PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 12, Fasc. 1 (1991), pp. 113–125

ON THE RATE OF CONVERGENCE TO BROWNIAN MOTION OF THE PARTIAL SUMS OF INFIMA OF INDEPENDENT RANDOM VARIABLES

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Abstract. Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent and positive random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$, with a common distribution function F. Put

$$Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m),\ m \geqslant 1, \quad S_n = \sum_{m=1}^{n} Y_m^*,\ n \geqslant 2,\ S_1 = 0.$$

In this paper a convergence rate in the invariance principle for the sums S_n, $n \geqslant 1$, is obtained.

1. Introduction and results. Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent and positive random variables (i.p.r.vs.) with a common distribution function F such that

$$(1) \qquad \int_0^1 |F(x)-x/b|\, x^{-2}\, dx < \infty \quad \text{for some } b,\ 0 < b < \infty.$$

Let us put

$$Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m),\ m \geqslant 1, \quad \text{and} \quad S_n = \sum_{m=1}^{n} Y_m^*,\ n \geqslant 2,\ S_1 = 0.$$

Several authors ([2]–[4], [6]–[10]) have investigated the asymptotic convergence S_n as $n \to \infty$ in probability, almost sure and in law. The almost sure and Donsker's invariance principles of sums S_n were investigated in [7] and [9]. In this paper we examine the rate of convergence in the Donsker's invariance principle for the sums S_n.

Let $\{Y_n, n \geqslant 1\}$ be a sequence of i.p.r.vs. with a common distribution function F such that (1) holds. Let us define

$$Y_{n,k}^* = (Y_k^* - b/k)/b(2\log n)^{1/2},\ 1 \leqslant k \leqslant n,\ n > 1, \qquad Y_{1,1}^* = 0,$$

and write

$$(2) \qquad S_{n,k} = \sum_{m=1}^{k} Y_{n,m}^*, \quad 1 \leqslant k \leqslant n,\ n \geqslant 1.$$

Put

(3) $$F_n(\lambda) = P[\max_{1 \leqslant k \leqslant n} |S_{n,k}| \leqslant \lambda].$$

Under the assumption (1) it follows from the Donsker's invariance principle that, for each $\lambda > 0$,

$$\lim_{n\to\infty} F_n(\lambda) = T(\lambda),$$

where

(4) $$T(\lambda) = P[\max_{0 \leqslant t \leqslant 1} |W(t)| \leqslant \lambda] = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\{-(2k+1)^2 \pi^2/8\lambda^2\},$$

and $\{W(t), 0 \leqslant t \leqslant 1\}$ is a standard Wiener process (cf. [7], Corollaries 1 and 2). The purpose of this paper is to study the rate of convergence of F_n to T. The main result is the following

THEOREM 1. *Under the assumption* (1) *we have*

(5) $$\sup_{\lambda} |P[\max_{1 \leqslant k \leqslant n} |S_{n,k}| \leqslant \lambda] - T(\lambda)| = O((\log n)^{-1/3}),$$

where $\{S_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *and* $T(\lambda)$ *are given by* (2) *and* (4), *respectively.*

2. Proof of the result. In the proof of Theorem 1 we apply some lemmas given by Dehéuvels [3], Höglund [10] and Sawyer [15]. Moreover, we use the Skorokhod representation theorem. For the sake of completeness we present them in Section 3.

Proof of Theorem 1. At the beginning suppose that $\{X_n, n \geqslant 1\}$ is a sequence of independent random variables uniformly distributed on [0, 1] (i.r.vs.u.d.). (In this case $b = 1$.) Put

$$X_m^* = \inf(X_1, X_2, \ldots, X_m),\ m \geqslant 1, \qquad \tilde{S}_n = \sum_{m=1}^{n} X_m^*,\ n \geqslant 1,$$

and define

(6) $$\tilde{S}_{n,k} = (\tilde{S}_k - \sum_{i=1}^{k} 1/i)/(2\log n)^{1/2},\ 1 \leqslant k \leqslant n,\ n > 1, \qquad \tilde{S}_{1,1} = 0.$$

We are going to prove that

(7) $$\sup_{\lambda} |P[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,k}| \leqslant \lambda] - T(\lambda)| = O((\log n)^{-1/3}).$$

Let us set

(8) $$V_{n,k} = [\tau_{k+1} - \tau_k - \mathrm{E}(\tau_{k+1} - \tau_k)]/k(2\log n)^{1/2}, \qquad 1 \leqslant k \leqslant n,\ n \geqslant 2,$$
$$V_{1,1} = 0,$$

and put

$$U_{n,k} = \sum_{m=1}^{k} V_{n,m}, \qquad 1 \leqslant k \leqslant n, \ n \geqslant 1,$$

where the random variables τ_n, $n \geqslant 1$, are given in Section 3 by (3.1) ($\varepsilon(n) = n^{-1}$).

