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ESTIMATES FOR TAIL PROBABILITIES OF QUADRATIC AND BILINEAR FORMS IN SUBGAUSSIAN RANDOM VARIABLES

WITH APPLICATIONS TO THE LAW OF THE ITERATED LOGARITHM

BY

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Abstract. We prove upper estimates for the tail probabilities of quadratic and bilinear forms in independent subgaussian random variables. These inequalities are used to get upper estimates in the law of iterated logarithm. It is shown that iterated logarithm behaviour in the class of random quadratic and bilinear forms is heterogeneous. Examples show that the results are sharp.

0. Introduction. We start with some notation and definitions. Let $A = (a_{ij})$, $i, j = 1, 2, \ldots$, be an array of reals, $A_n = (a_{ij})$, $i, j = 1, \ldots, n$, and (X_n), (Y_n) be independent sequences of independent random variables (r.v.'s). Put $X_n = (X_1, \ldots, X_n)^T$, $Y_n = (Y_1, \ldots, Y_n)^T$ and define the quadratic forms (q.f.'s) $Q_n = X_n^T A_n X_n$ and the bilinear forms (b.f.'s) $T_n = X_n^T A_n Y_n$. In the case of q.f.'s Q_n we may without loss of generality assume that A is symmetric. For a matrix $B = (b_{ij})$, $i, j = 1, \ldots, n$, $\|B\|$ and $\mu(B)$ denote the Frobenius and spectral norms of B, respectively, i.e.

$$\|B\|^2 = \sum_{i,j=1}^{n} b_{ij}^2$$

and $\mu^2(B)$ is the largest eigenvalue of $B^T B$. Moreover, $\operatorname{tr} B$ and $\operatorname{rk} B$ stand for trace of B and rank of B, respectively, and $\operatorname{diag}(a_1, \ldots, a_n)$ denotes the diagonal matrix with diagonal elements $a_1, \ldots, a_n$. For some further theory of matrices we refer to [4].

An r.v. X is called *subgaussian* if there exists a constant $\alpha > 0$ such that $Ee^{uX} \leqslant \exp(\alpha^2 u^2/2)$ for all real u. The minimum of such numbers α is denoted by $\alpha(X)$. Special cases of subgaussian r.v.'s include Gaussian r.v.'s with $EX = 0$ and centered r.v.'s which are bounded almost surely (a.s.) by a constant. These are two important classes of r.v.'s. A subgaussian r.v. X always satisfies the relations $EX = 0$, and $EX^2 \leqslant \alpha^2(X)$. If even $EX^2 = \alpha^2(X)$ then X is called *strictly subgaussian.* Further information about subgaussian r.v.'s is e.g. contained in [3].

Exponential estimates for q.f.'s in Gaussian r.v.'s are proved in [1, 2, 6–8, 11–13].

One of the main tools of these works is the representation

$$Q_n = \sum_{i=1}^{n} \lambda_i^{(n)} (Z_i^{(n)})^2,$$

where $(\lambda_i^{(n)})_{i=1,\ldots,n}$ are the eigenvalues of $\varrho_n A_n$, ϱ_n is the covariance matrix of X_n and $(Z_i^{(n)})_{i=1,\ldots,n}$ are independent identically distributed (i.i.d.) Gaussian r.v.'s with $EZ_1^{(n)} = 0$, $E(Z_1^{(n)})^2 = 1$ (in short, $N(0, 1)$ r.v.'s). Such a representation is not valid for q.f.'s in non-Gaussian r.v.'s, but for q.f.'s in subgaussian r.v.'s one can estimate the moment generating function of Q_n by that of an appropriate q.f. in Gaussian r.v.'s. This enables one to derive exponential estimates for $P(Q_n > x)$ and $P(T_n > x)$. These inequalities will be applied then to obtain results of bounded iterated logarithm type, i.e.

$$\overline{\lim}\, b_n^{-1}(Q_n - c_n) \leqslant 1 \text{ a.s.} \quad \text{and} \quad \overline{\lim}\, b_n^{-1}|Q_n - c_n| \leqslant 1 \text{ a.s.}$$

for suitable constants b_n, c_n.

1. Exponential estimates for tail probabilities. Throughout (X_n) is a sequence of independent subgaussian r.v.'s. Put $\alpha_i = \alpha(X_i)$, $V_n = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, $B_n = 2\|V_n A_n V_n\|^2$, $\mu_n = \mu(V_n A_n V_n)$, $Q_n^* = Q_n - \operatorname{tr}(V_n A_n V_n)$.

