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DRIVING NOISE OF A FINITE STATE MARKOV PROCESS

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Abstract. Given a finite state Markov process $\{X_t\}$, $t \geq 0$, a global "driving noise" process is constructed on a larger probability space using non-standard analysis.

Following ideas suggested by Kunita [3], forward and backward equations related to the Markov process are obtained.

1. INTRODUCTION

Given a finite state Markov process $\{X_t\}$, $t \geq 0$, a global "driving noise" process is constructed on a larger probability space using non-standard analysis. The global driving process describes what happens at all times to all states of the process, and a particular realization of the Markov process is obtained by specifying a particular starting time and state; the evolution of the process from then takes place under the action of the driving noise. The driving process, therefore, plays a similar role to the diffusion term in a stochastic differential equation. Following ideas suggested by Kunita [3], forward and backward equations related to the Markov process are obtained.

The forward equation is derived in the next section, and does not need the global driving process. The formulation of the backward equation needs something like a global process; the backward equation is derived in section 3, assuming the basic properties derived from the global process that is constructed in section 4.

2. FORWARD EQUATIONS

In this section we describe the finite state Markov process and derive some related "forward" equations. Write $e_i = (0, 0, \ldots, 1, \ldots, 0)'$ for the i^{th} unit column vector in $\mathbf{R}^q$. Consider a Markov process $\{X_t\}$, $t \geq 0$, defined on a probability space $(\Omega, \mathscr{G}, P)$ whose state space is $S = \{e_1, \ldots, e_q\}$. Write $p_t^i = P\{X_t = e_i\}$; we shall suppose that $p_t = (p_t^1, \ldots, p_t^q)'$ satisfies the forward equation $dp_t/dt = A_t p_t$, where the entries of the matrices A_t are uniformly bounded. The transition matrix associated with A_t will be denoted by $\Phi(t, s)$, so

$$\frac{d\Phi(t, s)}{dt} = A_t \Phi(t, s), \qquad \Phi(s, s) = I, \tag{2.1}$$

and

$$\frac{d\Phi(t, s)}{ds} = -\Phi(t, s) A_s, \qquad \Phi(t, t) = I, \tag{2.2}$$

where I is the $q \times q$ identity matrix. In fact, if $\Phi(t, s) = (\varphi_{ij}(t, s))$ then $\varphi_{ij}(t, s) = P(X_t = e_i | X_s = e_j)$ for $t > s$.

Consider now a Markov process with transition matrix A_t that starts in state e at time s, and write $X_t^{s,e}$ for its state at the time $t \geq s$. Then $\mathrm{E}[X_t^{s,e}] = \Phi(t, s) e$, and, more generally,

$$\mathrm{E}[X_t^{s,e} | X_u^{s,e}] = \Phi(t, u) X_u^{s,e} \quad \text{for } s \leq u \leq t. \tag{2.3}$$

Write $\mathscr{G}_t^s = \sigma\{X_u^{s,e}\colon s \leq u \leq t\}$ and define a process $N_t^s(e)$ by

$$N_t^s(e) = X_t^{s,e} - e - \int_s^t A_u X_u^{s,e} \, du.$$

Then we have the following "forward" equation for $X_t^{s,e}$ (in t):

LEMMA 2.4. *For $t \geq s$ the process $\{N_t^s(e)\}$ is a $\mathscr{G}_t^s$ martingale.*

Proof. Let $t \geq s$ and $h \geq 0$; writing N_t for $N_t^s(e)$ we have

$$N_{t+h} - N_t = X_{t+h}^{s,e} - X_t^{s,e} - \int_t^{t+h} A_u X_u^{s,e} \, du.$$

Then

$$\begin{aligned} \mathrm{E}[N_{t+h} - N_t | \mathscr{G}_t^s] &= \mathrm{E}[N_{t+h} - N_t | X_t^{s,e}] \quad \text{(by the Markov property)} \\ &= \Phi(t+h, t) X_t^{s,e} - X_t^{s,e} - \int_t^{t+h} A_u \Phi(u, t) X_t^{s,e} \, du \quad \text{(by (2.3))} \\ &= 0 \quad \text{(by (2.1))}. \end{aligned}$$

Remark 2.5. A scalar function of $X_t^{s,e}$ can be represented by a row vector $f(t) = (f_1(t), \ldots, f_q(t))$, so that $f(t, X_t^{s,e}) = \langle f(t), X_t^{s,e} \rangle$, where $\langle \; \rangle$ denotes the scalar product in $\mathbb{R}^q$. Similar calculations establish the following

LEMMA 2.6. *Suppose (the components of) $f(t)$ are differentiable in t. Then*

$$f(t, X_t^{s,e}) - f(s, e) = \int_s^t \langle f'(u), X_u^{s,e}\rangle du + \int_s^t \langle f(u), A_u X_u^{s,e}\rangle du + M_{s,t}(f),$$

where $M_{s,t}(f)$ is a forward $\mathscr{G}_t^s$ martingale (in t).

