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RANDOM MEASURE OF NON-COMPACTNESS IN PM-SPACES AND APPLICATIONS

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Abstract. The aim of this note is to extend some results of [5] by introducing a measure of non-compactness and a corresponding class of probabilistic condensing multivalued mappings. A characterization of condensing mappings in the sense of Himmelberg, Porter and Van Vleck [12] is given and some results on the existence of fixed points with applications in the random multivalued operator equations are obtained.

The notion of measure of non-compactness has been firstly introduced by Kuratowski [13] and subsequently axiomatically generalized by Sadovski [17], Goldenstein, Gohberg and Markus [8], Petryshyn and Fitzpatrick [14], Himmelberg, Porter and Van Vleck [12], and others. The probabilistic measures of non-compactness have been introduced by Bocşan and Constantin in [3, 4]. The interesting results in fix point theory and random operator equations have been given by Bocşan [1, 2], Hadzič [10, 11], Cain [5], Constantin and Istrăţescu [6, 7], and Radu [15].

Let Δ^+ be the set of the distribution functions of all non-negative real random variables. Let S be a linear space and

$$\mathscr{F}\colon S \to \mathscr{D}^+ = \{F \in \Delta^+ \colon \sup_{x \in R} F(x) = F(\infty) = 1\}$$

be a probabilistic norm such that $(S, \mathscr{F}, T)$ is a random normed space, i.e.,

1. $F_p = H_0$ iff $p = 0$ (H_0 is the characteristic function of $(0, \infty)$);
2. $F_{\lambda p}(x) = F_p(x/|\lambda|)$ for every $x > 0$, $\lambda \neq 0$ in the scalar field and $p \in S$;
3. $F_{p+q}(x+y) \geqslant T(F_p(x), F_q(y))$ for every $p, q \in S$ and $x, y > 0$, where T is a t-norm such that $T \geqslant T_m$.

A t-norm is a function $T\colon [0, 1] \times [0, 1] \to [0, 1]$ which is associative, commutative, non-decreasing in every place and such that $T(a, 1) = a$ for every $a \in [0, 1]$. T_m is the t-norm defined by $T_m(x, y) = \max\{x+y-1, 0\}$.

An interesting class of t-norms weaker than Min is introduced by Hadzič [9] as follows:

Definition 1. A t-norm T is of *H-type* (or $T \in \mathscr{H}$) if $\{T^n(t)\}_{n\in N}$ is equicontinuous at $t=1$, where $T^1(t) = T(t, t)$, $T^{n+1}(t) = T(T^n(t))$, $n \geqslant 1$, $t \in [0, 1]$.

Write $S(t, \lambda) = \{p \in S\colon F_p(t) > 1-\lambda\}$ and $\bar{S}(t, \lambda) = \{p \in S\colon F_p(t) \geqslant 1-\lambda\}$.

PROPOSITION 1. *Let* $(S, \mathscr{F}, T)$ *be a random normed space and let* T *be a continuous H-type t-norm. Let* Λ *be the set of all* $\lambda_n \in R^+$ *such that* $\{\lambda_n\}_{n\in N}$ *is a monotone decreasing, convergent to zero sequence and* $T(1-\lambda_n, 1-\lambda_n) = 1-\lambda_n$. *Then the family* $\{\bar{S}(t, \lambda_n)\}_{n\in N}$ *is a generalized basis of the neighbourhood system* $\mathscr{N}_0$ *of the origin. This neighbourhood system determines a Hausdorff locally convex topological vector space.*

Proof. It is known [9] that any random normed space with a continuous H-type t-norm T is in the (ε, λ)-topology a locally convex topological space.

The family $\{\bar{S}(t, \lambda_n)\}_{n\in N}$, having the properties mentioned in the hypothesis, is a generalized basis for $\mathscr{N}_0$ since for every $V \in \mathscr{N}_0$ there is an $S(\varepsilon, \lambda') \subset V$ and a $\lambda'_n < \lambda'$ such that

$$p \in \bar{S}(\varepsilon, \lambda_n) \Leftrightarrow F_p(\varepsilon) \geqslant 1-\lambda_n > 1-\lambda' \Rightarrow p \in S(\varepsilon, \lambda') \Rightarrow p \in V.$$

On the other hand, $\bar{S}(\varepsilon, \lambda) \supset S(\varepsilon, \lambda)$ implies that $\bar{S}(\varepsilon, \lambda) \in \mathscr{N}_0$.

