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CONDITIONS FOR CONVERGENCE OF NUMBER OF CROSSINGS TO THE LOCAL TIME

APPLICATION TO STABLE PROCESSES WITH INDEPENDENT INCREMENTS AND TO GAUSSIAN PROCESSES

BY

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Abstract. Let $X(t)$, $t \in \boldsymbol{R}$, be a real valued stochastic process admitting a local time and let $X_\varepsilon(t)$, $\varepsilon \in \boldsymbol{R}^+$, be a family of *smooth* processes which converge in some sense to $X(t)$. We exhibit sufficient conditions for L^2-convergence of the number of crossings of $X_\varepsilon(t)$ to the local time of $X(t)$, after normalization.

Two main cases are considered for $X(t)$, stable processes and Gaussian processes.

Two main cases are considered for $X_\varepsilon(t)$: $X_\varepsilon(t)$ being the convolution of $X(t)$ with a size ε approximate identity and $X_\varepsilon(t)$ being the size ε polygonal approximation of $X(t)$.

Such a convergence is shown to hold for both approximations when $X(t)$ is a stable process with independent increments with index $\alpha > 1$.

Convergence of crossings of the polygonal approximation is shown to hold for a Gaussian process under technical conditions.

1. INTRODUCTION

The concept of local time of a stochastic process $X(t)$ was introduced by Levy [19] in the case of the Brownian motion process. The general theory of local time has been developed in three different directions, those of Markov processes, Gaussian processes and semimartingale processes. Even the definition of the local time is different according to the considered class of processes. The paper by Geman and Horowitz [17] gives a general survey of the different approaches.

In this paper we define the local time $L(u, t)$ as the density of the occupation measure of $X(t)$. This definition was used for example by Berman [2] for Gaussian processes, but it can be applied to any other class of processes.

Apart from the work on the definition and study of the properties of local time some effort has been put into the construction of $L(u, t)$ by limiting processes based on the sample path properties of $X(t)$. Particularly, Levy's constructions [19, sect. 50] for Brownian motion have been extended to general Markov processes (see e.g. [16]).

In 1984, Wschebor [23, 24] proposed a construction which is quite different from those of Fristedt and Taylor. Let $W(t)$ be a Brownian motion, and let W_ε be the convolution approximation of W defined by

$$W_\varepsilon = W * \psi_\varepsilon, \quad \text{where } \psi_\varepsilon(t) = \frac{1}{t}\psi(t/\varepsilon),$$

ψ being a non-negative $\mathscr{C}^\infty$ function with compact support, and let $N_\varepsilon^u[a, b]$ be the number of crossings of level u by the process $W_\varepsilon(t)$ for $t \in [a, b]$. Wschebor showed that

$$(\pi/2)^{1/2} \|\psi\|_2^{-1} \varepsilon^{-1/2} N_\varepsilon^u[a, b] \xrightarrow{L^k} L(u, [a, b]) \quad \text{as } \varepsilon \to 0.$$

Let $X_\varepsilon(t)$ be a family of smooth processes which converge in some sense to $X(t)$. Such a family will here be called an *approximating family*. The aim of this paper is to exhibit general sufficient conditions for convergence of crossings of X_ε to the local time of $X(t)$. Two particular cases will be examined in sections 6 and 7:

the convolution case: $X_\varepsilon(t)$ is defined as for the Wiener process;

the polygonalization case: $X_\Delta(t)$ is the polygonal line going through the points $\{(k\Delta, X(k\Delta)); k \in N\}$.

General notation. Throughout this paper:

$p[X_1, \ldots, X_n; x_1, \ldots, x_n]$ – the joint density of random variables $X_1, \ldots, X_n$ at the point $x_1, \ldots, x_n$;

(const) – an arbitrary positive constant;

N_ε^u – the number of crossings of X_ε defined as

$$N_\varepsilon^u(I) = \#\{t \in I; X_\varepsilon(t) = u\};$$

$\dot{N}_\varepsilon(I)$ – the number of local extrema of X_ε on the interval I;

$\dot{X}_\varepsilon$ – the derivative of X_ε (in the sense of absolute continuity).

2. MAIN RESULTS

Let $\mathscr{D}$ be a bounded real interval and let $X(t)$, $t \in \mathscr{D}$, be a stochastic process having 2-dimensional continuous density functions

$$p[X(t_1), X(t_2); x_1, x_2]$$

with respect to the Lebesgue measure. It is assumed that these density functions verify the hypothesis

H_1. *There exists a function g, with $g(\varepsilon) = o(\varepsilon)$, as ε tends to zero, such that, for every $[a, b]$ included in $\mathscr{D}$,*

$$\int_{[a,b]^2} \sup_{(x_1,x_2)\in R^2} p[X(t_1), X(t_2); x_1, x_2]\, dt_1\, dt_2 \leqslant g(b-a). \tag{2.1}$$

The condition of finiteness of the integral in (2.1) with continuity of the density is a little stronger than Geman and Horowitz's [17] condition (23.4). So it implies the existence of local time in the sense that, almost surely, the occupation measure μ of the paths of $X(t)$ over $[a, b]$ is absolutely continuous with respect to Lebesgue measure: there exists a function $L(u, [a, b])$ a.s. μ a.e. finite which is the density of μ. Moreover, $L(u, dt)$ is atomless almost surely and for almost every u.

These two conditions also provide an easy way to prove that the functions

$$q_I(x_1, x_2) = \int_{I\times I} p[X(t_1), X(t_2); x_1, x_2]\, dt_1\, dt_2$$

are continuous. Arguments used by Berman [3, lemma 2.1] and Geman and Horowitz imply that the random variables

$$\eta_\delta^u[a, b] = (2\delta)^{-1}\mathbb{1}\{|X(t)-u| < \delta\}\, dt$$

converge in L^2-norm as δ tends to zero and that $q_{[a,b]}(u, u)$ is the L^2-norm of $L(u, [a, b])$.

Hypothesis H_1 is stronger than Geman and Horowitz's condition (23.4) but for non-periodic stationary Gaussian processes it is equivalent to their condition (22.7) which is the weakest known sufficient condition for the existence of the local time.

Hypothesis H_1 does not imply the joint continuity of the local time. But Berman [4] has given a set of sufficient conditions depending on the function q. For example, if the function g of (2.1) satisfies the additional hypothesis $g(\varepsilon) = o(\varepsilon^{1+\delta})$ with $\delta > 0$, then Berman's condition (2.9) is fulfilled for $k = 2$. If, in addition, q satisfies condition (2.8), then by Berman's theorem 2.1, the local time can be chosen jointly continuous.

