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REMARKS ON BANACH SPACES OF S-COTYPE p*

BY

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Abstract. This paper continues the investigations of [10]. There are examined relations between the class of Banach spaces of S-cotype p, the class of Banach spaces of M-cotype p in the sense of Mouchtari [7] and the class V_p of Banach spaces defined by Tien and Weron [11].

1. Introduction. Let E be a Banach space with dual E'. E is said to be of *stable type* p $(0 < p \leqslant 2)$ if, for every sequence (x_n) in E with $\sum \|x_n\|^p < \infty$, $\sum x_n \theta_n^{(p)}$ converges a.s., where $\theta_n^{(p)}$ are i.i.d. symmetric p-stable random variables. For $p = 2$ stable type 2 is equivalent to type 2. E is said to be of *cotype* 2 if, for every sequence (x_n) in E such that $\sum x_n \theta_n^{(2)}$ converges a.s., $\sum \|x_n\|^2 < \infty$. It is known that an analogous definition of stable cotype p $(0 < p < 2)$ by replacing the sequence $\{\theta_n^{(2)}\}$ by the sequence $\{\theta_n^{(p)}\}$ does not restrict the class of Banach spaces, since the a.s. convergence of $\sum x_n \theta_n^{(p)}$ implies that $\sum \|x_n\|^p$ is finite for $p < 2$.

In attempting to extend the results of [1] to p-stable measures, Tien and Weron [11] defined a class V_p $(1 \leqslant p < 2)$ of Banach spaces, and we have defined the notion of S-cotype p [10]. From another motivation, Mouchtari [7] has introduced the notion of M-cotype p.

Our aim is to study the relation between the class M_p of spaces of M-cotype p, the class S_p of spaces of S-cotype p and the class V_p. The main results of the paper are the inclusions $M_p \subset V_p \subset S_p \subset \bigcap_{\varepsilon > 0} M_{p+\varepsilon}$ $(1 \leqslant p < 2)$, from which we obtain the inclusion $V_p \subset V_q$ for $p < q$ (*going up phenomenon*). By this phenomenon we can refer to a Banach space in the class V_p as a Banach space of V-cotype p. It is interesting to know whether the three possible notions of cotype coincide.

2. Preliminaries and notation. Let E be a Banach space with dual E'. We say that E is a *Sazonov space* if there exists a topology $\mathscr{T}$ on E such that a positive definite function f with $f(0) = 1$ is $\mathscr{T}$-continuous iff it is a characteristic

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function (ch. f.) of a probability measure on E. It has been shown [6] that every Sazonov space can be embedded into L_0 and, conversely, if a Banach space with the metric approximation property embeds in L_0, then it is a Sazonov space. In particular, every closed subspace of $L_p (1 \leqslant p \leqslant 2)$ is a Sazonov space, while, for $p > 2$, L_p is not a Sazonov space.

For a real number p $(0 < p \leqslant 2)$ we denote by X_p a closed subspace of L_p. $\Lambda_p(E', X_p)$ denotes the set of linear continuous operators T from E' into X_p for which the function $f(a) = \exp\{-\|Ta\|^p\}$, $a \in E'$, is the ch. f. of a probability measure on E. An operator T in $\Lambda_p(E', X_p)$ for some X_p is called a *Λ_p-operator* on E'. Let $\mathscr{T}_p$ denote the coarsest topology on E for which all the ch. f.'s of symmetric p-stable measure are continuous. A Banach space E is said to be of *M-cotype* p $(0 < p \leqslant 2)$, provided that the function $f\colon E' \to C$ is the ch. f. of a probability measure on E, if it is positive definite, $\mathscr{T}_p$-continuous and $f(0) = 1$. Equivalently, a Banach space E is of M-cotype p iff any $\mathscr{T}_p$-continuous linear mapping A from E' into $L_0(\Omega, P)$ is decomposable. We remind that a linear mapping A from E' into $L_0(\Omega, P)$ is said to be *decomposable* if there exists an E-valued random variable φ such that

$$P\{\omega\colon\ Aa(\omega) = \langle\varphi(\omega), a\rangle\} = 1 \quad \text{for all } a \in E'.$$

Mouchtari [7] has shown that M-cotype 2 spaces are exactly cotype 2 spaces and M-cotype p spaces, for some $p < 1$, are exactly Sazonov spaces.

Following [11] we say that a Banach space E *is in the class* V_p $(0 < p \leqslant 2)$ if for every symmetric p-stable measure μ and for every symmetric p-stable cylindrical measure ν the inequality $|1-\hat{\nu}(a)| \leqslant |1-\hat{\mu}(a)|$ for all $a \in E'$ implies that ν is a Radon measure, where $\hat{\mu}(a)$ and $\hat{\nu}(a)$ are the ch. f.'s of μ and ν, respectively.