The existence for every $\lambda' > 0$ of a λ'_n such that $T(\lambda'_n, \lambda'_n) = \lambda'_n$ follows from the characterization of the H-type t-norm ([16], Lemma 1), i.e., since T is continuous and H-type, it follows that for every $a > 0$ there is a $b > a$ such that $T(b, b) = b < 1$.

To prove convexity, take $p, q \in \bar{S}(t, \lambda_n)$ and let $r = up+(1-u)q$, $0 \leqslant u \leqslant 1$, and then consider

$$\begin{aligned} F_r(t) &= F_{up+(1-u)q}(ut+(1-u)t) \geqslant T(F_{up}(ut), F_{(1-u)q}(1-u)t) \\ &= T(F_p(t), F_q(t)) \geqslant T(1-\lambda_n, 1-\lambda_n) = 1-\lambda_n. \end{aligned}$$

Thus $r \in \bar{S}(t, \lambda_n)$.

Remark. We write $\bar{S}(t_1, \lambda_n)+\bar{S}(t_2, \lambda_n) \subset \bar{S}(t_1+t_2, \lambda_n)$ since, for every $p \in \bar{S}(t_1, \lambda_n)$ and $q \in \bar{S}(t_2, \lambda_n)$,

$$F_{p+q}(t_1+t_2) \geqslant T(F_p(t_1), F_q(t_2)) \geqslant 1-\lambda_n.$$

LEMMA 1. *A subset* $A \subset S$ *is bounded iff* $A \subseteq \bar{S}(t, \lambda_n)$ *for some* t *and* λ_n.

Let $\mathscr{S} = \{\bar{S}(t, \lambda_n)\}_{n\in N}$, where $\{\lambda_n\}_{n\in N} = \Lambda$ has the form stated in Proposition 1. For a subset $A \subset S$ and for every $\lambda_k \in \Lambda$ define

$$\gamma_{\lambda_k}(A) = \inf\{t\colon A \subset F+\bar{S}(t, \lambda_k) \text{ for a finite set } F \subset S\}.$$

LEMMA 2. *A set $A \subset S$ is precompact iff $\gamma_{\lambda_k}(A) = 0$ for every $\lambda_k \in \Lambda$.*

Definition 2. Let $(S, \mathscr{F}, T)$ be a random normed space with a continuous H-type t-norm T. A multivalued mapping $f\colon C \to \mathscr{P}(S)$, defined on a subset C of S, is said to be *probabilistic condensing* if, for every bounded non-precompact set $A \subset C$, $\gamma_{\lambda_k}(f(A)) \leqslant \gamma_{\lambda_k}(A)$ for every $\lambda_k \in \Lambda$ and $\gamma_{\lambda_{k'}}(f(A)) < \gamma_{\lambda_{k'}}(A)$ for at least one $\lambda_{k'} \in \Lambda$.

THEOREM 1. *Let $(S, \mathscr{F}, T)$ be a random normed space with a continuous H-type t-norm T and let C be a complete convex subset of S. Suppose that $f\colon C \to \mathscr{P}(C)$ is a probabilistic condensing upper semicontinuous multivalued mapping such that $f(p)$ is closed and convex for every $p \in C$, and $f(C)$ is bounded. Then f has a fixed point in C.*

Proof. Let us recall the fixed point result of Himmelberg, Porter and Van Vleck [12]. The multivalued mapping f is closed valued and upper semicontinuous, so it has a closed graph. Then it is sufficient to show that f is condensing in the sense of [12] relative to the family $\mathscr{S}$.

Indeed, let $A \subset C$ be a bounded, but not precompact subset of C. Then

$$Q(A) = \{\bar{S}(t, \lambda_k) \in \mathscr{S}\colon A \subset K + \bar{S}(t, \lambda_k) \text{ for some precompact } K\}.$$

To prove that $Q(A)$ is properly contained in $Q(f(A))$, choose $\bar{S}(\varepsilon, \lambda) \in Q(A)$.