Our main result can heuristically be stated as follows. Let $\left(X_\varepsilon(t),\ \varepsilon\in \boldsymbol{R}^+\right)$ be a smooth almost sure approximation of $X(t)$, and let $n(\varepsilon)$ be a normalization such that

$$X, X_\varepsilon, n(\varepsilon)\dot{X}_\varepsilon \xrightarrow{\mathscr{D}} (X, X)\otimes \mathscr{S} \quad \text{as } \varepsilon\to 0, \tag{2.2}$$

where $\xrightarrow{\mathscr{D}}$ denotes the convergence of all finite distributions and $\mathscr{S}$ is a white noise independent of X with $E[|\mathscr{S}(t)|] = 1$; then we get theorem 2:

$$n(\varepsilon) N_\varepsilon^u(I) \xrightarrow{L^2} L(u, I). \tag{2.3}$$

Formally,

$$L(u, I) = \lim_{\delta\to 0}(2\delta)^{-1}\int_I \mathbb{1}\{|X(t)-u| < \delta\}\, dt,$$

and by Kac's formula [18], with appropriate conditions, we have

$$N_\varepsilon^u(I) = \lim (2\delta)^{-1}\int_I \mathbb{1}\{|X_\varepsilon(t)-u| < \delta\}\, |\dot{X}_\varepsilon(t)|\, dt.$$

Condition (2.2) implies the 2-dimensional cylindrical convergence of $n(\varepsilon)\dot{X}_\varepsilon(t)$ to a white noise independent of X or X_ε. Our theorems state that, when you integrate a function of $X_\varepsilon(t)$, $n(\varepsilon)|\dot{X}_\varepsilon(t)|\,dt$ converges in L^2-norm to the Lebesgue measure.

Sections 6 and 7 present applications for a given class of processes. Theorem 4 of section 6 proves the convergence in the convolution and polygonalization cases where $X(t)$ is a stable process with independent increments. Theorem 5 of section 7 proves the convergence in the polygonalization case for a general class of Gaussian processes. Using our method, Florens-Zmirou [15] has shown the convergence in the polygonalization case for recurrent diffusions. At last, our method could be applied to the convolution of stationary Gaussian processes, but in this last case this would need stronger conditions than those of Azaïs and Florens-Zmirou [1] which are obtained by explicit calculations.

L^2-convergence of the polygonal approximation crossings (theorem 5) seems to be a new result even for the Brownian motion. Using results by Borodin [9] or Csörgö and Révész [12], we can define a probability space on which the polygonal approximation crossings converge strongly to the local time of another Brownian motion. In our context this implies only weak convergence of the crossings of X_ε to the local time of X itself.

Almost sure convergence seems very hard to establish since the only calculations that can be done about crossings are moment calculations. In some cases, like convolution of particular stationary Gaussian processes, it can be proved that the L^2-norm between the normalized number of crossings and the local time is bounded by a power of ε. By the Borel–Cantelli lemma this implies almost sure convergence along any sequence ε_n decreasing to zero at geometrical rate. Unfortunately, nothing is known about the variation of N_ε as a function of ε, except that in the polygonalization case we have $N^u_{\Delta/n}(I) \geqslant N^u_\Delta(I)$. Anyway it is not sufficient to control the variation of the normalized number of crossings between two terms of the almost surely converging sequence. The problem remains open even for the Brownian case.

Section 3 gives a new proof of Rice's formulae for a given class of stochastic processes.

3. RICE'S FORMULAE

Let $Y(t)$, $t \in \mathscr{D}$, be a stochastic process. We say that it *satisfies Rice's formulae of order* n for $(t_1, \ldots, t_n)$ belonging to $\mathscr{D}_n \subset \mathscr{D}^n$ iff the following three conditions hold:

(3.1) $Y(t)$ has almost surely absolutely continuous sample paths.

(3.2) For $(t_1, \ldots, t_n)$ belonging to $\mathscr{D}_n$, $Y(t_1), \ldots, Y(t_n)$, $\dot{Y}(t_1), \ldots, \dot{Y}(t_n)$ have a joint density and the function

$$A_{t_1,\dots,t_n}(y_1,\dots,y_n) = \int_{R^n} |\dot{y}_1|\dots|\dot{y}_n|\, p\,[Y(t_1), Y(t_2), \dots, Y(t_n), \dot{Y}(t_1), \dots, \dot{Y}(t_n);$$

$$y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n]\, d\dot{y}_1 \dots d\dot{y}_n$$

is continuous in $(y_1, \dots, y_n)$; this function will be called the *order n Rice function* of process Y.

(3.3) Let

$$M^n[N^{u_1}(I_1), N^{u_2}(I_2), \dots, N^{u_n}(I_n)] = \#\{\prod_{i=1}^{n}[t_i \in I_i;\ Y(t_i) = u_i] - E\},$$

where $E = \{t_1, \dots, t_n \in \mathscr{D}_n | \exists\, i, j;\ t_i = t_j\}$. Then, for each set of levels u_i and intervals I_i such that $\prod_1^n I_i \subset \mathscr{D}_n$, we have

$$EM^n[N^{u_1}(I_1), N^{u_2}(I_2), \dots, N^{u_n}(I_n)] = \int_{\Pi I_i} A_{t_1,\dots,t_n}(u_1, \dots, u_n)\, dt_1 \dots dt_n.$$

Let us remark that:

1. When all I_i's and u_i's are equal, $E(M^n)$ is the factorial moment of order n.
2. The restriction "t belong to $\mathscr{D}_n$" and not the whole $\mathscr{D}^n$ has been introduced for two reasons. First, if Y is the polygonal approximation, Y and $\dot{Y}$ may not have joint densities for $|t_1 - t_2| < \varepsilon$. Second, in the same case ($|t_1 - t_2| < \varepsilon$) we do not know how to prove the existence of joint densities for Y or $\dot{Y}$, when Y is the convolution approximation.
3. The existence of joint densities is not really needed; it has just been supposed, for simplicity, only the existence and continuity of $A_{t_1,\dots,t_n}$ is needed [5].

Such formulae have been first established for a given class of Gaussian processes by Rice [21] in 1945. The non-Gaussian case has received contributions by Besson [5], Marcus [20], and Wschebor [22, 23]. These authors have given very general proofs of Rice formulae using weak but technical hypotheses. Here we give a proof for processes with not "too many" crossings – hypothesis K_2 – which prevents us from studying most of the pathological cases of the above-mentioned papers.

PROPOSITION. *Let $Y(t)$ satisfy the following hypotheses.*

K_1. *Conditions* (3.1) *and* (3.2) *hold and for any compact sets $K_1 \subset \mathscr{D}_n$ and $K_2 \subset R^n$ we have*

$$\int_{K_1} \sup_{(x_1,\dots,x_n)\in K_2} A_{t_1,\dots,t_n}(x_1, \dots, x_n)\, dt_1 \dots dt_n < \infty.$$

K_2. *The moment of order n of the number of local extrema $\dot{N}(\mathscr{D})$ of $Y(t)$ over $\mathscr{D}$ is finite.*

Then $Y(t)$ satisfies Rice's formulae of order n for $(t_1, \dots, t_n)$ belonging to $\mathscr{D}_n$.