3. BACKWARD EQUATIONS

In this section we derive a backward equation for $X_t^{s,e}$; i.e. an equation that describes the behaviour of this random function as s decreases over $[0, t]$ with t fixed. To do this, we must have a representation of all the processes $X^{s,e}$ on a single probability space $\boldsymbol{\Omega} = (\Omega, \mathscr{F}, P)$ say. The joint behaviour of these processes will, of course, affect the backward equation. We will obtain a reverse martingale description of $X_t^{s,e}$ under the following natural assumptions on the joint behaviour.

(a) Consistency. Suppose that $s < u$, and $X_u^{s,e} = e'$. Then we require the future behaviour of $X^{s,e}$ for $t > u$ to be the same as $X^{u,e'}$; i.e., for $t > u$, $X_t^{u,e'} = X_t^{s,e}$. Thus we impose the condition

(C) $$X_t^{u, X_u^{s,e}} = X_t^{s,e} \quad \text{for all } e, \text{ and } s \leqslant u \leqslant t.$$

(b) Independence of the future. We assume that

(I) $X_u^{s,e}$ and $X_v^{t,e'}$ are independent, for all e, e' and $s \leqslant u \leqslant t \leqslant v$.

The intuition underlying these assumptions is as follows. Given a space $\boldsymbol{\Omega}$ carrying all the processes $X^{s,e}$, it is natural to imagine some underlying driving force that determines their behaviour. Specifically, we envisage the jumps from $X_t^{s,e}$ to $X_u^{s,e}$ to be given by a random function $g_{u,t}\colon S \to S$ such that $P(g_{u,t}(e_i) = e_j) = \varphi_{ji}(u, t)$. Then we would have

$$X_u^{s,e} = g_{u,t}(X_t^{s,e}) \quad \text{for all } e;\ s < t < u.$$

From this the consistency condition (C) follows immediately.

The Markov property means that $g_{u,t}$ should be independent of the past (i.e. what has happened before time u) and the future (i.e. what will happen after time t). This then gives the independence condition (I).

In the next section we will describe a construction of a space $\boldsymbol{\Omega}$ carrying processes $X^{e,s}$ for all e, s, satisfying (C) and (I), and functions $g_{u,t}$ as described above. We shall also show that there is a process $g_t = \lim_{s \uparrow t} g_{t,s}$ such that

$$X_t^{s,e} = g_t(X_{t-}^{s,e}).$$

The process g_t can be thought of as a gremlin who at each time t, independently of past and future, tosses a coin to determine independently for each possible state where to jump to at time t. We shall see that g_t is an analogue of the driving white noise db_t and $g_{u,t}$ is an analogue of the difference $b_u - b_t$ in stochastic integration.

Let us now assume that we are given a space Ω carrying processes $X^{s,e}$ for all s, e. To formulate our backward equation, we define

$$X_t^s = (X_t^{s,e_1}, \ldots, X_t^{s,e_q}) = \text{matrix with columns } X_t^{s,e_i}.$$

Then $X_t^{s,e} = X_t^s e$ for all e. Notice that $\mathrm{E}(X_t^s) = \Phi(t, s)$.

It is easy to see that the consistency condition (C) is equivalent to

$$\text{(C')} \qquad X_t^s = X_t^u X_u^s \quad \text{for all } s \leqslant u \leqslant t.$$

Now define the σ-algebras $\mathscr{F}_t^s$ by $\mathscr{F}_t^s = \sigma\{X_v^u\colon\ s \leqslant u \leqslant v \leqslant t\}$. Then we have the following backward equation:

LEMMA 3.1. *Assume conditions* (C) *and* (I). *Fix t; for $s \leqslant t$ define the process*

$$\tilde{N}_t^s = X_t^s - \int_s^t X_t^u A_u\, du.$$

Then $\tilde{N}_t^s$ is a backward $\mathscr{F}_t^s$ martingale.