Now, let us observe that $V_{n,k}$, $1 \leqslant k \leqslant n$, are independent random variables (Lemma 3.2) and

(9) $$\max_{1 \leqslant k \leqslant n} |V_{n,k}| \leqslant (1+A)\log_2 n/(2\log n)^{1/2} \text{ a.s.}$$

for sufficiently large n, where $\log_p n = \log_{p-1}(\log n)$, $p > 2$, $\log_2 n = \log(\log n)$.

In fact, by (3.12) for all $A > 0$ we have

$$\tau_{k+1} - \tau_k - \mathrm{E}(\tau_{k+1} - \tau_k) \leqslant (1+A)k\log_2 k \text{ a.s.}$$

for sufficiently large k, so by the definition (8) we get

$$V_{n,k} \leqslant \frac{(1+A)\log_2 k}{\sqrt{2\log n}} \leqslant \frac{(1+A)\log_2 n}{\sqrt{2\log n}} \text{ a.s.}, \qquad 1 \leqslant k \leqslant n,$$

for sufficiently large n.

By the Skorokhod representation result applied to the sequence $V_n = (V_{n,1}, V_{n,2}, \ldots, V_{n,n})$ there is a standard Wiener process $\{W(t), t \in \langle 0, 1\rangle\}$ together with a sequence of nonnegative independent random variables $z_1, z_2, \ldots, z_n$ on a new probability space such that

(10) $$\{U_{n,1}, U_{n,2}, \ldots, U_{n,n}\} \stackrel{\mathrm{d}}{=} \{W(z_1), W(z_1+z_2), \ldots, W(\sum_{i=1}^{n} z_i)\},$$

$n > 1$, where $\stackrel{\mathrm{d}}{=}$ means the equivalence in joint distribution,

(11) $$\mathrm{E}z_k = \mathrm{E}Y_{n,k}^2,$$

for each real number $r \geqslant 1$

(12) $$\mathrm{E}|z_k|^r \leqslant C_r \mathrm{E}(Y_{n,k})^{2r}, \qquad 1 \leqslant k \leqslant n,$$

where $C_r = 2(8/\pi^2)^{r-1}\Gamma(r+1)$, and

(13) $$Y_{n,k} = W(\sum_{i=1}^{k} z_i) - W(\sum_{i=1}^{k-1} z_i) \stackrel{\mathrm{d}}{=} V_{n,k}.$$

Now we shall prove that

(14) $$\sup_{\lambda} |F_n^{(1)}(\lambda) - T(\lambda)| = O((\log n)^{-1/3}),$$

where

(15) $$F_n^{(1)}(\lambda) = P[\max_{1 \leqslant k \leqslant n} |U_{n,k}| \leqslant \lambda], \qquad \lambda \geqslant 0.$$

Let us observe that by (13) and (9) we obtain

$$F_n^{(1)}(\lambda) = P[\max_{1\leqslant k\leqslant n} |W(z_1+\ldots+z_k)| \leqslant \lambda]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} |W(z_1+\ldots+z_{k-1})| - \max_{1\leqslant k\leqslant n} |W(\sum_{i=1}^{k} z_i) - W(\sum_{i=1}^{k-1} z_i)| \leqslant \lambda]$$

$$\leqslant P[\max\{|W(t)| \leqslant \lambda + (1+A)(\log_2 n)(2\log n)^{-1/2};\ 0 \leqslant t \leqslant \sum_{i=1}^{n-1} z_i\}]$$

and, analogously,

$$F_n^{(1)}(\lambda) \geqslant P[\max\{|W(t)| \leqslant \lambda - (1+A)(\log_2 n)(2\log n)^{-1/2};\ 0 \leqslant t \leqslant \sum_{i=1}^{n-1} z_i\}].$$