Remark. Hypothesis K_2 is much stronger than those of Besson, Marcus or Wschebor, but it is trivially satisfied in the polygonalization case. In the convolution case, K_2 is a consequence of lemma 1.

In the general case it must be considered as a condition of smoothness on the family X_ε.

Proof. Since both members of (3.3) are additive, it is sufficient to prove it for $\operatorname{dist}(\prod_i I_i, E) > 0$. M^n is thus an ordinary product. By K_2 the number $\dot{N}$ of local extrema of $Y(t)$ for t belonging to any subinterval of $\mathscr{D}$ is almost surely finite; we are allowed to use "the Kac's crossings counter" [18]: original Kac's proof can be extended without problems to absolutely continuous functions. As $Y(t)$ is supposed to have 1-dimensional densities, it does not take — with probability 1 — any of the values u_i at extremities of intervals and is not constantly equal to any u_i on any interval. Thus

$$\prod_{i=1}^{n} N^{u_i}(I_i) = \lim_{\delta\to 0}(2\delta)^{-n} \int_{\prod_i I_i} \mathbf{1}\{|Y(t_1)-u_1| < \delta\} \ldots \mathbf{1}\{|Y(t_n)-u_n| < \delta\} \times$$
$$\times |\dot{Y}(t_1)| \ldots |\dot{Y}(t_n)|\, dt_1 \ldots dt_n \quad \text{a.s.}$$

Kac's result can be improved by noting that, for every $\delta > 0$,

$$(3.4)\quad (2\delta)^{-n} \int_{\prod_i I_i} \mathbf{1}\{|Y(t_1)-u_1| < \delta\} \ldots \mathbf{1}\{|Y(t_n)-u_n| < \delta\} \times$$
$$\times |\dot{Y}(t_1)| \ldots |\dot{Y}(t_n)|\, dt_1 \ldots dt_n$$
$$\leqslant \prod_i \{N^{u_i}(I_i) + \dot{N}(I_i)\} \leqslant (\text{const}) \{\dot{N}\cdot(\bigcup_{i=1}^{n} I_i)\}^n + (\text{const}).$$

The last inequality in (3.4) is an easy application of the Rolle Theorem. The first inequality may be proved as follows.

It is sufficient to prove it for $n = 1$. Let $J = \{t \in I:\ |Y(t)-u| < \delta\}$, where J is the union of finitely many open intervals:

$$J = \bigcup_{k=1}^{K} L_k.$$

Divide J according to local extrema of Y. Since Y is monotonous within each subinterval of the partition, the integral of $|\dot{Y}|$ over such a subinterval is less than 2δ in the general case and less than δ when L_k contains no crossing.

Now, by K_2 and the dominated convergence theorem we get

$$E\{\prod_{i=1}^{n} N^{u_i}(I_i)\}$$
$$= \lim_{\delta\to 0}(2\delta)^{-n} \int_{\prod_{i=1}^{n} I_i} \int_{u_1-\delta}^{u_1+\delta} \ldots \int_{u_n-\delta}^{u_n+\delta} A_{t_1,\ldots,t_n}(y_1, \ldots, y_n)\, dy_1 \ldots dy_n\, dt_1 \ldots dt_n.$$

Since $A_{t_1,\dots,t_n}$ is continuous and dominated by K_1, a second application of the dominated convergence theorem gives

$$E\{\prod_{i=1}^{n} N^{u_i}(I_i)\} = \int_{\prod_{i=1}^{n} I_i} A_{t_1,\dots,t_n}(u_1, \dots, u_n)\, dt_1 \dots dt_n.$$

4. SUFFICIENT CONDITIONS FOR CONVERGENCE OF NUMBER OF CROSSINGS TO THE LOCAL TIME

THEOREM 1. *Let $X(t)$, $t \in \mathscr{D}$, satisfy H_1, and let $X_\varepsilon(t)$, $\varepsilon \in R^+$, be a family of processes defined on the same probability space. Suppose there exists a normalization $n(\varepsilon)$ such that (X, X_ε) satisfies the following hypotheses H_2, H_3, and H_4.*

H_2. *Rice formulae.* (a) *For every ε, $X_\varepsilon(t)$ satisfies order two Rice's formulae for (t_1, t_2) belonging to $[\mathscr{D}^2 - \{|t_1 - t_2| < 2\varepsilon\}]$.*

(b) *For every pair of intervals I_1, I_2 with* $\operatorname{dist}(I_1, I_2) > 2\varepsilon$ *and for every level u we have*

$$E\{\int_{I_1} 1\{|x(t) - u| < \delta\}\, N_\varepsilon^u(I_2)\}$$

$$= \int_{I_1 \times I_2} \int_{u-\delta}^{u+\delta} \int_{R} |\dot{x}|\, p[X(t_1), X_\varepsilon(t_2), \dot{X}_\varepsilon(t_2); v, u, x]\, d\dot{x}\, dv\, dt_1\, dt_2.$$

The last quantity will be denoted by

$$\int_{I_1 \times I_2} B^\varepsilon_{t_1,t_2}(u - \delta, u + \delta, u)\, dt_1\, dt_2.$$

H_3. *Convergence and domination of Rice functions. Let $A_t^\varepsilon(u)$ $A^\varepsilon_{t_1,t_2}(u_1, u_2)$ be the order one and two Rice functions of X_ε. We have, for all considered u,*

(a) $(n(\varepsilon))^2 A^\varepsilon_{t_1,t_2}(u, u) \to p[X(t_1), X(t_2); u, u]$ *as $\varepsilon \to 0$, the convergence being pointwise for all different t_1 and t_2;*

(b) $\sup_{\varepsilon \leqslant |t_1 - t_2|/2} (n(\varepsilon))^2 A^\varepsilon_{t_1,t_2}(u, u) \leqslant H_u(t_1, t_2)$ *with* $\iint_{\mathscr{D}^2} H_u(t_1, t_2) < \infty$;

(c) *for all* $\delta > 0$

$$n(\varepsilon) B^\varepsilon_{t_1,t_2}(u - \delta, u + \delta, u) \to \int_{u-\delta}^{u+\delta} p[X(t_1), X(t_2); v, u]\, dv \quad \text{as } \varepsilon \to 0,$$

the convergence being pointwise;

(d) *there exists an $\varepsilon_0 > 0$ such that*

$$\operatorname*{Sup}_{t \in \mathscr{D}, \varepsilon < \varepsilon_0} \{n(\varepsilon) A_t^\varepsilon(u)\} < \infty.$$

H_4. *For each considered u*

$$\operatorname*{Sup}_{I \subset \mathscr{D}, l(I) < \varepsilon} E[n(\varepsilon) N^u(I)]^2 = o(\varepsilon),$$

$l(I)$ being length of the interval I.

