A(i_1, i)$ $(m_2+1 \leqslant i \leqslant n_2)$ of the interval $D(i_1)\backslash L(i_1)$ according to a partition $\{D''(i_1), I''(i_1)\}$. Set

$$D(i_1, i) = \begin{cases} A(i_1, i) \cap D'(i_1) & \text{if } 1 \leqslant i \leqslant m_2, \\ A(i_1, i) \cap D''(i_1) & \text{otherwise.} \end{cases}$$

For every i with $1 \leqslant i \leqslant n_2$, let $\mathscr{F}(i_1, i)$ be the σ-algebra generated by $\mathscr{G}_1$ and $A(i_1, i)$. Let $\mathscr{G}_2$ be the σ-algebra generated by all the intervals introduced in the two first steps.

Step $k+1$. Fix $k+1$ numbers $P_{k+1}(j)$ with $0 < P_{k+1}(j) < 1$ for $j = 2, \dots, k+2$. Let $A(i_1, \dots, i_k)$ be a fixed interval obtained at the step k; $A(i_1, \dots, i_k)$ belongs to a $(j, 2^{-(k+1)})$-family, where $j = 2$ or 3 or ... or $k+2$. Let $L(i_1, \dots, i_k)$ be the extreme left interval of $D(i_1, \dots, i_k)$ with

$$P[L(i_1, \dots, i_k)] = P_{k+1}(j)P[D(i_1, \dots, i_k)].$$

Set $m_{k+1} = 3(k+2)\cdot 2^{k+2}$ and $n_{k+1} = 2m_{k+1}$. Define a $(j, 2^{-(k+2)})$-family of sets $A(i_1, \dots, i_k, i)$ $(1 \leqslant i \leqslant m_{k+1})$ of $L(i_1, \dots, i_k)$ according to a partition $\{D'(i_1, \dots, i_k), I'(i_1, \dots, i_k)\}$. Define a $(j+1, 2^{-(k+2)})$-family of sets $A(i_1, \dots, i_k, i)$ $(m_{k+1}+1 \leqslant i \leqslant n_{k+1})$ of $D(i_1, \dots, i_k)\backslash L(i_1, \dots, i_k)$ according to a partition $\{D''(i_1, \dots, i_k), I''(i_1, \dots, i_k)\}$. Set

$$D(i_1, \dots, i_k, i) = \begin{cases} A(i_1, \dots, i_k, i) \cap D'(i_1, \dots, i_k) & \text{if } 1 \leqslant i \leqslant m_{k+1}, \\ A(i_1, \dots, i_k, i) \cap D''(i_1, \dots, i_k) & \text{otherwise.} \end{cases}$$

For every i with $1 \leqslant i \leqslant n_{k+1}$, let $\mathscr{F}(i_1, \dots, i_k, i)$ be the σ-algebra generated by $\mathscr{G}_k$ and $A(i_1, \dots, i_k, i)$. Let $\mathscr{G}_{k+1}$ be the σ-algebra generated by all the intervals introduced in the previous steps.

Suppose that

$$\sum_k \sum_{j=2}^{k+1} [1-p_k(j)] < \infty.$$

For any $\varepsilon > 0$, choose M such that

$$2^{-M} < \varepsilon, \quad \sum_{k=M}^{\infty} \Big[\sum_{j=2}^{k+1} (1-p_k(j))\Big] < \varepsilon$$

and

$$\prod_{k=M}^{\infty} \left[\prod_{j=2}^{k+1} p_k(j)\right] > 1-\varepsilon.$$

For every i denote by $\bar{A}_i$ the union of all sets $A(s_1, \dots, s_i)$ which are elements of a $(j, 2^{-(i+1)})$-family for some j with $2 \leqslant j \leqslant M+1$. The sequence $\bar{A}_i$ is decreasing, and for every $i \geqslant M$ we have

$$P(\bar{A}_i) \geqslant \prod_{k=M}^{i} \left[\prod_{j=2}^{k+1} p_k(j)\right] \geqslant 1-\varepsilon;$$

therefore $P(\bigcap \bar{A}_i) \geqslant 1-\varepsilon$.