Let us put

$$a_n = (1+A)(\log_2 n)(2\log n)^{-1/2}, \qquad n > 2.$$

Thus from the above we get

$$(16) \quad F_n^{(1)}(\lambda) \leqslant P[\max\{|W(t)| \leqslant \lambda + a_n;\ 0 \leqslant t \leqslant \sum_{i=1}^{n-1} z_i\}, |\sum_{i=1}^{n-1} z_i - 1| < g(n)] + P[|\sum_{i=1}^{n-1} z_i - 1| \geqslant g(n)]$$

$$\leqslant P[\max_{0\leqslant t\leqslant 1-g(n)} |W(t)| \leqslant \lambda + a_n] + P[|\sum_{i=1}^{n-1} z_i - 1| \geqslant g(n)],$$

where $g(\cdot)$ is a positive function decreasing to zero as $n \to \infty$, slower than $(\log n)^{-1/2}$.

We first estimate the second part of the extreme right-hand side of (16). From the construction of z_i and the relations (11)–(13) and (3.2), (3.7) we have

$$P[|\sum_{i=1}^{n-1} z_i - 1| \geqslant g(n)] \leqslant P[|\sum_{i=1}^{n-1} (z_i - \mathrm{E}z_i)| + |\sum_{i=1}^{n-1} \mathrm{E}z_i - 1| \geqslant g(n)]$$

$$= P[|\sum_{i=1}^{n-1} (z_i - \mathrm{E}z_i)| + |\sum_{i=1}^{n-1} \mathrm{E}Y_{n,i}^2 - 1| \geqslant g(n)]$$

$$= P[|\sum_{i=1}^{n-1} (z_i - \mathrm{E}z_i)| + |(\log n)^{-1} \sum_{i=1}^{n-1} i^{-1} - 1| \geqslant g(n)]$$

$$\leqslant P[|\sum_{i=1}^{n-1} (z_i - \mathrm{E}z_i)| \geqslant g(n) - O(1)/\log n]$$

$$\leqslant \mathrm{E}[\sum_{i=1}^{n-1} (z_i - \mathrm{E}z_i)]^2 (g(n) - O(1)/\log n)^{-2}$$

$$= \sum_{i=1}^{n-1} [\mathrm{E}z_i^2 - (\mathrm{E}z_i)^2]\big(g(n) - O(1)/\log n\big)^{-2}$$

$$\leqslant \sum_{i=1}^{n-1} [C_2 \mathrm{E}V_{n,i}^4 - (\mathrm{E}V_{n,i}^2)^2]\big(g(n) - O(1)/\log n\big)^{-2}$$

$$\leqslant 4!\, C_2 (\log n)\big[(2\log n)\big(g(n) - O(1)/\log n\big)\big]^{-2}$$

$$= O\Big(\big((\log n)(g(n))^2\big)^{-1}\Big),$$

where C_2 is a positive constant defined in (12). Putting $g(n) = (\log n)^{-1/3}$ we obtain

$$P\Big[\Big|\sum_{i=1}^{n-1} z_i - 1\Big| \geqslant (\log n)^{-1/3}\Big] = O\big((\log n)^{-1/3}\big). \tag{17}$$

As for the first term of the right-hand side of (16), from the scaling property of the Wiener process we get

$$P\Big[\max_{0\leqslant t\leqslant 1-g(n)} |W(t)| \leqslant \lambda + a_n\Big] = P\Big[\max_{0\leqslant t\leqslant 1} \big|W\big((1-g(n))t\big)\big| \leqslant \lambda + a_n\Big]$$

$$= P\Big[\max_{0\leqslant t\leqslant 1} |W(t)| \leqslant (\lambda + a_n)\big(1-g(n)\big)^{-1/2}\Big].$$

Thus from (16) and (17) we have

$$F_n^{(1)}(\lambda) \leqslant T\Big((\lambda + a_n)\big(1-g(n)\big)^{-1/2}\Big) + O\big((\log n)^{-1/3}\big). \tag{18}$$

We can also obtain, by a similar argument, the relation

$$F_n^{(1)}(\lambda) \geqslant T\Big((\lambda - a_n)\big(1+g(n)\big)^{-1/2}\Big) + O\big((\log n)^{-1/3}\big). \tag{19}$$