Then for all considered levels u and every interval I, the random variables $n(\varepsilon)N_\varepsilon^u(I)$ *and*

$$(1/2\delta)\int_I \mathbf{1}\{|X(t)-u|<\delta\}\,dt$$

converge in L^2-norm to the same limit as ε and δ tend to zero.

If $X(t)$ admits a.s. a space continuous local time $L(u, I)$, then it is a.s. the limit of the last variable and we have the following

THEOREM 2. *Let X and X_ε satisfy* H_1-H_3, *and suppose that $X(t)$ admits a space continuous local time $L(u, I)$. Then, for every interval I in $\mathscr{D}$,*

$$n(\varepsilon)N_\varepsilon^u(I) \xrightarrow{L^2} L(u, I) \quad \text{as } \varepsilon \to 0.$$

Put

$$\xi_\varepsilon^u(I) = n(\varepsilon)N_\varepsilon^u(I) \quad \text{and} \quad \eta_\delta^u(I) = \frac{1}{2\delta}\int_I \mathbf{1}\{|X(t)-u|<\delta\}\,dt.$$

Remarks. A. Hypothesis H_2 seems very difficult to avoid, because the only tool for studying crossings numbers are Rice formulae.

B. Hypothesis H_3 seems very difficult to avoid, because in the general case it is impossible to prove the Rice formulae validity without excluding a strip around the diagonal.

C. $n(\varepsilon)$ may depend additionally on time t — as in section 7 — and even on the level u. The proof of the theorem is then the same, but the statement of conditions is more complicated. In the most general case, where $n(\varepsilon) = n(\varepsilon, t, u)$, the main modifications are the following:

1° $n(\varepsilon)\dot{X}_\varepsilon(t)$ must be replaced, in (2.2), by $n(\varepsilon, t, X_\varepsilon(t))\cdot\dot{X}_\varepsilon(t)$;

2° $\xi_\varepsilon^u(I)$ must be written as

$$\sum_{\{t\in I; X_\varepsilon(t)=u\}} n(\varepsilon, t, u);$$

3° H_3 must be written as $E\{\xi_\varepsilon^u(I)\} = o(\varepsilon)$.

Proof of theorem 1. It is a direct consequence of the following result:

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} E|\xi_\varepsilon^u(I)-\eta_\delta^u(I)|^2 = 0, \tag{4.1}$$

$$E|\xi_\varepsilon^u(I)-\eta_\delta^u(I)|^2 = E[\xi_\varepsilon^u(I)]^2 + 2E\,\eta_\delta^u(I)\,\xi_\varepsilon^u(I) + E[\eta_\delta^u(I)]^2. \tag{4.2}$$

The quantity $[\xi_\varepsilon^u(I)]^2$ is divided in two terms:

$$\xi_A = (n(\varepsilon))^2\,\#\{(t_1, t_2)\in I^2;\ |t_1-t_2| > 2\varepsilon;\ X_\varepsilon(t_1) = X_\varepsilon(t_2) = u\},$$

$$\xi_B = (n(\varepsilon))^2\,\#\{(t_1, t_2)\in I^2;\ |t_1-t_2| \leqslant 2\varepsilon;\ X_\varepsilon(t_1) = X_\varepsilon(t_2) = u\}.$$

$E(\xi_A)$ can be calculated by Rice formulae. The convergence and domination of Rice functions and the dominated convergence theorem imply

$$\lim_{\varepsilon\to 0} E(\xi_A)^2 = \lim_{\varepsilon\to 0}\iint_{I^2} p[X(t_1), X(t_2); u, u]\,dt_1\,dt_2. \tag{4.3}$$

$E(\xi_B)$ can be bounded as follows. Let n be the integer part of $((b-a)/\varepsilon+1)$. Divide I into n ordered equal intervals $I_1, \ldots, I_n$ and put $\xi_i = n(\varepsilon) N^u_\varepsilon(I_i)$, $i = 1, \ldots, n$. Then

$$E(\xi_B) \leqslant \sum_{|i-j| \leqslant 3} E(\xi_i, \xi_j) \leqslant 7n \sup_{i=1,n} E(\xi_i^2).$$

By H_4, $\sup_{i=1,\ldots,n} E(\xi_i^2) = o(\varepsilon)$, thus

$$E(\xi_B) \to 0. \tag{4.4}$$

The quantity $\eta^u_\delta(I)\,\xi^u_\varepsilon(I)$ is divided into the same two parts as $[\xi^u_\varepsilon(I)]^2$:

$$\eta_A = \sum_{t_1 \in I, X_\varepsilon(t_1)=u} \frac{1}{2\delta} \int\limits_{\substack{t_2 \in I, \\ |t_2 - t_1| > 2\varepsilon}} \mathbf{1}\{|X(t_2)-u| < \delta\}\, dt_2,$$

$$\eta_B = \sum_{t_1 \in I, X_\varepsilon(t_1)=u} \frac{1}{2\delta} \int\limits_{\substack{t_2 \in I, \\ |t_1 - t_2| \leqslant 2\varepsilon}} \mathbf{1}\{|X(t_2)-u| < \delta\}\, dt_2.$$

$E(\eta_A)$ can be calculated using the "Rice like" formula H_2 (b); its limit in ε is given through the dominated convergence theorem by H_3 (c) and H_3 (d):

$$\lim_{\varepsilon \to 0} E(\eta_A) = \iint_{I^2} \int_{u-\delta}^{u+\delta} p[X(t_1), X(t_2); u, v]\, dv\, dt_1\, dt_2,$$

and now, using H_1,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} E(\eta_A) = \iint_{I^2} p[X(t_1), X(t_2), u, u]\, dt_1\, dt_2. \tag{4.5}$$

$E(\eta_B)$ can be bounded by the same proof as $E(\xi_B)$, since H_1 implies

$$\sup_{i=1,\ldots,n} E[\eta^u_\delta(I_i)]^2 = o(\varepsilon),$$

so

$$\lim_{(\delta,\varepsilon) \to 0} E(\eta_B) = 0. \tag{4.6}$$

The expectation $E[\eta^u_\delta(I)]^2$ can be calculated by the Fubini theorem:

$$E[\eta^u_\delta(I)]^2 = \left(\frac{1}{2\delta}\right)^2 \iint_{I^2} \int_{u-\delta}^{u+\delta} \int_{u-\delta}^{u+\delta} p[X(t_1), X(t_2); u_1, u_2]\, du_1\, du_2\, dt_1\, dt_2;$$

and now, by H_1,

$$\lim_{\delta \to 0} E[\eta^u_\delta(I)]^2 = \iint_{I^2} p[X(t_1), X(t_2); u, u]\, dt_1\, dt_2. \tag{4.7}$$

Now, relations (4.2) to (4.7) imply (4.1).