There is a precompact K for which $A \subset K + \bar{S}(\varepsilon, \lambda)$ and $\gamma_\lambda(A) \leqslant \varepsilon$. Indeed, if $\gamma_\lambda(A) > \varepsilon$, we choose t such that $\varepsilon + t < \gamma_\lambda(A)$. Since the set K is precompact, there is a finite set F for which $K \subset F + \bar{S}(t, \lambda)$. Hence

$$A \subset K + \bar{S}(\varepsilon, \lambda) \subset F + \bar{S}(t, \lambda) + \bar{S}(\varepsilon, \lambda) \subset F + \bar{S}(t+\varepsilon, \lambda),$$

which contradicts $t + \varepsilon < \gamma_\lambda(A)$. Thus $\gamma_\lambda(A) \leqslant \varepsilon$ and from the condensing hypothesis it follows that $\gamma_\lambda(f(A)) \leqslant \gamma_\lambda(A) \leqslant \varepsilon$. This, obviously, implies $\bar{S}(\varepsilon, \lambda) \in Q(f(A))$. Hence $Q(A) \subset Q(f(A))$. This inclusion is proper since, if λ is chosen such that $\gamma_\lambda(f(A)) < \gamma_\lambda(A)$ and $\bar{t} \in (\gamma_\lambda(f(A)), \gamma_\lambda(A))$, we have $\bar{S}(\bar{t}, \lambda) \in Q(f(A))$, but $\bar{S}(\bar{t}, \lambda) \notin Q(A)$.

The existence of the fixed point follows from the following

THEOREM A [12]. *Let C be a non-empty complete convex subset of a separated locally convex space E, and let $f\colon C \to \mathscr{P}(C)$ be a condensing multivalued mapping with convex values, closed graph, and bounded range. Then f has a fixed point.*

This result inclines one to study the connection between condensing mappings as defined by Himmelberg and Van Vleck and γ-condensing mappings in general, where γ is a random measure of non-compactness. To this purpose let us note that Theorem 2.2 of [1] takes place also for more general situations, i.e., THEOREM 2. *Let $(S, \mathcal{F}, T)$ be a random normed space with a continuous H-type t-norm T. Then there is a random measure of non-compactness γ such that every mapping is condensing in the sense of Himmelberg, Porter and Van Vleck relative to $\mathcal{S}$ iff it is probabilistic γ-condensing.*

Proof. Let $f\colon (S, \mathcal{F}, T) \to \mathcal{P}(S)$ be a condensing mapping relative to $\mathcal{S}$. Then there is a (t_0, λ_0) and a non-precompact set $A \subset S$ such that $f(A) \subseteq \bar{S}(t_0, \lambda_0)+K_0$ for a precompact set $K_0 \subset S$, but $A \not\subseteq \bar{S}(t_0, \lambda_0)+K$ whichever be the precompact set $K \subset S$. Hence there exists a $t \in \boldsymbol{R}$ such that

$$\sup_{t'<t}\sup\{\lambda \in \Lambda\colon \exists K = \text{precompact subset}, A \subseteq \bar{S}(t', \lambda)+K\}$$

$$< \sup_{t'<t}\sup\{\lambda \in \Lambda\colon \exists K = \text{precompact subset}, f(A) \subseteq \bar{S}(t', \lambda)+K\}.$$

Let

$$\gamma_A(t) = \sup_{t'<t}\sup\{\lambda \in \Lambda, \exists K = \text{precompact subset}, A \subseteq \bar{S}(t', \lambda)+K\}.$$

We will prove that γ_A is a random measure of non-compactness on $\mathcal{P}(S)$, whereas f is a probabilistic γ-condensing mapping.