Proof. Let $v < s \leqslant t$. Then

$$\begin{aligned}
\mathrm{E}(\tilde{N}_t^v - \tilde{N}_t^s \mid \mathscr{F}_t^s) &= \mathrm{E}\Big(X_t^v - X_t^s - \int_v^s X_t^u A_u\, du \,\Big|\, \mathscr{F}_t^s\Big)\\
&= X_t^s \mathrm{E}\Big(X_s^v - I - \int_v^s X_s^u A_u\, du \,\Big|\, \mathscr{F}_t^s\Big) && \text{(using (C'))}\\
&= X_t^s\Big(\Phi(s, v) - I - \int_v^s \Phi(s, u) A_u\, du\Big)\\
& && \text{(since } X_s^\tau \text{ is independent of } \mathscr{F}_t^s \text{ if } \tau \leqslant s)\\
&= X_t^s\left(\Phi(s, v) - I + \int_v^s \frac{\partial \Phi}{\partial u}(s, u)\, du\right) && \text{(by 2.2)}\\
&= 0.
\end{aligned}$$

As in Lemma 2.6 we can prove

LEMMA 3.2. *Suppose that $f(t, e)$ is a differentiable, scalar function given as in Remark* 2.5. *Then*

$$f(s, X_t^{s,e}) - f(t, e) = -\int_s^t \langle f'(u), X_t^{u,e}\rangle du + \int_s^t \langle f(u), X_t^{u,A_u e}\rangle du + \tilde{M}_{s,t}(f),$$

where $\tilde{M}_{s,t}(f)$ is a backward $\mathscr{F}_t^s$ martingale (in s).

4. CONSTRUCTION OF THE GLOBAL PROCESS

In this section we use the methods of nonstandard analysis to construct processes X_t^s with the properties described in the previous section. We shall also describe the processes $g_{u,t}$ and g_t discussed there. We assume given and fixed an ω_1-saturated nonstandard universe, as for example in [1]. (This reference should be consulted for background.)

Fix an infinite natural number H, and let T be the discrete time set $T = \{k/H\colon 0 \leqslant k < H^2\}$. We use symbols $\underline{s}$, $\underline{t}$, and $\underline{u}$ to range over T, and write $\Delta\underline{t} = H^{-1}$.

Let $\Lambda = S^S = \{(e_{j_1}, e_{j_2}, \ldots, e_{j_q})\colon j_1, \ldots, j_q \in \{1, \ldots, q\}\}$.

The gremlin of the previous section will choose an element λ from Λ at each $\underline{t} \in T$, $\underline{t} > 0$. Thus we define (internally) $\Omega = \Lambda^{T - \{0\}}$.

We wish to put an internal probability measure ν on Ω. Define, for each $\underline{t} > 0$, a probability $\nu_{\underline{t}}$ on Λ by

$$\nu_{\underline{t}}(e_{j_1}, \ldots, e_{j_q}) = \prod_{i=1}^{q} \varphi_{j_i,i}(\underline{t}, \underline{t} - \Delta\underline{t}).$$

Then define ν on Ω by

$$\nu(\{\omega\}) = \prod_{\underline{t}>0} \nu_{\underline{t}}(\{\omega(\underline{t})\}).$$

Let $\mathscr{A}$ be the internal algebra on Ω consisting of *all* (internal) subsets of Ω. Then on the space $(\Omega, \mathscr{A}, \nu)$ we have internal processes $Y_{\underline{s}}^{s,e}$ for $0 \leqslant \underline{s} \leqslant \underline{t}$ and $e \in S$ as follows:

$$Y_{\underline{s}}^{s,e} = e, \qquad Y_{\underline{t}+\Delta\underline{t}}^{s,e} = \omega(\underline{t} + \Delta\underline{t})(Y_{\underline{t}}^{s,e}).$$

It is clear from the construction that

LEMMA 4.1. $\nu(Y_{\underline{t}}^{s,e} = e_j \mid Y_{\underline{u}}^{s,e} = e_i) = \varphi_{j,i}(\underline{t}, \underline{u})$ *if* $\underline{t} \geqslant \underline{u} \geqslant \underline{s}$.