Let B be an adapted family of sets; define an adapted family of sets C by

$$C(s_1, \dots, s_i) = B(s_1, \dots, s_i) \cap \bar{A}_{i-1},$$

and for every number i put

$$\bar{C}_i = \bigcup \{C(s_1, \dots, s_i)\colon 1 \leqslant s_j \leqslant n_j,\ 1 \leqslant j \leqslant i\}.$$

We show that for any fixed i there exists a stopping time $\tau_i \in IM$ such that

$$e_{\tau_i} \leqslant M, \quad P(\bar{C}_i \backslash C(\tau_i)) \leqslant \sum_{j=2}^{i+1} [1-p_i(j)] + \frac{M+2}{m_i}$$

and τ_i only takes on values among the subsets of $\{(s_1, \dots, s_i)\colon 1 \leqslant s_j \leqslant n_j,$ $1 \leqslant j \leqslant i\}$. Fix i; for any $(s_1, \dots, s_i)$ set

$$C(s_1, \dots, s_i) = G(s_1, \dots, s_i) + H(s_1, \dots, s_i),$$

where $G(s_1, \dots, s_i)$ is the largest subset of $C(s_1, \dots, s_i)$ which is $\mathscr{G}_{i-1}$-measurable. Since $\mathscr{G}_{i-1} \subset \mathscr{F}(s_1, \dots, s_i)$, we can assume that sets $G(s_1, \dots, s_i)$ to be disjoint; if $G(s_1, \dots, s_i) \neq \emptyset$, put $\tau_i = (s_1, \dots, s_i)$ on this set. Let

$$\bar{G}_i = \bigcup \{G(s_1, \dots, s_i)\colon 1 \leqslant s_j \leqslant n_j,\ 1 \leqslant j \leqslant i\};$$

each set of the form $C(s_1, \dots, s_i) \backslash \bar{G}_i$ is either $\emptyset$, or $A(s_1, \dots, s_i)$, or $D(s_1, \dots, s_{i-1}) \backslash A(s_1, \dots, s_i)$. Fix $s_1, \dots, s_{i-1}$; if one of the sets $C(s_1, \dots, s_i) \backslash \bar{G}_i$ is $D(s_1, \dots, s_{i-1}) \backslash A(s_1, \dots, s_i)$, then put $\tau_i = (s_1, \dots, s_i)$ on this set. If all non-empty sets $C(s_1, \dots, s_i) \backslash \bar{G}_i$ are of the form $A(s_1, \dots, s_i)$, then let $(s_1, \dots, s_i) \in \tau_i(\omega)$, ω belonging to one of the above sets $A(s_1, \dots, s_i)$. Since $\bar{C}_i \subset \bar{A}_{i-1}$, we have $e_{\tau_i} \leqslant M$. Furthermore,

$$P(\bar{C}_i \backslash C(\tau_i)) \leqslant \sum_{j=2}^{i+1} [1-p_i(j)] + 2^{-i-1} + \frac{1}{m_i}.$$

For $i_0 < i_1$, set

$$\tau = \tau_{i_0} \text{ on } \bar{C}_{i_0}, \quad \tau = \tau_{i_0+1} \text{ on } \bar{C}_{i_0+1} \backslash \bar{C}_{i_0}, \quad \dots,$$
$$\tau = \tau_{i_1} \text{ on } \bar{C}_{i_1} \backslash \Big(\bigcup_{i_0 \leqslant j < i_1} \bar{C}_j\Big).$$