PROPOSITION 1.1. *Assume that (A_n) is a sequence of positive semidefinite symmetric matrices. Given $\delta \in (0, 1)$, the following inequalities are true for all n:*

$$\text{(A)} \qquad P(Q_n^* > y) \leqslant \exp\left(-\frac{y^2}{2B_n}\left(1 - \frac{2}{3}\frac{2y\mu_n}{B_n}\left(1 - \frac{2y\mu_n}{B_n}\right)^{-1}\right)\right)$$

for $0 \leqslant y \leqslant ((1-\delta)/2\mu_n)B_n$,

$$\text{(B)} \qquad P(Q_n^* > y) \leqslant \exp\left(-\frac{y(1-\delta)}{4\mu_n}\left(1 - \frac{2}{3}\frac{1-\delta}{\delta}\right)\right)$$

for $y \geqslant ((1-\delta)/2\mu_n)B_n$,

$$\text{(B}'\text{)} \qquad P(Q_n^* > y) \leqslant C(\delta)\exp\left(-\frac{y(1-\delta)}{\sqrt{2B_n}}\right)$$

for some constant $C(\delta) > 0$ and all $y > 0$.

Proof. We start with an auxiliary result.

LEMMA 1.2. *Assume that A_n is positive semidefinite and $\alpha_i = 1$ for all i. Then*

$$E\exp(hQ_n) \leqslant \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \ln(1 - 2h\lambda_i)\right)$$

for $0 \leqslant h < 1/2\mu_n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A_n.

Proof of Lemma 1.2. Assume that $\operatorname{rk} A_n = k > 0$. There exists an orthogonal matrix U such that $A_n = U^T L U$, $L = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$, where $\lambda_1, \ldots, \lambda_k$ are positive eigenvalues of A_n. Then

$$X_n^T A_n X_n = V^T L_0 V$$

with

$$V = \big((UX_n)_1, \ldots, (UX_n)_k\big)^T, \quad L_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_k).$$

For a positive definite matrix B of dimension $k \times k$ the relation

$$e^h z^T B z = \int_{R^k} e^{\sqrt{2h} z^T y} q(y)\,dy, \quad z \in R^k,\ h > 0,$$

is satisfied, where $q(y)$ is the density of a k-dimensional Gaussian vector with expectation zero and variance matrix B. Hence

$$I = E\exp(hQ_n) = \int_{R^k} E e^{\sqrt{2h} V^T y} q(y)\,dy$$

for sufficiently small $h > 0$, where

$$q(y) = (\det L_0)^{-1/2} (2\pi)^{-k/2} \exp(-\tfrac{1}{2} y^T L_0^{-1} y), \quad y \in R^k.$$

Using the subgaussivity of X_n with $\alpha_n = 1$ we get

$$E e^{\sqrt{2h} V^T y} = \prod_{j=1}^{n} E\exp\Big(\sqrt{2h} X_j \big(\sum_{i=1}^{k} y_i u_{ij}\big)\Big)$$

$$\leqslant \prod_{j=1}^{n} \exp\Big(h\big(\sum_{i=1}^{k} y_i u_{ij}\big)^2\Big) \leqslant e^{h y^T y},$$

$$I \leqslant \int_{R^k} q(y) e^{h y^T y}\,dy$$

$$= (\det L_0)^{-1/2} (2\pi)^{-k/2} \int_{R^k} \exp\left(-\frac{1}{2} \sum_{i=1}^{k} y_i^2 \frac{1 - 2h\lambda_i}{\lambda_i}\right) dy$$

for $0 \leqslant 2h \max_{i \leqslant k} \lambda_i < 1$. The matrix

$$B = \operatorname{diag}\left(\frac{\lambda_1}{1 - 2h\lambda_1}, \ldots, \frac{\lambda_k}{1 - 2h\lambda_k}\right)$$

is positive definite. It follows that

$$I \leqslant \left(\frac{\det B}{\det L_0}\right)^{1/2} \Big((\det B)^{-1/2} (2\pi)^{-k/2} \int_{R^k} e^{-y^T B^{-1} y/2}\,dy\Big)$$

$$= \left(\frac{\det B}{\det L_0}\right)^{1/2} = \prod_{i=1}^{k} (1 - 2h\lambda_i)^{-1/2},$$

which proves the lemma.