Finally, a Banach space E is said to be of *S-cotype* p $(0 < p \leqslant 2)$ if for every sequence (x_n) in E and every symmetric p-stable measure μ on E the inequality

$$1-\exp\{-\sum|\langle x_n, a\rangle|^p\} \leqslant 1-\hat{\mu}(a) \quad \text{for all } a \in E'$$

implies that $\sum\|x_n\|^p$ is finite.

In [10] it was shown that E is of S-cotype 2 iff it is of cotype 2. A Banach space with the approximation property is of S-cotype p for $p < 1$ iff it is a Sazonov space.

3. Relation between spaces of M-cotype p, spaces of S-cotype p and spaces in the class V_p.

1. THEOREM. *Let M_p and S_p denote the class of spaces of M-cotype p and the class of spaces of S-cotype p, respectively. Then*

$$M_p \subset V_p \subset S_p \subset \bigcap_{\varepsilon>0} M_{p+\varepsilon} \quad (1 \leqslant p < 2).$$

Proof. (a) $M_p \subset V_p$. Let E be a Banach space of M-cotype p and suppose that μ is a symmetric p-stable measure on E, and ν is a symmetric p-stable cylindrical measure on E such that $|1-\hat{\nu}(a)| \leqslant |1-\hat{\mu}(a)|$ for all $a \in E'$. From this inequality it follows that $\hat{\nu}(a)$ is $\mathcal{T}_p$-continuous. Since E is of M-cotype p, $\hat{\nu}(a)$ is a ch. f. of a Radon measure on E. This shows that E belongs to the class V_p.

(b) $V_p \subset S_p$. Let E be in the class V_p and let (x_n) be a sequence in E such that

$$1-\exp\{-\sum |\langle x_n, a\rangle|^p\} \leqslant 1-\hat{\mu}(a) \quad \text{for all } a \in E',$$

where μ is a symmetric p-stable measure on E.

Let ν be the p-stable cylindrical measure with the ch. f.

$$\hat{\nu}(a) = \exp\{-\sum |\langle x_n, a\rangle|^p\}.$$

By the assumption that E belongs to V_p, $\hat{\nu}(a)$ is a ch. f. of a Radon measure on E. From the Itô–Nisio theorem it follows that the series $\sum x_n \theta_n^{(p)}$ converges a.s. Since $p < 2$, we have $\sum \|x_n\|^p < \infty$. Hence E is of S-cotype p.

(c) $S_p \subset \bigcap_{\varepsilon > 0} M_{p+\varepsilon}$. We split the proof into two steps.

(i) *Suppose that E is of S-cotype p $(1 \leqslant p < 2)$. Then every symmetric q-stable $(q > p)$ measure on E is the continuous image of a symmetric q-stable measure on some Sazonov space.*

Indeed, let μ be a symmetric q-stable measure on E $(q > p)$ with the ch. f. $\hat{\mu}(a) = \exp\{-\|Ta\|^q\}$, where $T \in \Lambda_q(E', L_q)$. Because of $q > p$, by Theorem 2 in [7], the function $\exp\{-\|Ta\|^p\}$ is also the ch. f. of a Radon measure on E. Thus $T \in \Lambda_p(E', L_q)$. Since E is of S-cotype p by Theorem 3.3 in [10], the adjoint T': $L'_q \to E$ is p-summing. By the Pietsch factorization theorem, there exists a factorization T^*: $L'_q \xrightarrow{U} S \xrightarrow{V} E$, where S is a closed subspace of L_p, V: $S \to E$ is a linear continuous operator and U: $L'_q \to S$ is a p-summing operator.

The operator U, being p-summing, is also r-summing for $1 \leqslant p < r < q$. Let γ_q be the canonical cylindrical q-stable measure on L'_q with the ch. f. $\exp\{-\|x\|^q\}$, $x \in L_q$. γ_q is of the scalar order r, i.e.

$$\sup_{\|x\| \leqslant 1} \int_{L'_q} |\langle x, y\rangle|^r d\gamma_q(y) < \infty.$$

Because U is r-summing $(r > 1)$ in view of the Schwartz radonification theorem [9], $\nu = U(\gamma_q)$ is a Radon measure on S. We have $\mu = T^*(\gamma_q) = V[U(\gamma_q)] = V(\nu)$.

ν is a symmetric q-stable measure on S and S is a Sazonov space (since every closed subspace of L_p $(1 \leqslant p \leqslant 2)$ is a Sazonov space).

(ii) *Suppose that every symmetric p-stable measure on a Banach space E is a continuous image of a symmetric p-stable measure on some Sazonov space. Then E is of M-cotype p.*

Indeed, let A be a $\mathcal{T}_p$-continuous linear mapping from E' into $L_0(\Omega, P)$. Then, given $\varepsilon > 0$, there exists a Λ_p-operator T_ε on E' such that $\|T_\varepsilon a\| \leqslant 1$ implies $\|Aa\|_0 < \varepsilon$, where $\|\cdot\|_0$ is the F-norm in $L_0(\Omega, P)$ metrizing the topology of convergence in probability.