Theorem 1 holds by noting that (4.1) implies that $\xi_\varepsilon^u(I)$ is a Cauchy sequence and, again by (4.1), $\eta_\delta^u(I)$ converges to the same limit.

5. APPROXIMATING FAMILIES WITH NOT "TOO MANY" LOCAL EXTREMA

In this section we assume that the approximating family $\{X_\varepsilon;\ \varepsilon \in \boldsymbol{R}^+\}$ satisfies the following hypothesis:

H_5. *For every integer m there exists a constant c_m such that, for all intervals of size less than ε included in $\mathscr{D}$, we have*

$$E[\dot{N}_\varepsilon(I)]^m \leqslant c_m[\varepsilon^{-1} n(\varepsilon)+1].$$

H_5 can be interpreted as a condition of "smoothness" of the approximating family. It is trivially met for polygonal approximation and is a consequence of lemma 1 below for convolution approximation. Note that H_5 implies that, for every ε, X_ε satisfies hypothesis K_2 of the proposition of section 3 about Rice formulae.

THEOREM 3. *The hypothesis H_5 on X_ε and the following hypothesis H_6 imply the hypotheses H_2, H_3, and H_4 of theorem* 1.

H_6 (a) *to* H_6 (c) *are* H_3 (a) *to* H_3 (c).

H_6 (d). *For every ε and every compact set K_2 in $\boldsymbol{R}^2$*

$$\iint_{\substack{(t_1,t_2)\in\mathscr{D}^2\\ |t_1-t_2|>2\varepsilon}} \sup_{(x_1,x_2)\in K_2} A^\varepsilon_{t_1,t_2}(x_1, x_2)\,dt_1\,dt_2 < \infty.$$

H_6 (e). *There exists an ε_0 such that, for every compact $Q_2 \subset \boldsymbol{R}$,*

$$\sup_{\varepsilon<\varepsilon_0,t\in\mathscr{D},x\in Q_2} n(\varepsilon) A^\varepsilon_t(x) < \infty.$$

H_6 (f). *$A^\varepsilon_{t_1,t_2}$, A^ε_t and $B^\varepsilon_{t_1,t_2}(u-\delta, u+\delta, \cdot)$ are continuous for given ε if $|t_1-t_2| > 2\varepsilon$.*

H_6 (g). *There exists $\beta > 0$ such that $n(\varepsilon) = O(\varepsilon^\beta)$.*

Consequently, if X satisfies H_1, conclusions of theorem 1 hold.

Proof of theorem 3. H_6 (a) and the continuity of $A^\varepsilon(t_1, t_2)$ on $\mathscr{D}_2^\varepsilon = \{(t_1, t_2)\in\mathscr{D}^2;\ |t_1-t_2| > 2\varepsilon\}$ imply K_1 of the proposition of section 3. Since H_5 implies K_2, X_ε satisfies Rice formulae of order 2 on $\mathscr{D}_2^\varepsilon$.

$B^\varepsilon_{t_1,t_2}(u-\delta, u+\delta, x)$ is less than $A^\varepsilon_{t_2}(x)$ which is bounded by H_6 (e); the proof of the proposition implies that X_ε satisfies the "Rice like" formula H_2 (b). Thus H_2 hold.

H_6 (e) is stronger than H_3 (d), thus H_3 hold.

We now prove H_4. Let u be fixed; H_6 (e) and H_5 imply that X_ε satisfies Rice formulae of order 1 on $\mathscr{D}$ and that, for every $[a, b] \subset \mathscr{D}$, every u, and every $\varepsilon < \varepsilon_0$, $n(\varepsilon)E(N^u_\varepsilon[a, b]) \leqslant (\text{const})(b-a)$.

Let now $(b-a) < \varepsilon$. By Bienayme's inequality,

$$P[N_\varepsilon^u[a, b] > 0] \leqslant (\text{const})\,\varepsilon\,[n(\varepsilon)]^{-1}. \tag{5.1}$$

By the Rolle theorem,

$$E[(N_\varepsilon^u[a, b])^2] \leqslant E[(\dot{N}_\varepsilon[a, b]+1)^2 \mathbf{1}\{N_\varepsilon^u[a, b] > 0\}].$$

Take now q an integer and p its conjugate: $p^{-1}+q^{-1} = 1$. Then apply the Hölder inequality. From H_5 and (5.1) we get

$$E(n(\varepsilon))^2 (N_\varepsilon^u[a, b])^2 \leqslant c_q'[\varepsilon^{1/p-1/q} n(\varepsilon)^{2+1/q-1/p} + \varepsilon^{1/p} n(\varepsilon)^{2-1/p}], \tag{5.2}$$

where c_q' is a constant depending on q.

An elementary calculation shows that q can be chosen large enough for the right-side term of (5.2) being bounded — by H_6 (g) — by $\varepsilon^{1+\delta}$ with δ positive. This proves H_4 and theorem 3.

The following lemma gives sufficient conditions for H_5.

LEMMA 1. *Suppose that for every ε the process $X_\varepsilon(t)$ verifies the following hypotheses*:

(a) *$X_\varepsilon(t)$ has a.s. $\mathscr{C}^\infty$ sample paths;*

(b) *for t belonging to $\mathscr{D}$, $\dot{X}_\varepsilon(t)$ and $\ddot{X}_\varepsilon(t)$ have a joint density bounded by* $(\text{const})\,\varepsilon\,[n(\varepsilon)]^2$, *where $n(\varepsilon)$ is a given function;*

(c) *for $p > 1$ we have $E[\sup_{t\in\mathscr{D}} |X_\varepsilon^{(p)}(t)|] < c_p \varepsilon^{-p}$, $X^{(p)}$ denoting the pth derivative, and c_p a constant depending on p.*

Then H_5 holds.

In the convolution case (a) holds if ψ is $\mathscr{C}^\infty$, and (c) holds if $E[\sup_{t\in\mathscr{D}^+} |X(t)|] < \infty$, where $\mathscr{D}^+$ is a domain larger than $\mathscr{D}$, but (b) has to be proved in each case.

Proof. Apply Besson and Wschebor's condition [6], [24] for finiteness of moments of crossings. Consider the process $Y_\varepsilon(t) = n(\varepsilon)\dot{X}_\varepsilon(\varepsilon t)$.

Apply Wschebor's [24] corollary 4, section 3.2, with $p = 2m+2$ to get that the order m moment of Y_ε level zero crossings over an interval of length less than 1 is bounded by (const) $\varepsilon^{-1} n(\varepsilon)$ + (const), where both constants depend on m. Returning to $X_\varepsilon(t)$, H_5 holds.