Now we start the proof of the proposition. In virtue of the transformation $X_i \leftrightarrow X_i/\alpha_i$ we may assume that $\alpha_i = 1$ for all i.

By Lemma 1.2,

$$P(Q_n^* > y) \leqslant e^{-hy} E e^{hQ_n^*} \leqslant \exp\left(-h(y+\operatorname{tr} A_n) - \frac{1}{2}\sum_{i=1}^{k} \ln(1-2h\lambda_i)\right),$$

$$E\exp\left(-\tfrac{1}{2}\ln(1-2h\lambda_i) - h\lambda_i\right) \leqslant \exp\left(\lambda_i^2 h^2 \left(1 + \tfrac{2}{3}(2h\mu_n + (2h\mu_n)^2 + \ldots)\right)\right)$$
$$= \exp\left(\lambda_i^2 h^2 \left(1 + \frac{2}{3}\frac{2h\mu_n}{1-2h\mu_n}\right)\right),$$

for $0 \leqslant h < 1/2\mu_n$. Hence

$$P(Q_n^* > y) \leqslant \exp\left(-hy + \frac{h^2 B_n}{2}\left(1 + \frac{2}{3}\frac{2h\mu_n}{1-2h\mu_n}\right)\right),$$

where

$$B_n = 2\sum_{i=1}^{n} \lambda_i^2, \quad \operatorname{tr} A_n = \sum_{i=1}^{n} \lambda_i.$$

Put $h = y/B_n$ or $h = (1-\delta)/2\mu_n$ according as $y \leqslant ((1-\delta)/2\mu_n)B_n$ or $y \geqslant ((1-\delta)/2\mu_n)B_n$. Then (A) is immediate. In (B) take $h = (1-\delta)/\sqrt{2B_n}$. Then

$$2h\mu_n = (1-\delta)\frac{\mu_n}{\|A_n\|} \leqslant 1-\delta$$

and

$$h^2 \frac{B_n}{2}\left(1 + \frac{2}{3}\frac{2h\mu_n}{1-2h\mu_n}\right) \leqslant \frac{(1-\delta)^2}{4}\left(1 + \frac{2}{3}\frac{1-\delta}{\delta}\right) = \ln C(\delta).$$

This proves the proposition.

The following is immediate from (A).

COROLLARY 1.3. *Let* (A_n) *be as in Proposition* 1.1. *If* (y_n) *is a sequence of positive reals with* $(y_n\mu_n/B_n) \to 0$, *then, given* $\delta \in (0, 1)$,

$$P(Q_n^* > y_n) \leqslant \exp\left(-\frac{(1-\delta)y_n^2}{2B_n}\right)$$

for sufficiently large n.

Putting $\delta = \frac{1}{2}$ in (A) we get

COROLLARY 1.4. *Let* (A_n) *be as in Proposition* 1.1. *Then*

$$P(Q_n^* > y) \leqslant \exp\left(-\min\left(\frac{y}{12\mu_n}, \frac{1}{6}\frac{y^2}{B_n}\right)\right) \quad \text{for all } y > 0.$$

Remarks. 1. Lower estimates for $P(Q_n^* > x)$ with Gaussian X_n show that the inequalities of Proposition 1.1, Corollaries 1.3 and 1.4 are sharp [2, 7, 12].

2. It should be noted that the inequalities of (A) in Proposition 1.1 are very much like the upper estimates for tail probabilities of sums of independent r.v.'s in Kolmogorov's LIL [10].

3. It is an immediate consequence of (B) and of one-sided versions of maximal inequalities with quasi-superadditive structure (e.g. [8, 9]) that

$$P(\max_{i\leqslant n} Q_i^* > y) \leqslant C(\delta)\exp\left(-\frac{y(1-\delta)}{\sqrt{2B_n}}\right)$$

for each fixed $\delta\in(0, 1)$, some $C(\delta)$, all $y > 0$ and $n \geqslant 1$.