First we shall show that $\gamma_A = H_0$ *iff* A is a precompact set. Indeed, let $t \in \boldsymbol{R}^+$ and $\lambda \in \Lambda$. Choose $t' \in \boldsymbol{R}^+$ and $\lambda' \in \Lambda$ such that

$$\bar{S}(t', \lambda')+\bar{S}(t', \lambda') \subseteq \bar{S}(t, \lambda).$$

From $\gamma_A = H_0$ it follows that there exists a precompact subset K' such that $A \subseteq \bar{S}(t', \lambda')+K'$. Since K' is a precompact subset, there exists also a finite set F' such that $K' \subseteq S(t', \lambda')+F'$, hence $A \subseteq \bar{S}(t, \lambda)+F'$, i.e. A is a precompact subset.

Conversely, if A is a precompact subset, then for every (t, λ) there is a finite set F such that $A \subseteq \bar{S}(t, \lambda)+F$; hence $\gamma_A = H_0$.

It is also true that $\gamma_A \in \Delta^+$. Furthermore, $\gamma_A \in \mathcal{D}^+$ if A is a probabilistic bounded set, as $A \subset \bar{S}(t, \lambda)$ for some t and λ.

Moreover, γ is even monotone, subadditive, invariant with respect to the closure and to a convex hull.

The first three properties follow immediately. For invariance with respect to a convex hull it is sufficient to prove that $\gamma_{\mathrm{co}A} \geqslant \gamma_A$. So let $\bar{S}(t, \lambda) \in \mathcal{S}$ and let K be a precompact subset such that $A \subseteq \bar{S}(t, \lambda)+K$. Then $\mathrm{co}\,K$ is also precompact. Since $\bar{S}(t, \lambda)$ is convex, $\bar{S}(t, \lambda)+\mathrm{co}\,K$ is also convex and $\bar{S}(t, \lambda)+\mathrm{co}\,K \supseteq \mathrm{co}\,A$. Hence

$$\gamma_{\mathrm{co}A}(t) = \sup_{t'<t}\sup\{\lambda \in \Lambda, \exists K = \text{precompact}, \mathrm{co}\,A \subseteq \bar{S}(t', \lambda)+K\} \geqslant \gamma_A(t)$$

and thus we have proved that γ_A is a random measure of non-compactness associated to the condensing mapping f.

f is γ-condensing since there exists a pair (t_0, λ_0) such that there exists a precompact set K_0 for which $f(A) \subseteq \bar{S}(t_0, \lambda_0)+K_0$, but $A \not\subseteq \bar{S}(t_0, \lambda_0)+K$, whatever be the precompact set $K \subset S$. Indeed, in this case $\gamma_{f(A)}(t_0) > \gamma_A(t_0)$ and, generally, $\gamma_{f(A)}(t) \geqslant \gamma_A(t)$.

Conversely, let f be a probabilistic γ-condensing mapping. If $A \in \mathscr{P}(S)$ is such that $Q(A) \neq \mathscr{S}$, then A is a non-precompact set, i.e. $\gamma_A < H_0$. Since f is γ-condensing, we have $\gamma_{f(A)}(t) > \gamma_A(t)$, hence there exists a $t' < t$ such that $f(A) \subseteq \bar{S}(t', \lambda)+K_0$ for a precompact subset K_0, but $A \not\subseteq \bar{S}(t', \lambda)+K$, whatever be the precompact set K. Therefore $\bar{S}(t', \lambda) \in Q(f(A))$, but $\bar{S}(t', \lambda) \notin Q(A)$, i.e. f is a condensing mapping relative to $\mathscr{S}$, and the proof of the theorem is completed.

Remark. This result allows us to state the fixed point theorems with respect to the condensing mapping as defined by Himmelberg, Porter and Van Vleck relative to $\mathscr{S}$, which are also true for random normed spaces relative to the γ-condensing mapping. Conversely, the fixed point theorem of Himmelberg, Porter and Van Vleck in random normed spaces can be derived from the previous theorem for a probabilistic γ-condensing mapping, where γ is as stated above.