6. APPLICATION TO A STABLE PROCESS WITH INDEPENDENT INCREMENTS

THEOREM 4. *Let $X(t)$ be a stable process with independent increments and index $\alpha > 1$ defined by*

$$X(0) = 0, \qquad E\exp[i\lambda(X(a)-X(b))] = \exp[-(b-a)s(\lambda)],$$

where $s(\lambda) = r[1-i\,\text{sign}(\lambda)\beta\,\text{tg}(\pi\alpha/2)|\lambda|^\alpha]$, $r\in \mathbf{R}^+$, $\beta\in[-1, 1]$, $\alpha\in(1, 2]$ *(see* [14]).

When $\alpha = 2$ and $\beta = 0$, $X(t)$ is a Brownian motion.

Let ψ be a $\mathscr{C}^\infty$ function with support in $[-1/2, 1/2]$, $\psi_\varepsilon(t) = (1/\varepsilon)\psi(t/\varepsilon)$ and X_ε obtained by convolution with ψ_ε. Then for every level u and every interval I included in $\mathscr{D} = [a_0, b_0]$, with $a_0 > 0$, we have

$$(E|X(1)|)^{-1} \|\psi\|_\alpha^{-1} \varepsilon^{1-1/\alpha} N_\varepsilon^u(I) \xrightarrow{L^2} L(u, I).$$

Let X_Δ be the size Δ polygonal approximation of X and N_Δ the number of crossings of X_Δ. Then

$$(E|X(1)|)^{-1} \Delta^{1-1/\alpha} N_\Delta^u(I) \xrightarrow{L^2} L(u, I).$$

Proof. Boylan [10] has shown that such processes admit a bicontinuous local time $L(u, I)$. We know that $X(1)$ has a bounded continuous density and that $X(\lambda t)$ has the same distribution as $\lambda^{1/\alpha} X(t)$, hence H_1 is met. We give the proof only for X_ε, which is the more complicated case, the proof for X_Δ being very similar except that we do not use lemma 1. The following lemma gives the n-dimensional distribution of $(X_\varepsilon(t), \dot{X}(t))$:

LEMMA 2. *Let g be a continuous function with compact support in $\boldsymbol{R}^+$ and let*

$$G(t) = \int_u^{+\infty} g(t)\,dt, \qquad Y = \int_{\boldsymbol{R}^+} g(t) X(t)\,dt.$$

Then

$$E[e^{iY}] = \exp\left[-\int_{\boldsymbol{R}} s[G(t)]\,dt\right]. \tag{6.1}$$

Proof. Since gX is right continuous with left limits, $\int g(t)X(t)\,dt$ is equal to the limit of its Riemann's sums, so it suffices to prove (6.1) for $g(t)$ being a sum of "Dirac functions". In this case, by a triangular operation (which is in fact a hidden integration by parts), $\int g(t)X(t)\,dt$ can be written as a linear combination of increments of X over disjoint intervals. Since these increments are independent, both members of (6.1) are additive and it suffices to prove this for $g(t)$ defining an increment over an interval, say $(a, b]$. In this case $g(t) = \delta_b - \delta_a$, δ being the Dirac distribution, $G(t) = \mathbf{1}\{(a, b]\}$, and (6.1) is trivial.

Continuation of the proof of Theorem 4. ε will be supposed to be less than a_0. Take $g(t) = \sum \lambda_i \psi_\varepsilon(t_i - t) + \sum \mu_i \psi'_\varepsilon(t_i - t)$. Then lemma 2 gives the characteristic function of $X_\varepsilon(t_1), \ldots, X_\varepsilon(t_n), \dot{X}_\varepsilon(t_1), \ldots, \dot{X}_\varepsilon(t_n)$.

If every two t_i's differ from each other more than of 2ε, we can see that this characteristic function is integrable over $\boldsymbol{R}$; thus the density of the $2n$-tuplet exists, is bounded and continuous. From the expression of the characteristic function we can also see that (2.2) holds with

$$n(\varepsilon) = (E|X(1)|)^{-1} \|\psi\|_\alpha^{-1} \varepsilon^{1-1/\alpha}. \tag{6.2}$$

We prove now that the hypotheses of lemma 1 are met.

(a) is trivial.

(b) comes from the fact that $(\dot{X}_\varepsilon(t), \ddot{X}_\varepsilon(t))$ has the same law as $\varepsilon^{1/\alpha-1}\dot{X}_1(t)$, $\varepsilon^{1/\alpha-2}\ddot{X}_1(t)$, which has a bounded density by lemma 2.

(c) $X(t)-tE[X(1)]$ is a martingale with finite order β $(1<\beta<\alpha)$ moments. Apply the Doob inequalities [13] to get

$$E[\sup_{t\in[0,b_0+a_0]} |X(t)|] < \infty.$$

Since $\varepsilon < a_0$, (c) follows and H_5 is proved.

We prove now that the hypotheses of theorem 3 are met.

(6.2) implies H_6 (g).

$X(t)$ is a Markov process. Let ${}^z p$ denote densities for the distribution $X(0) = z$ and define

$${}^z B_t^\varepsilon = n(\varepsilon) \int_R |\dot{x}| \sup_x {}^z p[X_\varepsilon(t), \dot{X}_\varepsilon(t); x, \dot{x}]\, dx.$$

Then

$$\sup_{z\in R, \varepsilon<t} {}^z B_t^\varepsilon \leqslant (\text{const})\, t^{-1/\alpha}. \tag{6.3}$$

To prove (6.3) use a conditioning by $X(t/2)$:

$${}^z B_t^\varepsilon = n(\varepsilon) \int_R |\dot{x}| \sup_x \int_R {}^z p[X(t/2); w]\, {}^w p[X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x, \dot{x}]\, dw\, d\dot{x}.$$

${}^z p[X(t/2); w]$ is bounded by $(\text{const})(t/2)^{-1/\alpha}$.

Since X has independent increments,

$${}^w p[X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x, \dot{x}] = {}^0 p[X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x-w, \dot{x}],$$

we obtain

$${}^z B_t^\varepsilon \leqslant (\text{const})\, t^{-1/\alpha} n(\varepsilon) \int |\dot{x}| \sup_x \int {}^0 p[X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x-w, \dot{x}]\, dw\, d\dot{x}.$$

Noting that the integral over w does not depend on x, we get

$${}^z B_t^\varepsilon \leqslant (\text{const})\, t^{-1/\alpha} n(\varepsilon) E|\dot{X}_\varepsilon(t/2)| = (\text{const})\, t^{-1/\alpha}.$$

Using the Markov property, relation (6.3) gives all the required estimations:

H_6 (c) with $H_u(t_1, t_2) = (\text{const})\, a_0^{-1/\alpha} |t_1 - t_2|^{-1/\alpha}$;

H_6 (d) and H_6 (e).