4. By the martingale maximal inequality we get

$$P\big(\max_{i\leqslant n}(Q_i - EQ_i) > y\big) \leqslant e^{-hy} E\exp\big(h(Q_n - EQ_n)\big), \qquad y > 0,$$

for sufficiently small $h > 0$. Thus, modifying the proof of Proposition 1.1, we get

$$P\big(\max_{i\leqslant n}(Q_i - EQ_i) > y\big) \leqslant \exp\left(-\frac{y^2}{2B_n}\left(1-\frac{2}{3}\frac{2y\mu_n}{B_n}\left(1-\frac{2y\mu_n}{B_n}\right)^{-1}\right)+\frac{y}{B_n}t_n\right)$$

for $0 \leqslant y \leqslant ((1-\delta)/2\mu_n)B_n$, and

$$P\big(\max_{i\leqslant n}(Q_i - EQ_i) > y\big) \leqslant \exp\left(-\frac{y(1-\delta)}{4\mu_n}\left(1-\frac{2}{3}\frac{1-\delta}{\delta}\right)+\frac{1-\delta}{2\mu_n}t_n\right)$$

for $y \geqslant ((1-\delta)/2\mu_n)B_n$, where

$$t_n = \sum_{i=1}^{n} a_{ii}(\alpha_i^2 - \sigma_i^2), \qquad \sigma_n^2 = EX_n^2 \quad \text{for all } n.$$

Note that $t_n = 0$ for strictly subgaussian X_n.

5. We could not prove sharp estimates for $P(-Q_n^* > y)$ if A_n is positive semidefinite and for general symmetric matrices A_n. But following the ideas of Wright [14] and Hanson and Wright [5] one can derive the estimate

$$P(\max_{i\leqslant n}|Q_i - EQ_i| > y) \leqslant 2\exp\left(-C\min\left(\frac{y}{\mu(|V_nA_nV_n|)}, \frac{y^2}{\|V_nA_nV_n\|^2}\right)\right)$$

for some $C > 0$, all $y > 0$ and symmetric A, where $|V_nA_nV_n| = (\alpha_i|a_{ij}|\alpha_j)_{i,j=1,\ldots,n}$.

The estimates of Hanson and Wright are applicable to independent r.v.'s X_i satisfying

$$P(|X_i| > y) \leqslant M\int_y^{\infty} e^{-\gamma t^2}\,dt$$

for all i, $y \geqslant 0$ and some γ, $M > 0$. For subgaussian X_i with $\alpha_i = 1$ this estimate is true with $0 < \gamma < 1/2$ and some $M > 0$.

Note that $\mu(|V_nA_nV_n|) \leqslant \|V_nA_nV_n\|$ such that

$$P(\max_{i\leqslant n}|Q_i - EQ_i| > y) \leqslant 2\exp\left(-C\frac{y}{\|V_nA_nV_n\|}\right), \qquad y > 0,$$

and

$$P(\max_{i\leqslant n}|Q_i-EQ_i|>y_n)\leqslant 2\exp\Big(-C\frac{y_n^2}{\|V_nA_nV_n\|^2}\Big)$$

for sufficiently large n and positive y_n satisfying the condition

$$\frac{y_n\mu(|V_nA_nV_n|)}{\|V_nA_nV_n\|^2}\to 0.$$

As another consequence of Proposition 1.1 we obtain exponential estimates for b.f.'s T_n. In this case the symmetry assumption about A would be restrictive. Assume that (Y_n) is a sequence of independent subgaussian r.v.'s. Put

$$C_n=\operatorname{diag}(\alpha(Y_1),\ldots,\alpha(Y_n)),\qquad S_n=\|V_nA_nC_n\|^2,\qquad v_n=\mu(V_nA_nC_n).$$

PROPOSITION 1.5. *Given $\delta\in(0,1)$, the following inequalities are true for all n:*

(A) $$P(\max_{i\leqslant n}T_i>y)\leqslant\exp\Big(-\frac{y^2}{2S_n}\Big(1-\frac{1}{2}\Big(\frac{y}{S_n}v_n\Big)^2\Big(1-\Big(\frac{yv_n}{S_n}\Big)^2\Big)^{-1}\Big)\Big)$$

for $y\leqslant((1-\delta)/v_n)S_n,\ y\geqslant 0$,

$$P(\max_{i\leqslant n}T_i>y)\leqslant\exp\Big(-\frac{(1-\delta)}{2v_n}y\Big(1-\frac{1}{2}\frac{(1-\delta)^2}{1-(1-\delta)^2}\Big)\Big)$$

for $y\geqslant((1-\delta)/v_n)S_n$,

(B) $$P(\max_{i\leqslant n}T_i>y)\leqslant C(\delta)\exp\Big(-\frac{(1-\delta)y}{\sqrt{S_n}}\Big)$$

for some constant $C(\delta)>0$ *and all* $y>0$,

(C) (A) *and* (B) *remain true if* $P(\max_{i\leqslant n}T_i>y)$ *is replaced by*