In order to give an example of how to utilize the probabilistic property to derive new fixed point theorems, let us remember

Definition 3. A multivalued mapping (multifunction) $f\colon C \subset S \to \mathscr{P}(S)$ is a *probabilistic contraction* if there exists a constant k, $0 < k < 1$, such that, for $p, q \in C$ and $r \in f(p)$, there exists a point $s \in f(q)$ with

$$F_{r-s}(kt) \geqslant F_{p-q}(t) \quad \text{for all } t > 0.$$

Let us also recall that a multivalued mapping $h\colon C \to \mathscr{P}(S)$ is called *completely continuous* if $h(B)$ is precompact, whenever B is a bounded subset of C.

A probabilistic and multifunction analogy of Krasnoselskii's theorem for random normed spaces with the H-type t-norm T can be stated as

THEOREM 3. *Suppose C is a complete convex subset of the random normed space $(S, \mathscr{F}, T)$. Let $f\colon C \to \mathscr{P}(C)$ be an upper semicontinuous compact convex valued mapping of C into itself. If $f = g+h$, where g is a compact valued contraction and h is completely continuous, then f has a fixed point.*

Proof. Let us first consider a result which is of interest also by itself.

LEMMA 3. *Let $f\colon C \to \mathscr{P}(S)$ be a multivalued mapping such that $f = g+h$, where g is a compact valued contraction and h is completely continuous. Then f is probabilistic condensing.*

Proof. Let B be a bounded non-precompact subset of the domain of f. We shall show that, for every $\lambda \in \Lambda$, there is a $t' \leqslant \gamma_\lambda(B)$ such that $f(B) \subseteq$

$F+\bar{S}(t', \lambda)$ for some finite set F. This will prove that f is probabilistic condensing.

Let $\varepsilon > 0$ be an arbitrary positive number and let K be the contraction constant of g. We choose t and t' such that $kt < t' < \gamma_\lambda(B)+\varepsilon = t$.

Since $t > \gamma_\lambda(B)$, there exists a finite set G such that $B \subseteq G+\bar{S}(t, \lambda)$. Moreover, since $h(B)$ is contained in a precompact set, there exists a finite set H such that $h(B) \subseteq H+\bar{S}((t'-2k)/2, \lambda)$. Let I_r be a finite set, for every $r \in G$, such that $g(r) \subset I_r+\bar{S}((t'-kt)/2, \lambda)$.

Now define $J = H+\bigcup\{I_r\colon r \in G\}$ which is clearly finite. We shall prove next that $f(B) \subseteq J+\bar{S}(t', \lambda)$.

Let $p \in B$ and q be an arbitrary element of $g(p)$. Let $r \in G$ be such that $F_{p-r}(t) > 1-\lambda$, and choose $s \in g(r)$ such that $F_{s-q}(kt) \geqslant F_{p-r}(t) > 1-\lambda$. There is a $u \in I_r$ for which it is true that $F_{u-s}((t'-kt)/2) \geqslant 1-\lambda$. Thus

$$F_{u-q}((t'+kt)/2) \geqslant \mathrm{T}\left(F_{u-s}((t'-kt)/2),\ F_{s-q}(kt)\right) \geqslant T(1-\lambda,\ 1-\lambda) = 1-\lambda,$$

i.e. $g(B) \subseteq \bigcup\{I_r\colon r \in G\}+\bar{S}((t'+kt)/2, \lambda)$.

We can now conclude that

$$f(B) = g(B)+h(B) \subseteq \bigcup\{I_r,\ r \in G\}+\bar{S}((t'+kt)/2, \lambda)+H+$$
$$+\bar{S}((t'-kt)/2, \lambda) \subseteq J+\bar{S}(t', \lambda).$$

Thus $\gamma_\lambda(f(B)) \leqslant t' < \gamma_\lambda(B)+\varepsilon$, which, obviously, means that $\gamma_\lambda(f(B)) \leqslant \gamma_\lambda(B)$.

To see that $\gamma_\lambda(f(B)) < \gamma_\lambda(B)$ for at least one $\lambda \in \Lambda$, note that if $\gamma_\lambda(B) > 0$, then ε can be chosen to be zero. Hence f is condensing.

Lemma 3 and Theorem 1 show that f has a fixed point. Hence the proof of Theorem 3 is completed.

Some other applications of condensing multivalued mappings on random normed spaces to the problem of stability of solutions of some classes of multivalued random operator equations will be given in a subsequent work.

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