Moreover:

LEMMA 4.2. *For all* e, *if* $\underline{s} \leqslant \underline{u} \leqslant \underline{t}$, *then for all* ω,

$$Y_{\underline{t}}^{u,e}\, Y_{\underline{u}}^{s,e} = Y_{\underline{t}}^{s,e}.$$

LEMMA 4.3. $Y_{\underline{u}}^{s,e}$ *and* $Y_{\underline{v}}^{t,e'}$ *are independent for all* e, e' *and* $\underline{s} \leqslant \underline{u} \leqslant \underline{t} \leqslant v$.

The number of jumps in each path of these processes is bounded by the internal process $N(\underline{t}, \omega)$ defined by $N(0, \omega) = 0$ and

$$\Delta N(\underline{t}+\Delta\underline{t}, \omega) = N(\underline{t}+\Delta\underline{t}, \omega) - N(\underline{t}, \omega) = \begin{cases} 0 & \text{if } \omega(\underline{t}+\Delta\underline{t}) = \iota, \\ 1 & \text{otherwise.} \end{cases}$$

(Here ι = the identity function $= (e_1, \ldots, e_q)$).

The behaviour of $N(\underline{t})$ is given by the next lemmas.

LEMMA 4.4. *There is an internal function* $(k_{\underline{t}})_{\underline{t}>0}$, *finitely bounded, with* $\nu(\Delta N(\underline{t})=0)=1-k_{\underline{t}}\Delta \underline{t}$ *for all* $\underline{t}>0$.

Proof. By transfer of standard theory

$$\Phi(\underline{t}+\Delta \underline{t}, \underline{t}) = I + \int_{\underline{t}}^{\underline{t}+\Delta t} A_u \Phi(u, \underline{t})\, du.$$

Thus $\varphi_{ii}(\underline{t}+\Delta \underline{t}, \underline{t}) = 1 - l_{\underline{t},i}\Delta t$, where $l_{\underline{t},i} \leqslant K$, a finite uniform bound on the entries of A_u. Now

$$\nu(\Delta N(\underline{t})=0) = \prod_{i=1}^{q} \varphi_{i,i}(\underline{t}+\Delta \underline{t}, \underline{t}) = \prod_{i=1}^{q} (1 - l_{\underline{t},i}\Delta \underline{t}) = 1 - k_{\underline{t}}\Delta \underline{t}$$

for uniformly bounded $k_{\underline{t}}$. (In fact $k_{\underline{t}} \leqslant qK+1$).

Now we can prove the following

LEMMA 4.5. (a) *For* $\underline{s}>0$ *let* $M_{\underline{s}} = \sum_{\underline{t}\leqslant \underline{s}} k_{\underline{t}}\Delta \underline{t}$. *Then for finite* $\underline{s}>0$ *and* $p \in N$

(4.6) $$\nu(N(\underline{s})=p) \approx M_{\underline{s}}^p \exp(-M_{\underline{s}})/p!.$$

(b) *More generally, writing*

$$M_{\underline{s},\underline{u}} = M_{\underline{s}} - M_{\underline{u}} = \sum_{\underline{u}<\underline{t}\leqslant \underline{s}} k_{\underline{t}}\Delta \underline{t},$$

then for finite $\underline{u}<\underline{s}$ *and* p

$$\nu(N(\underline{s})-N(\underline{u})=p) \approx M_{\underline{s},\underline{u}}^p \exp(-M_{\underline{s},\underline{u}})/p!.$$

Proof. (a) Fix $\underline{s} = L\Delta \underline{t} > 0$, finite. Then

(4.7) $$\nu(N(\underline{s})=p) = \sum_{0<\underline{t}_1<\ldots<\underline{t}_p\leqslant \underline{s}} \nu\Big(\Lambda_i(\Delta N(\underline{t}_i)=1)\ \Lambda_{\substack{\underline{t}\neq \underline{t}_i\\ \underline{t}\leqslant \underline{s}}} (\Delta N(\underline{t})=0)\Big)$$

$$= \sum_{0<\underline{t}_1<\ldots<\underline{t}_p\leqslant \underline{s}} \Big[\prod_{i=1}^{p} k_{\underline{t}_i}\Delta \underline{t} \prod_{\substack{\underline{t}\neq \underline{t}_i\\ \underline{t}\leqslant \underline{s}}} (1-k_{\underline{t}}\Delta \underline{t})$$

$$= \sum_{0<\underline{t}_1<\ldots<\underline{t}_p\leqslant \underline{s}} \Big[\prod_{i=1}^{p} k_{\underline{t}_i}\Delta \underline{t}(1-k_{\underline{t}_i}\Delta \underline{t})^{-1} \prod_{\underline{t}\leqslant \underline{s}} (1-k_{\underline{t}}\Delta \underline{t})\Big].$$

Now if L is finite, this is infinitesimal, as is the right-hand side of 4.6. So we may assume that L is infinite.