(6.3) implies also the continuity of Rice functions in H_6 (f) by the dominated convergence theorem.

Convergence of normalized Rice functions. A change of variable gives

$$[n(\varepsilon)]^2 A_{t_1,t_2}^\varepsilon(x_1, x_2)$$
$$= \int_R |\dot{x}_1 \dot{x}_2|\, p[X_\varepsilon(t_1), X_\varepsilon(t_2), n(\varepsilon)\dot{X}_\varepsilon(t_1), n(\varepsilon)\dot{X}_\varepsilon(t_2); x_1, x_2, \dot{x}_1, \dot{x}_2]\, d\dot{x}_1\, d\dot{x}_2.$$

As soon as $\varepsilon \leqslant \inf(t_1/2, (t_1-t_2)/3)$, conditioning by $X(t_1/2)$ and the increments of X over $[(2t_1+t_2)/3, (t_1+2t_2)/3]$ gives

$$[n(\varepsilon)]^2 A^{\varepsilon}_{t_1,t_2}(x_1, x_2)$$

$$= \int_{R^2} |\dot{x}_1||\dot{x}_2| \int_{R^2} p[X(a), X(b), z_1, z_2]\, p[X_\varepsilon(t_1'), X_\varepsilon(t_2'), n(\varepsilon)\dot{X}_\varepsilon(t_1'), n(\varepsilon)\dot{X}_\varepsilon(t_2');$$

$$x_1 - z_1, x_2 - z_2, \dot{x}_1, \dot{x}_2]\, dz_1\, dz_2\, d\dot{x}_1\, d\dot{x}_2,$$

where $a = t_1/2$, $b = t_1/2 + (t_2 - t_1)/3$, $t_1' = t_1 - a$, and $t_2' = t_2 - b$.

Now $p[X(a), X(b); \ldots]$ is bounded and continuous, and the quadruplet $(X_\varepsilon(t_1'), X_\varepsilon(t_2'), n(\varepsilon)\dot{X}_\varepsilon(t_1'), n(\varepsilon)\dot{X}_\varepsilon(t_2'))$ converges in distribution, but $|\dot{x}_1||\dot{x}_2|$ is not bounded. Noting that $[n(\varepsilon)\dot{X}_\varepsilon(t_1'), n(\varepsilon)\dot{X}_\varepsilon(t_2')]$ has the same distribution as $\mathscr{S} \otimes \mathscr{S}$, we get for $1 < \gamma < \alpha$

$$\sup_\varepsilon E|n(\varepsilon)\dot{x}_\varepsilon(t_1') n(\varepsilon)\dot{x}_\varepsilon(t_2')|^\gamma < \infty,$$

and now, by a uniform integrability argument [7], the limit of (6.4) is equal to

$$\int_{R^2} p[X(a), X(b); z_1, z_2]\, p[X(t_1'), X(t_2'); x_1 - z_1, x_2 - z_2]\, dz_1\, dz_2$$

$$= p[X(t_1), X(t_2); x_1, x_2].$$

Convergence of $n(\varepsilon) B^{\varepsilon}_{t_1,t_2}(u-\delta, u+\delta, u)$ follows by a similar proof. H_6 (a) and H_6 (c), and thus theorem 4, are proved.

7. POLYGONAL APPROXIMATION OF GAUSSIAN PROCESSES

In this section we prove the following

THEOREM 5. *Let $X(t)$ be a zero-mean Gaussian process with covariance function $r(t, s)$. We suppose that r is twice continuously differentiable outside the diagonal $(t = s)$ and satisfies the following hypotheses on a compact set $\mathscr{D}$.*

There exist $\alpha < 2$, $\beta > \alpha/2$ and d positive such that, for $|t-s| < d$,

$$r(s, s)r(t, t) - r^2(s, t) > (\text{const})|t-s|^\alpha, \tag{7.1}$$

$$|r(s, t) - r(t, t)| < (\text{const})|t-s|^\beta. \tag{7.2}$$

Condition (7.1) implies that $X(t)$ is non-differentiable in quadratic mean [11].

Outside the diagonal the process satisfies a non-determinism condition:

$$\inf_{|t-s| \geq d} \{r(s, s)r(t, t) - r^2(s, t)\} = e > 0. \tag{7.3}$$

The variance is bounded:

$$0 < \inf_{t\in\mathscr{D}} r(t, t) \leq \sup_{t\in\mathscr{D}} r(t, t) < \infty. \tag{7.4}$$

Let r_Δ be the covariance function of the size $-\Delta$ polygonal approximation. There exists a constant f such that in a strip $|t-s| <$ (const) we have, for $\Delta < |t-s|/f$,

$$r_\Delta(t, t)r_\Delta(s, s) - r_\Delta^2(t, s) \geq (\text{const})|t-s|^\gamma \quad \textit{with } \gamma < 2. \tag{7.5}$$

Then the number of crossings over $[a, b] \subset \mathscr{D}$ *of the size* $-\Delta$ *polygonal approximation* $X(t)$, *normalized by*

$$(7.6)\quad n(\Delta, t) = \sqrt{\pi/2}\,\Delta\left[r(k\Delta, k\Delta)+r((k+1)\Delta, (k+1)\Delta)-2r(k\Delta, (k+1)\Delta)\right]^{-1/2} \quad \text{for } k\Delta \leqslant t < (k+1)\Delta$$

converges in L^2*-norm to the same limit as* $\eta_\delta^u[a, b]$.

Proof. Let r' denote the derivative with respect to the second variable and r'' the cross second derivative.

Put $t_1 = k\Delta + s_1\Delta$ $(0 \leqslant s_1 < 1)$, $t_2 = h\Delta + s_2\Delta$ $(0 \leqslant s_2 < 1)$ and suppose that $\Delta \leqslant d$. Then

$$r_\Delta(t_1, t_2) = (1-s_1)(1-s_2)r(k\Delta, h\Delta)+(1-s_1)s_2 r(k\Delta, (h+1)\Delta)+ \\ +(1-s_2)s_1 r((k+1)\Delta, h\Delta)+s_1 s_2 r((k+1)\Delta, (h+1)\Delta).$$

Differentiation gives:

$$(7.7)\quad r'_\Delta(t_1, t_2) = E(X_\Delta(t_1)\dot{X}_\Delta(t_2)) \\ = \Delta^{-1}\{(1-s_1)[r(k\Delta, (h+1)\Delta)-r(k\Delta, h\Delta)]+ \\ +s_1[r((k+1)\Delta, (h+1)\Delta)-r((k+1)\Delta, h\Delta)]\},$$

$$(7.8)\quad r''_\Delta(t_1, t_2) = E[\dot{X}_\Delta(t_1)\dot{X}_\Delta(t_2)] \\ = \Delta^{-2}[r(k\Delta, h\Delta)+r((k+1)\Delta, (h+1)\Delta)-r(k\Delta, (h+1)\Delta)-r((k+1)\Delta, h\Delta)].$$

The convergence of the 2-dimensional distribution (2.2) is obtained with the normalization $\sqrt{\pi/2}\cdot r_\Delta(t, t)$, which explains (7.6).