If $(\log n)^{-1/3} < \frac{1}{2}$, then we easily find that

$$(\lambda + a_n)\big(1-g(n)\big)^{-1/2} - (\lambda - a_n)\big(1+g(n)\big)^{-1/2} \leqslant 2\lambda g(n) + 4a_n$$

$$= 2\lambda (\log n)^{-1/3} + 4(1+A)(\log_2 n)(2\log n)^{-1/2}.$$

Hence, by Lemma 3.7 (cf. [15]), for $(\log n)^{-1/3} < \frac{1}{2}$ we have

$$T\Big((\lambda + a_n)\big(1-g(n)\big)^{-1/2}\Big) - T\Big((\lambda - a_n)\big(1+g(n)\big)^{-1/2}\Big)$$

$$\leqslant \sqrt{8/\pi}\,\big(2\lambda g(n) + 4a_n\big)\exp\Big\{-\tfrac{1}{2}\Big((\lambda - a_n)/\big(1+g(n)\big)^{1/2}\Big)^2\Big\}$$

$$\leqslant \sqrt{8/\pi}\,\big(2\lambda g(n) + 4a_n\big)\exp\big\{-\tfrac{1}{3}(\lambda - a_n)^2\big\}$$

$$\leqslant \sqrt{3/\pi e}\,4g(n) + \sqrt{8/\pi}\,\big(4 + 2g(n)\big)a_n = O\big((\log n)^{-1/3}\big).$$

Combining this with (18) and (19) we obtain (14).

Now let us put

$$\tilde{S}_{n,\tau_k} = \Big(\sum_{i=1}^{\tau_k} X_i^* - \sum_{i=1}^{k} i^{-1}\Big)(2\log n)^{-1/2}, \quad 1 \leqslant k \leqslant n,\ n \geqslant 2, \quad \tilde{S}_{1,\tau_1} = 0,$$

where τ_k is defined in (3.1).

Let us denote

$$F_n^{(2)}(\lambda) = P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k}| \leqslant \lambda].$$

By (3.9) and the fact that $\tau_1 = 1$ and $\tilde{S}_{\tau_1} = X_1^* \leqslant 1$ a.s. we obtain

$$F_n^{(2)}(\lambda) \leqslant P[\max_{1\leqslant k\leqslant n} |U_{n,k}| - \max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \leqslant \lambda]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} |U_{n,k}| - \tilde{S}_{\tau_1}(2\log n)^{-1/2} \leqslant \lambda] \leqslant F_n^{(1)}(\lambda + (2\log n)^{-1/2}).$$

Analogously, by (3.4), (3.9) and (3.8), we obtain

$$F_n^{(2)}(\lambda) \geqslant P[\max_{1\leqslant k\leqslant n} |U_{n,k}| + \max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \leqslant \lambda]$$

$$\geqslant P\left[\max_{1\leqslant k\leqslant n} |U_{n,k}| + \frac{|2-\tilde{S}_{\tau_1}|}{\sqrt{2\log n}} + \max_{1\leqslant k\leqslant n} \frac{|U_k' - U_k|}{\sqrt{2\log n}} \leqslant \lambda\right]$$

$$= P\left[\max_{1\leqslant k\leqslant n} |U_{n,k}| + \frac{U_n - U_n'}{\sqrt{2\log n}} \leqslant \lambda - \frac{2}{\sqrt{2\log n}}\right]$$

$$\geqslant P\left[\max_{1\leqslant k\leqslant n} |U_{n,k}| + \frac{U_n - U_n'}{\sqrt{2\log n}} \leqslant \lambda - \frac{2}{\sqrt{2\log n}}, \frac{U_n - U_n'}{\sqrt{2\log n}} < \frac{C}{(\log n)^{1/3}}\right]$$

$$-P\left[\frac{U_n - U_n'}{\sqrt{2\log n}} \geqslant \frac{C}{(\log n)^{1/3}}\right]$$

$$\geqslant F_n^{(1)}\left(\lambda - \frac{C}{(\log n)^{1/3}} - \frac{2}{(2\log n)^{1/2}}\right) - \frac{\sigma^2(U_n - U_n')}{2C^2(\log n)^{1/3}}$$

$$\geqslant F_n^{(1)}(\lambda - C'(\log n)^{-1/3}) - O((\log n)^{-1/3}),$$

where C and C' are positive constants independent of n such that

$$C(\log n)^{-1/3} + 2(2\log n)^{-1/2} \leqslant C'(\log n)^{-1/3}.$$

Hence, by (14), we get

$$\sup_{\lambda} |F_n^{(2)}(\lambda) - T(\lambda)| = O((\log n)^{-1/3}). \tag{20}$$

Now, let $\{\tilde{S}_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of sums of random variables defined by (6).