Examining expression (4.7) we see that

$$\prod_{i=1}^{p} (1-k_{\underline{t}_i}\Delta \underline{t})^{-1} \approx 1 \quad \text{for all } 0<\underline{t}_1<\ldots<\underline{t}_p\leqslant \underline{s},$$

and

$$\prod_{\underline{t} \leqslant \underline{s}} (1 - k_{\underline{t}} \Delta \underline{t}) \approx \exp(-\sum_{\underline{t} \leqslant \underline{s}} k_{\underline{t}} \Delta \underline{t}) = \exp(-M_{\underline{s}}).$$

Thus

$$v(N(\underline{s}) = p) \approx \exp(-M_{\underline{s}}) \sum_{0 < \underline{t}_1 < \ldots < \underline{t}_p \leqslant \underline{s}} \prod_{i=1}^{p} k_{\underline{t}_i} \Delta \underline{t}.$$

We now show that $R \approx 0$, where

$$M_{\underline{s}}^p = (\sum_{\underline{t} \leqslant \underline{s}} k_{\underline{t}} \Delta \underline{t})^p = p! \sum_{0 < \underline{t}_1 < \ldots < \underline{t}_p \leqslant \underline{s}} \prod_{i=1}^{p} k_{\underline{t}_i} \Delta \underline{t} + R \tag{4.8}$$

$$= A + R, \text{ say.}$$

If this holds, we are done.

The expanded product on the left of (4.8) has L^p terms, of which $L!/(L-p)!$ are accounted for in A. Thus R consists of $L^p - L!/(L-p)!$ terms of the form $\prod_{i=1}^{p} k_{\underline{t}_i} \Delta \underline{t}$ with each $\underline{t}_i \leqslant \underline{s}$. So, if k is a uniform finite bound for $k_{\underline{t}}$, we have

$$R \leqslant (L^p - L!/(L-p)!) k^p \left(\frac{\underline{s}}{L}\right)^p = (k\underline{s})^p \left(1 - \frac{L(L-1)\ldots(L-p+1)}{L^p}\right) \approx 0$$

since L is infinite.

The proof of (b) is identical.

Now let $P = v_L =$ the Loeb measure on Ω given by v, and let $\mathscr{F} =$ the Loeb σ-algebra on Ω (= the P-completion of $\sigma(\mathscr{A})$). We also put on Ω the right continuous filtration $(\mathscr{F}_t)_{t \geqslant 0}$ defined in the usual way,

$$\mathscr{F}_t = \sigma(\bigcup_{\underline{s} \approx t} \mathscr{A}_{\underline{s}}) \vee \mathscr{N},$$

where $\mathscr{N} = P$-null sets, and $\mathscr{A}_{\underline{s}} =$ the internal algebra generated by $\{\omega(\underline{u}), \underline{u} \leqslant \underline{s}\}$. Let $\boldsymbol{\Omega} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, P)$.

COROLLARY 4.9. *$N(\underline{s})$ is finite for all finite $\underline{s}$, a.s.* (P).

Proof. We have

$$\mathrm{P}(N(\underline{s}) \text{ is finite}) = \sum_{p=0}^{\infty} \mathrm{P}(N(\underline{s}) = p)$$

$$= \exp(-{}^\circ M_{\underline{s}}) \sum_{p=0}^{\infty} {}^\circ M_{\underline{s}}^p/p! = 1.$$

Now we can define a standard process n_t from $N_{\underline{t}}$, in a manner identical to the construction of a Poisson process described in [1].

Definition 4.10. For $t \in \mathbf{R}^+$, (a) $n_t = \sup_{\underline{s} \approx t} N(\underline{s})$ (so n: $\mathbf{R}^+ \times \Omega \to \mathbf{N} \cup \{\infty\}$); (b) $m_t = {}^0 M_{\underline{s}}$ for $\underline{s} \approx t$.

Then $m_0 = 0$, m_t is increasing, continuous, and $m_t \leqslant kt$, where k is a bound for $k_{\underline{t}}$.