$$\tfrac{1}{2}P(\max_{i\leqslant n}|T_i|>y).$$

Proof. Without loss of generality assume that $\alpha_i=1$ and $\alpha(Y_i)=1$ for all i. By the martingale maximal inequality for sufficiently small $h>0$,

$$P(\max_{i\leqslant n}T_i>y)\leqslant e^{-hy}E\exp(hT_n).$$

By the subgaussivity of X_n, Y_n and by Lemma 1.2, for $0\leqslant h^2<1/\mu(A_n^TA_n)$ we obtain

$$\begin{aligned}E\exp(hT_n)&=E\,E\big(\exp(h\sum_{i=1}^n X_i\sum_{j=1}^n a_{ij}Y_j)\,|\,Y_1,\ldots,Y_n\big)\\&\leqslant E\exp\Big(\frac{h^2}{2}\sum_{i=1}^n(\sum_{j=1}^n a_{ij}Y_j)^2\Big)\\&=E\exp\Big(\frac{h^2}{2}Y_n^TA_n^TA_nY_n\Big)\leqslant\exp\Big(-\frac{1}{2}\sum_{i=1}^n\ln(1-h^2\lambda_i)\Big),\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A_n^T A_n$. Note that $v_n^2 = \mu(A_n^T A_n)$. Proceeding as in the proof of Proposition 1.1 we get

$$\exp(-\tfrac{1}{2}\ln(1-h^2\lambda_i)) \leqslant \exp\left(\frac{h^2\lambda_i}{2}\left(1+\tfrac{1}{2}((hv_n)^2+(hv_n)^4+\ldots)\right)\right)$$

$$= \exp\left(\frac{h^2\lambda_i}{2}\left(1+\frac{1}{2}\frac{(hv_n)^2}{1-(hv_n)^2}\right)\right)$$

and

$$P(\max_{i\leqslant n} T_i > y) \leqslant \exp\left(-hy+\frac{h^2}{2}S_n\left(1+\frac{1}{2}\frac{(hv_n)^2}{1-(hv_n)^2}\right)\right),$$

where $S_n = \|A_n\|^2 = \operatorname{tr} A_n^T A_n$.

Now put $h = y/S_n$ or $h = (1-\delta)/v_n$ according as $y \leqslant ((1-\delta)/v_n)S_n$ or $y \geqslant ((1-\delta)/v_n)S_n$. Then (A) is immediate. For (B) put $h = (1-\delta)/\sqrt{S_n}$. In order to obtain (C) note that

$$P(\max_{i\leqslant n}|T_i| > y) \leqslant P(\max_{i\leqslant n} T_i > y)+P(\max_{i\leqslant n}(-T_i) > y)$$

and that (A) remains true for $P(\max_{i\leqslant n}(-T_i) > y)$ instead of

$$P(\max_{i\leqslant n} T_i > y).$$

COROLLARY 1.6. *If (y_n) is a sequence of positive reals with $(y_n v_n/S_n) \to 0$, then, given $\delta \in (0, 1)$,*

$$P(\max_{i\leqslant n}|T_i| > y_n) \leqslant 2\exp(-y_n^2/2S_n)$$

for sufficiently large n.

COROLLARY 1.7. *For all $y > 0$*

$$P(\max_{i\leqslant n}|T_i| > y) \leqslant 2\exp\left(-\min\left(\frac{5}{24}\frac{y}{\mu_n}, \frac{5}{12}\frac{y^2}{S_n}\right)\right).$$

2. Some applications to the LIL. Maximal inequalities of exponential type permit to derive results of bounded LIL-type. The proofs of such results are standard, and therefore omitted (see [10], Chapter 10, for some foundations).

We consider sequences (X_n) and (Y_n) of subgaussian r.v.'s and use the notation of Section 1. Put

$$\chi_\alpha(n) = (2B_n)^{1/2}\log_2^{1/\alpha} B_n$$

with $\log_2 x = \log\log x$, $\log x = \max(1, \ln x)$ for $x > 0$ and $\alpha \in \{1, 2\}$.