By a similar argument to that in the proof of Theorem 1 (see [7]) (relations (9)–(12)) we obtain

$$P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| \geqslant C'(\log n)^{-1/3}]$$

$$= P[\max_{1\leqslant k\leqslant n} |\sum_{i=1}^{k} X_i^* - \sum_{i=1}^{\tau_k} X_i^*| \geqslant \sqrt{2}\, C'(\log n)^{1/6}]$$

$$\leqslant P\left[\max_{1\leqslant k\leqslant N(n)} |\tilde{S}_k-\tilde{S}_{\tau_k}| \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\right]$$

$$+P\left[\max_{N(n)<k\leqslant n} |\tilde{S}_k-\tilde{S}_{\tau_k}| \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\right],$$

where

$$\tilde{S}_k = \sum_{i=1}^{k} X_i^*, \qquad \tilde{S}_{\tau_k} = \sum_{i=1}^{\tau_k} X_i^*,$$

and $N(n)$ is a sequence of integers.

We note that for k such that $k \geqslant \tau_k$ by the definition (3.1) we have

$$\inf(X_1, X_2, \ldots, X_{\tau_k+i}) \leqslant \varepsilon(k) = 1/k \quad \text{for all } i \geqslant 0.$$

In this case we get

$$\tilde{S}_k = \tilde{S}_{\tau_k} + \sum_{i=\tau_k+1}^{k} X_i^*,$$

and so $|\tilde{S}_k-\tilde{S}_{\tau_k}| \leqslant k\varepsilon(k) = 1$. If $k < \tau_k$, then

$$\tilde{S}_k = \tilde{S}_{\tau_k} - \sum_{i=k+1}^{\tau_k} X_i^*.$$

Put $N(n) = (\log n)^{1/6-\delta}$, $0<\delta<1/6$. Then

$$(21)\qquad P\left[\max_{1\leqslant k\leqslant N(n)} |\tilde{S}_k-\tilde{S}_{\tau_k}| \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\right]$$

$$\leqslant P\left[\max_{1\leqslant k\leqslant \tau_k\leqslant N(n)} \sum_{i=k+1}^{\tau_k} X_i^* \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\right]$$

$$\leqslant P\left[N(n)X_1^* \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\right] = P\left[X_1 \geqslant \frac{\sqrt{2}C'}{2}\frac{(\log n)^{1/6}}{N(n)}\right] = 0$$

for all n such that $n \geqslant n_0$, where n_0 is the largest integer such that

$$\frac{\sqrt{2}C'}{2}(\log n)^{1/6}/N(n) > 1.$$

Now we are going to estimate

$$P[\max_{N(n)<k\leqslant n} |\tilde{S}_{n,k}-\tilde{S}_{n,\tau_k}| \geqslant C(\log n)^{-1/3}].$$

Analogously as previously and by Lemmas 3.4 and 3.5 for sufficiently large n, we have

$$P\Big[\max_{N(n)<k\leqslant n} |\tilde{S}_{n,k}-\tilde{S}_{n,\tau_k}| \geqslant \frac{C'}{2}(\log n)^{-1/3}\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} \sum_{i=k+1}^{\tau_k} X_i^* \geqslant \frac{\sqrt{2}\,C'}{2}(\log n)^{1/6}\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} (\tau_k-k)\frac{(1+A)\log_2 k}{k} \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} \Big[(\tau_k-\tau_{k-1})\frac{(1+A)\log_2 k}{k}+\tau_{k-1}\frac{(1+A)\log_2 k}{k} -(1+A)\log_2 k\Big] \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} \Big[(\tau_k-\tau_{k-1})\frac{(1+A)\log_2 k}{k}+\frac{\tau_{k-1}}{k\log_2 k}(1+A)(\log_2 k)^2\Big] \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}+(1+A)\log_2 N(n)\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} \frac{\tau_k-\tau_{k-1}}{(1+A)k\log_2 k}(1+A)^2(\log_2 k)^2 \geqslant \frac{\sqrt{2}C'}{2}(\log n)^{1/6}+(1+A)\log_2 N(n)-(1+A)^2(\log_2 n)^2\Big]$$

$$\leqslant P\Big[\max_{N(n)<k\leqslant\tau_k\leqslant n} \frac{\tau_k-\tau_{k-1}}{(1+A)k\log_2 k} \geqslant \frac{C_1(\log n)^{1/6}}{(1+A)^2(\log_2 n)^2}\Big]$$

$$\leqslant \sum_{k=N(n)+1}^{n} P[\tau_k-\tau_{k-1} \geqslant (1+A)k(\log_2 k)A_n],$$

where

$$A_n = C_1(\log n)^{1/6}/(1+A)^2(\log_2 n)^2,$$

and C_1 is a positive constant such that

$$\frac{\sqrt{2}C'}{2}(\log n)^{1/6}+(1+A)\log_2 N(n)-(1+A)^2(\log_2 n)^2 \geqslant C_1(\log n)^{1/6}.$$