We have, by routine techniques using lemmas 4.5 and 4.9, and the principle of overflow:

LEMMA 4.11. (a) *n_t is right continuous, and adapted to $\mathcal{F}_t$, with increments $n_u - n_t$ independent of $\mathcal{F}_t$ when $u > t$.*

(b) $P(n_t = p) = m_t^p e^{-m_t}/p!$.

More generally,

(c) $P(n_u - n_t = p) = m_{u,t}^p e^{-m_{u,t}}/p!$ *where* $m_{u,t} = m_u - m_t$ *for* $u \geqslant t$.

Finally, before defining the processes $X_t^{s,e}$, we have the following facts about the jumps of n_t. Let $\Delta n_t = n_t - n_{t-}$ (where we define $n_{0-} = 0$). Then

LEMMA 4.12. (a) *For each fixed t, $\Delta n_t = 0$, a.s.*

(b) *Almost surely in Ω, $\Delta n_t \leqslant 1$ for all t.*

Proof. (a) Follows easily from 4.11 (b).

(b) It is sufficient to show that this holds on each time interval $[0, p]$ for each $p \in \mathbf{N}$; taking $p = 1$, for example, we have

$$\mathrm{P}(\Delta n_t \geqslant 2, \text{some } t \leqslant 1) = \mathrm{P}\Big(\bigcap_{q \in \mathbf{N}} \bigcup_{j=1}^{q} \{n_{j/q} - n_{(j-1)/q} \geqslant 2\}\Big) = \mathrm{P}\Big(\bigcap_{q \in \mathbf{N}} A_q\Big), \text{ say.}$$

Now $\mathrm{P}(n_u - n_t \leqslant 1) = e^{-m_{u,t}}(1 + m_{u,t}) \geqslant e^{-k(u-t)}(1 + k(u-t))$. Thus

$$\mathrm{P}(A_q) \leqslant q\big(1 - e^{-k/q}(1 + k/q)\big) \to 0 \quad \text{as } q \to \infty,$$

and we are done.

Remarks. 1. Lemma 4.11 shows that n_t is a time changed Poisson process; specifically, putting $c_t = \sup\{u\colon m_u < t\}$, $c_0 = 0$, we have $\tilde{\mathrm{n}}_t = \mathrm{n}_{c_t}$ is a Poisson process with rate 1. The facts in Lemma 4.12 follow from this also.

2. In terms of the process $N_{\underline{t}}$, Lemma 4.12 (b) means that almost surely both $N_{\underline{t}}$, and all the processes $Y^{\underline{s},e}$ jump at most once on every monad. It is useful to fix a set Ω_1 of full measure on which n_t is finite and $\Delta n_t \leqslant 1$ for all t.

The processes $X_t^{s,e}$ can now be defined on Ω as follows:

Definition 4.13. Let $t \geqslant s \geqslant 0$. We have

$$X_t^{s,e} = \lim_{\substack{\underline{u}\uparrow, \underline{v}\uparrow \\ \underline{u} \approx s, \underline{v} \approx t}} Y_{\underline{v}}^{\underline{u},e} \quad (= Y_{\underline{v}}^{\underline{u},e} \text{ for all sufficiently large } \underline{u} \approx s,\ \underline{v} \approx t).$$

This makes sense on the set Ω_1 by Lemma 4.12 and the remark following it.

The following facts about the processes $X_t^{s,e}$ are routinely established:

LEMMA 4.14. (a) *On* Ω_1, *for all* s, e, (i) $X_s^{s,e} = e$; (ii) $X^{s,e}$ *is right continuous, and has only finitely many jumps on each finite time interval*; (iii) $X_t^{s,e}$ *is right continuous in* s, *for fixed* t.

(b) *For each* $t \geqslant s$, *and all* e, $X_t^{s,e} = Y_{\underline{t}}^{\underline{s},e}$ *all* $\underline{s} \approx s$, $\underline{t} \approx t$, *a.s.*

The next result is the main point of this section.

THEOREM 4.15. *The processes* $X^{s,e}$ *are* $\mathscr{F}_t$-*Markov, with transition probabilities given by* Φ, *satisfying the consistency and independence properties* (C) *and* (I) *on* Ω_1 *for all* s, t, u, v.