PROPOSITION 2.1. *Assume that (A_n) is a sequence of positive semidefinite symmetric matrices and that $B_n \to \infty$. Then*

$$\overline{\lim}\frac{Q_n^*}{\chi_1(n)} \leqslant 1 \quad a.s.$$

Moreover, if

(2.1) $$\mu_n = o\big((B_n/\log_2 B_n)^{1/2}\big)$$

and

(2.2) $$t_n = o\big((B_n/\log_2 B_n)^{1/2}\big)$$

then

$$\overline{\lim}\,\frac{Q_n - EQ_n}{\chi_2(n)} \leqslant 1 \ \ a.s.$$

Remarks. 1. Condition (2.2) is satisfied for strictly subgaussian r.v.'s. In this case, $t_n = 0$ for all n. It should be noted that (2.1) is very similar to Kolmogorov's LIL-condition [10].

2. Proposition 2.1 is a consequence of Remarks 3 and 4 in Section 1. In fact, by Remark 3,

$$P\big(\max_{i \leqslant n} Q_i^* > d\chi_1(n)\big) \leqslant C(\delta)(\log B_n)^{-d(1-\delta)}$$

for each $\delta \in (0, 1)$, $d > 0$, and by Remark 4, (2.1) and (2.2),

$$P\big(\max_{i \leqslant n}(Q_i - EQ_i) > d\chi_2(n)\big) \leqslant (\log B_n)^{-d(1-\delta)}$$

for each $\delta \in (0, 1)$, $d > 0$ and sufficiently large n.

3. In virtue of Remark 5, in the same way one gets

$$\overline{\lim}\,\frac{|Q_n - EQ_n|}{\chi_1(n)} < \infty \ \ \text{a.s.}$$

and if, in addition, $\mu(|V_n A_n V_n|) = o\big((B_n/\log_2 B_n)^{1/2}\big)$, then

$$\overline{\lim}\,\frac{|Q_n - EQ_n|}{\chi_2(n)} < \infty \ \ \text{a.s.}$$

4. The estimates of Proposition 2.1 are sharp. This is shown by the examples

$$\overline{\lim}\,\sum_{i=1}^{n} (X_i^2 - 1)/(2^2 n \log_2 n)^{1/2} = 1 \ \ \text{a.s.}$$

and

$$\overline{\lim}\,\Big(\sum_{i=1}^{n} X_i\Big)^2/(2n\log_2 n) = 1 \ \ \text{a.s.}$$

for i.i.d. $N(0, 1)$ r.v.'s.

Next we consider b.f.'s T_n. Put

$$\psi_1(n) = (S_n)^{1/2}\log_2 S_n, \qquad \psi_2(n) = (2S_n)^{1/2}\log_2^{1/2} S_n.$$

From Proposition 1.5, (B) and (C), and Corollary 1.6 we get the following

PROPOSITION 2.2. *Assume that* $S_n \to \infty$. *Then*

$$\overline{\lim} \frac{|T_n|}{\psi_1(n)} \leqslant 1 \text{ a.s.}$$

Moreover, if

$$v_n = o\big((S_n/\log_2 S_n)^{1/2}\big) \tag{2.3}$$

then

$$\overline{\lim} \frac{|T_n|}{\psi_2(n)} \leqslant 1 \text{ a.s.}$$

Remarks. 5. The sharpness of the inequalities in Proposition 2.2 is shown by such examples as

$$\overline{\lim} \Big(\sum_{i=1}^{n} X_i\Big)\Big(\sum_{i=1}^{n} Y_i\Big)/(n\log_2 n) = 1 \text{ a.s.}$$

and

$$\overline{\lim} \sum_{i=1}^{n} X_i Y_i/(2n\log_2 n)^{1/2} = 1 \text{ a.s.}$$

for i.i.d. $N(0, 1)$ r.v.'s X_n and Y_n.

6. As an immediate consequence of Proposition 2.2 it follows that if $|X_i| \leqslant 1$ a.s. and $|Y_i| \leqslant 1$ a.s. for all i, then $\alpha_i \leqslant 1$, $\alpha(Y_i) \leqslant 1$ and $S_n = \|V_n' A_n C_n\|^2 \leqslant \|A_n\|^2$. Hence

$$\overline{\lim} \frac{|T_n|}{\|A_n\| \log_2 \|A_n\|} \leqslant 1 \text{ a.s.}$$

and if, in addition, (2.3) is satisfied, then

$$\overline{\lim} \frac{|T_n|}{\|A_n\| (2\log_2 \|A_n\|)^{1/2}} \leqslant 1 \text{ a.s.}$$

In particular, these relations are true for i.i.d. Rademacher r.v.'s X_i and Y_i (i.e. $P(X_1 = \pm 1) = 1/2$), since $\alpha_1 = 1$ in this case.

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