For a fixed $i_0 \geqslant M$, by a suitable choice of i_1 we have

$$P(B^*\backslash B(\tau)) \leqslant P[\bigcup_{i \geqslant i_0} \bar{B}_i \backslash B(\tau)] \leqslant P[\bigcup_{i_0 \leqslant i \leqslant i_1} \bar{B}_i \backslash B(\tau)] + \varepsilon$$

$$\leqslant P[\bigcup_{i_0 \leqslant i \leqslant i_1} \{(\bar{B}_i \backslash B(\tau)) \cap \bar{A}_{i-1}\}] + 2\varepsilon$$

$$\leqslant \sum_{i=M}^{\infty} \sum_{j=2}^{i+1} [1-p_i(j)] + \sum_{i=M}^{\infty} 2^{-i-1} +$$

$$+3^{-1} \sum_{i=M}^{\infty} (i+1)^{-1} \cdot 2^{-i-1} + 2\varepsilon \leqslant 5\varepsilon.$$

Hence $(\mathscr{F}_t)$ satisfies C.

We now show that $(\mathscr{F}_t)$ does not satisfy sigma-SV. Let $A \in \mathscr{A} = \bigcup \mathscr{F}_t$; there exists k such that $A \in \mathscr{G}_k$. Suppose that A is not included in

$$\bar{I} = \bigcup \{I'(s_1, \ldots, s_i) \cup I''(s_1, \ldots, s_i) \colon 1 \leqslant s_j \leqslant n_j, j \geqslant 1\},$$

and notice that $F^* = \limsup F_t = A \backslash \bar{I}$ for the adapted family F defined by

$$F(s_1, \ldots, s_i) = \begin{cases} A(s_1, \ldots, s_i) \cap A & \text{if } i > k, \\ \emptyset & \text{otherwise.} \end{cases}$$

We show that, given any fixed $M > 0$, there exist $\varepsilon > 0$ and $s \in J$ such that for every $\tau \in IM$ the relations $\tau > s$ and $e_\tau \leqslant M$ imply $P(F^* \backslash F(\tau)) \geqslant \varepsilon$; this shows that none of the conditions $\mathrm{SV}(M)$ holds on A. Indeed, fix $M > 0$; by definition of the family A, there exists $k > 0$ such that if $\bar{F}_k$ is the union of sets $F(s_1, \ldots, s_k) = A(s_1, \ldots, s_k) \cap A$ such that $A(s_1, \ldots, s_k)$ belongs to an $(M+1, 2^{-k-1})$-family, then $P(F^* \backslash \bar{F}_k) > 0$. Then for every $i > k$ all the sets $A(s_1, \ldots, s_i)$ included in $A \backslash \bar{F}_k$ belong to $(j, 2^{-i-1})$-families for some $j \geqslant M+2$. Fix $\varepsilon < P(F^* \backslash \bar{F}_k)/2$ and fix $m > k$ such that $2^{-m} < \varepsilon$. By definition of A, the optimal way to cover $F^* \backslash \bar{F}_k$ by means of multivalued stopping times τ such that $e_\tau \leqslant M$ and $\tau > (n_1, \ldots, n_m)$ is to set $\{\tau = (s_1, \ldots, s_i)\} = F(s_1, \ldots, s_i)$ at each level $i > m$ and in each set $D(s_1, \ldots, s_{i-1})$ for $M+1$ distinct values of s_i with $1 \leqslant s_i \leqslant n_i$. Therefore, if $e_\tau \leqslant M$ and $\tau > (n_1, \ldots, n_m)$, we have

$$P[(F^* \backslash \bar{F}_k) \cap F(\tau)] \leqslant \sum_{i>m} 2^{-m-1} \leqslant 2^{-m} \leqslant \varepsilon \leqslant P(F^* \backslash \bar{F}_k) \cdot 2^{-1},$$

which implies

$$P[F^* \backslash F(\tau)] > P(F^* \backslash \bar{F}_k) \cdot 2^{-1} \geqslant \varepsilon.$$

If $(\mathscr{F}_t)$ were satisfying the condition sigma-SV, then since $P(\bar{I}) \leqslant 1/2$, at least one set $A_k \in \mathscr{A}$ such that $\mathrm{SV}(k)$ holds on A_k, say $A_k \in \mathscr{G}_{n_k}$, would have to intersect $\bar{I}^c$, which would contradict the result above.

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