Proof. The Markov property follows routinely from the following observations:

(1) for $A \in \mathscr{F}_t$, there is a $B \in \mathscr{A}_{\underline{t}}$ for some $\underline{t} \approx t$ with $A \Delta B$ null [2];

(2) for any $s < t < u$, if $\underline{s} \approx s$, $\underline{t} \approx t$, $\underline{u} \approx u$, then

$$\mathrm{P}(Y_{\underline{u}}^{\underline{s},e} = e_j \mid Y_{\underline{t}}^{\underline{s},e}) = \mathrm{P}(X_u^{s,e} = e_j \mid X_t^{s,e}),$$

which is an immediate corollary of Lemma 4.12(a);

(3) $Y_{\underline{u}}^{\underline{s},e}$ is $(\mathscr{A}_{\underline{t}})$ Markov, by construction.

From (2) we see that

$$\begin{aligned}\mathrm{P}(X_u^{s,e} = e_j \mid X_t^{s,e} = e_i) &= {}^\circ\varphi_{j,i}(\underline{u}, \underline{t}) \text{ (for } \underline{t} \approx t < u \approx \underline{u}) \\ &= \varphi_{ji}(u, t), \text{ by continuity.}\end{aligned}$$

Conditions (C) and (I) for the processes $X^{s,e}$ follow immediately from the corresponding properties of $Y^{\underline{s},e}$ (Lemmas 4.1 and 4.2).

Remark. Condition (C′) (i.e. $X_t^s = X_t^u X_u^s$) holds for all $s \leqslant u \leqslant t$ and all $\omega \in \Omega_1$ (see remark following Lemma 4.12). This completes the construction of the global processes $X^{s,e}$ needed for § 3.

We conclude this section with a brief discussion of the processes $g_{u,t}$ and g_t described intuitively in § 3. We can immediately define $G_{\underline{t},\underline{s}}(\omega) \in \Lambda$ for $\underline{t} \geqslant \underline{s}$ by

$$G_{\underline{s},\underline{s}} = \iota, \qquad G_{\underline{t}+\Delta t,\underline{s}}(\omega) = \omega(\underline{t}+\Delta \underline{t}) \circ G_{\underline{t},\underline{s}}.$$

Clearly we have $Y_{\underline{t}}^{\underline{s},e} = G_{\underline{t},\underline{s}}(e)$, and in fact putting $Y_{\underline{t}}^{\underline{s}} = (Y_{\underline{t}}^{\underline{s},e_1}, \ldots, Y_{\underline{t}}^{\underline{s},e_q})$, we have

$$Y_{\underline{t}}^{\underline{s}} = G_{\underline{t},\underline{s}}(I) = G_{\underline{t},\underline{s}}(e_1, \ldots, e_q),$$

so $Y_{\underline{t}}^{\underline{s}}$ and $G_{\underline{t},\underline{s}}$ are essentially the same thing. Likewise we may define $g_{t,s}$ for $t \geqslant s$ by

$$g_{t,s} = \lim_{\substack{\underline{u}\uparrow, \underline{v}\uparrow \\ \underline{u} \approx s, \underline{v} \approx t}} G_{\underline{v},\underline{u}} (= G_{\underline{v},\underline{u}} \text{ for sufficiently large } \underline{u} \approx s \text{ and } \underline{v} \approx t),$$

which makes sense on the set Ω_1. Then $X_t^{s,e} = g_{t,s}(e)$ and $X_t^s = g_{t,s}(I)$, so X_t^s and $g_{t,s}$ are essentially the same.

Clearly we have

LEMMA 4.16. *On* Ω_1, (a) $g_{s,s} = \imath$; (b) $g_{t,s} = g_{t,u} g_{u,s}$ *for* $t \geq u \geq s$; (c) $X_t^{s,e} = g_{t,u}(X_{\underline{u}}^{s,e})$.

Finally, on the set Ω_1 we may define the driving noise (or gremlin process) g_t by

$$g_t = \lim_{\substack{u\downarrow, v\uparrow \\ \underline{u} \approx t \approx \underline{v}}} G_{\underline{v},\underline{u}}.$$

The following is immediate from the definitions and earlier lemmas:

LEMMA 4.17. *On* Ω_1, (a) $g_t = \lim_{s \to t-} g_{t,s} = g_{t,t-} = g_{t+,t-}$; (b) $X_t^{s,e} = g_t(X_{t-}^{s,e})$ (*or* $X_t^s = g_t(X_{t-})$) *for* $t > s$.

Equation (b) is an analogue of a stochastic differential equation.

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