Boundedness and continuity of Rice functions. As soon as $\Delta \leqslant |t-s|/f$, the distribution of $X_\Delta(t_1)$ and $X_\Delta(t_2)$ is non-degenerated and the Rice functions are trivially continuous. Thus H_6 (f) holds by putting for example $\varepsilon = f\Delta/2$.

Conditions H_6 (d) and H_6 (e) are easy to prove.

We turn now to give bounds to $A^\Delta_{t_1,t_2}(u, u)$ (hypothesis H_6 (b)):

$$A^\Delta_{t_1,t_2} = E(|\dot{X}_\Delta(t_1)\dot{X}_\Delta(t_2)|/X_\Delta(t_1) = u;\ X_\Delta(t_2) = u)\,p[X_\Delta(t_1), X_\Delta(t_2); u, u].$$

By the Schwarz inequality we see that in order to prove H_6 (b) we have to bound

$$E[(\dot{X}_\Delta(t_1))^2/X_\Delta(t_1) = X_\Delta(t_2) = u]\,p[X_\Delta(t_1), X_\Delta(t_2); u, u] \\ = \{\mathrm{Var}[\dot{X}_\Delta(t_1)/X_\Delta(t_1), X_\Delta(t_2)]+\{E[\dot{X}_\Delta(t_1)/X_\Delta(t_1) = X_\Delta(t_2) = u]\}^2\}\times \\ \times p[X_\Delta(t_1), X(t_2); u, u].$$

Let t_1, t_2 and Δ be such that (7.5) holds. We have

$$\mathrm{Var}[\dot{X}_\Delta(t_1)/X_\Delta(t_1), X_\Delta(t_2)] \leqslant \mathrm{Var}[\dot{X}_\Delta(t_1)] \leqslant (\text{const})[n(\Delta, t_1)]^{-2}$$

and

$$E[\dot{X}_\Delta(t_1)/X_\Delta(t_1) = X_\Delta(t_2) = u]^2 = a^2 u^2.$$

Put $(Y, S) = (\dot{X}_\Delta(t_1), X_\Delta(t_1)+X_\Delta(t_2))$ under the conditional distribution $X_\Delta(t_1)-X_\Delta(t_2) = 0$. Then $\mathrm{Var}(Y) \leqslant (\mathrm{const})(n(\Delta, t_1))^{-2}$.

Let $\{\sigma_{ij};\ i = 1, 2;\ j = 1, 2\}$ be the variance matrix of $X_\Delta(t_1)$ and $X_\Delta(t_2)$. Then

$$\mathrm{Var}(S) = 4\frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{11}+\sigma_{22}-2\sigma_{12}}.$$

Now

$$a^2 = \left[\frac{\mathrm{Cov}(Y, S/2)}{\mathrm{Var}(S/2)}\right]^2 \leqslant 4\frac{\mathrm{Var}(Y)}{\mathrm{Var}(S)};$$

then

$$\begin{aligned} &a^2 u^2 p[X_\Delta(t_1), X_\Delta(t_2); u, u] \\ &\quad\leqslant (\mathrm{const})(n(\Delta, t_1))^{-2} u^2 \frac{\sigma_{11}+\sigma_{22}-2\sigma_{12}}{[\sigma_{11}\sigma_{22}-\sigma_{12}^2]^{3/2}} \exp\left[-\frac{u^2}{2}\frac{\sigma_{11}+\sigma_{22}-2\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}\right] \\ &\quad\leqslant (\mathrm{const})(n(\Delta, t_1))^{-2} |t_1-t_2|^{-\gamma/2} \end{aligned}$$

because of (7.5) and the fact that $ze^{-z/2}$ is bounded. Gathering the pieces we get

$$\begin{aligned} E\left[(\dot{X}_\Delta(t_1))^2/X_\Delta(t_1) = X_\Delta(t_2) = u\right] p[X_\Delta(t_1), X_\Delta(t_2); u, u] \\ \leqslant (\mathrm{const})(n(\Delta, t_1))^{-2}(t_1-t_2)^{-\gamma/2}, \end{aligned}$$

which proves H_6 (b) with $\varepsilon = (f\Delta)/2$ and $H_u(t_1, t_2) = (\mathrm{const})|t_1-t_2|^{-\gamma/2}$.

We turn now to the proof of the convergence of normalized Rice functions (hypotheses H_6 (a) and H_6 (c)). Formulae (7.2) and (7.7) imply

$$r'_\Delta(t_1, t_1) = E(X_\Delta(t_1), \dot{X}_\Delta(t_1)) < (\mathrm{const})\,\Delta^{\beta-1}.$$

Let t_1 and t_2 be any time points. Suppose that Δ is less than $(t_1-t_2)/2$,

$$r'_\Delta(t_1, t_2) = (1-s_1)r'(k\Delta, \xi)+s_1 r'((k+1)\Delta, \xi) \quad \text{with } \xi\in(h\Delta, (h+1)\Delta);$$

but the derivative r' is bounded on the compact set $\{s, t\in\mathscr{D};\ |s-t| > (t_1-t_2)/2\}$, so $|r'_\Delta(t_1, t_2)|$ is bounded by a constant as Δ varies.

We have $r''_\Delta(t_1, t_2) = r''(\xi_1, \xi_2)$, $\xi_1\in[k\Delta, (k+1)\Delta]$, $\xi_2\in[h\Delta, (h+1)\Delta]$, so that $|r''_\Delta(t_1, t_2)|$ is bounded by a constant as a function of Δ. On the other hand, (7.4) and (7.2) imply that r_Δ converges uniformly to r.

The relations above imply the convergence of the variance matrix of $X_\Delta(t_1)$, $X_\Delta(t_2)$, $n(\Delta, t_1)\dot{X}_\Delta(t_1)$, $n(\Delta, t_2)\dot{X}_\Delta(t_2)$ to the limit of (2.2). This implies H_6 (a). H_6 (c) follows by a similar proof, and theorem 5 is proved.

The following are two examples of processes satisfying the hypotheses of theorem 5.

1. The fractional Wiener process defined by

$$r(t, s) = \frac{|t|^\alpha + |s|^\alpha - |t-s|^\alpha}{2} \quad \text{with } 0 < \alpha < 2.$$

The limit is actually the local time.
2. Any stationary process with

$$r(t) = 1 - |t|^\alpha + O\,|t|^\alpha \quad \text{and} \quad 0 < \alpha < 2.$$

If the process satisfies the hypothesis of theorem 7.1 of [3] the limit is the local time again.

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