By Lemma 5.2 in [3] we can choose a single Λ_p-operator T on E' satisfying the following condition:

(1.1) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|Aa\|_0 < \varepsilon$, whenever $\|Ta\| < \delta$.

Let μ be a symmetric p-stable measure generated by T, i.e. $\hat{\mu}(a) = \exp\{-\|Ta\|^p\}$, $a \in E'$. By the assumption, there exist a Sazonov space S, a linear continuous operator V: $S \to E$ and a symmetric p-stable measure ν on S such that $\mu = V(\nu)$. Without loss of the generality we can assume that V is 1-1. Let H be a Λ_p-operator on S' generating ν, i.e. $\hat{\nu}(b) = \exp\{-\|Hb\|^p\}$, $b \in S'$. We have $\hat{\mu}(a) = V\hat{}(\nu)(a) = \hat{\nu}(V^* a) = \exp\{-\|HV^* a\|^p\}$. Hence

$$\|Ta\| = \|HV^* a\| \quad \text{for all } a \in E'. \tag{1.2}$$

Define a linear mapping G from $V^*(E')$ into $L_0(\Omega, P)$ by $G(V^* a) = Aa$.

G is well-defined on $V^*(E')$. Indeed, if $V^* a_1 = V^* a_2$, then by (1.2) we have $\|T(a_1 - a_2)\| = 0$, which, together with (1.1), enables us to conclude that $\|A(a_1 - a_2)\|_0 = 0$, i.e. $Aa_1 = Aa_2$ in $L_0(\Omega, P)$. In view of (1.1), for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|G(b)\|_0 < \varepsilon$, whenever $\|Hb\| < \delta$ for all $b \in V^*(E')$. In other words, G is $\mathscr{T}_p$-continuous on $V^*(E') \subset S'$. The linearity of G is obvious. Since $V^*(E')$ is dense in S', G admits a $\mathscr{T}_p$-continuous linear extension on the whole S'. Because S is of M-cotype p (every Sazonov space is of M-cotype p for all p), G is decomposed by an S-valued random variable φ, i.e. $G(b)(\omega) = \langle \varphi(\omega), b \rangle$ P-a.s. for all $b \in S'$. Hence, for all $a \in E'$, $A(a)(\omega) = G(V^* a)(\omega) = \langle \varphi(\omega), V^* a \rangle = \langle V\varphi(\omega), a \rangle$ P-a.s., which shows that A is decomposable, as desired.

Thus the proof of Theorem 1 is completed.

From Theorem 1 we derive:

2. Corollary. *If a Banach space E belongs to the class V_p, then it also belongs to the class V_q for $1 \leqslant p < q$.*

3. Corollary. *The space l_s (l_t), where $1 \leqslant p < t < s < q$, is in the class V_q, but not in the class V_p.*

Proof. By Theorem 7 in [7], l_s (l_t) is of M-cotype q, hence it is in the class V_q by Theorem 1. Assume that l_s (l_t) is in the class V_p. By Proposition 8 in [7], l_s (l_t) is of stable type p, so it imbeds in L_p by Theorem 4.5 in [10]. But this contradicts Proposition 9 in [7].

Thus, it is reasonable to refer to a Banach space in the class V_p as a Banach space of V-cotype p.

4. Concluding remarks. 1. If E is of stable type p $(1 \leqslant p < 2)$, then, by Proposition 4.8 in [10] and Theorem 1, the following statements are equivalent:

(1) E is of M-cotype p.
(2) E is of V-cotype p.
(3) E is of S-cotype p.

It is natural to ask

PROBLEM 1. Are the three possible notions of cotype equivalent in general?

2. Garling [2] characterized spaces of cotype 2 by the following property: a Banach space E is of cotype 2 iff every symmetric Gaussian measure on E is the continuous image of a symmetric Gaussian measure on a Hilbert space.

It is known that every Hilbert space is a Sazonov space. On the other hand, because every Sazonov space S is of cotype 2, every symmetric Gaussian measure on S is the continuous image of a symmetric Gaussian measure on a Hilbert space. Then Garling's theorem can be stated as follows:

A Banach space E is of cotype 2 iff every symmetric Gaussian measure on E is the continuous image of a symmetric Gaussian measure on a Sazonov space.

We want to extend this fact to spaces of S-cotype p.

PROBLEM 2. Is it true that a Banach space E is of S-cotype p iff every symmetric p-stable measure on E is the continuous image of a symmetric p-stable measure on a Sazonov space?

In the proof of Theorem 1 we have shown that:

1° if every symmetric p-stable measure on E is the continuous image of a symmetric p-stable measure on a Sazonov space, then E is of S-cotype p;

2° if E is of S-cotype $(p-\varepsilon)$, $p > 1$, then every symmetric p-stable measure on E is the continuous image of a symmetric p-stable measure on a Sazonov space.

It should be noted that if the answer to Problem 2 is positive, then the answer to Problem 1 is also positive.

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