Now, by (3.3), we obtain

$$(22)\quad P\left[\max_{N(n)<k\leqslant n}|\tilde{S}_{n,k}-\tilde{S}_{n,\tau_k}|\geqslant\frac{C'}{2}(\log n)^{-1/3}\right]$$
$$\leqslant\sum_{k=N(n)+1}^{n}\frac{1}{k}\left(1-\frac{1}{k}\right)^{(1+A)k(\log_2 k)A_n-1}$$
$$\leqslant\left(1+\frac{1}{N(n)}\right)\sum_{k=N(n)+1}^{n}\frac{1}{k}\exp\{-(1+A)A_n\log_2 k\}$$
$$=\left(1+\frac{1}{N(n)}\right)\sum_{k=N(n)+1}^{n}\frac{1}{k(\log k)^{(1+A)A_n}}=O\big((\log n)^{-1/3}\big).$$

The last equality is a consequence of the integrable type criterion of series convergence. Hence, by (20)–(22), we get (7).

Now, let $\{Y_n, n\geqslant 1\}$ be a sequence of i.p.r.vs. with the same distribution function F satisfying (1) and let, as previously, $\{X_n, n\geqslant 1\}$ be a sequence of i.r.vs.u.d. on $[0, 1]$.

Put

$$G(t)=\inf\{x\geqslant 0\colon\ F(x)\geqslant t\}.$$

Then, by [4], the sequences $\{G(X_n), n\geqslant 1\}$ and $\{Y_n, n\geqslant 1\}$ are the same in law. Furthermore, the sums

$$S_n=\sum_{k=1}^{n}Y_k^*,\qquad\text{where } Y_k^*=\inf(Y_1, Y_2, \ldots, Y_k),\ k\geqslant 1,$$

can be represented as

$$\bar{S}_n=\sum_{k=1}^{n}G(X_k^*),\qquad\text{where } X_k^*=\inf(X_1, X_2, \ldots, X_k),\ k\geqslant 1.$$

Let us define $\{\bar{S}_{n,k}, 1\leqslant k\leqslant n, n\geqslant 1\}$ as follows:

$$\bar{S}_{n,k}=(\bar{S}_k-b\log k)/b(2\log n)^{1/2},\qquad 1\leqslant k\leqslant n,\ n\geqslant 2,\ \bar{S}_{1,1}=0.$$

By Lemma 3.6 we can deduce that

$$(23)\qquad \frac{\bar{S}_n-b\tilde{S}_n}{b_n}=O(1)\ \text{a.s.}$$

for all sequences $\{b_n, n\geqslant 1\}$ of real numbers such that $b_n\nearrow\infty$, as $n\to\infty$. In fact, for some δ, $0<\delta<1$, and $n\geqslant 1$, putting $\delta_n=1$ if $X_n\leqslant\delta$ and $\delta_n=0$ if $X_n>\delta$, we get

$$|\bar{S}_n-b\tilde{S}_n|\leqslant\sum_{i=1}^{n}\delta_i|G(X_i^*)-bX_i^*|+\sum_{i=1}^{n}(1-\delta_i)|G(X_i^*)-bX_i^*|.$$

With probability one all but finitely many δ_i are equal to one, so if $b_n \nearrow \infty$, as $n \to \infty$, then

$$\sum_{i=1}^{n} (1-\delta_i)|G(X_i^*)-bX_i^*|/b_n \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

Moreover, if M is a positive constant, we see that

$$P\left[\sup_{n\geqslant k} \frac{\sum_{i=1}^{n} \delta_i|G(X_i^*)-bX_i^*|}{b_n} \geqslant M\right]$$
$$\leqslant P\left[\sum_{i=1}^{\infty} \delta_i|G(X_i^*)-bX_i^*| \geqslant Mb_k\right] = O\big((b_k)^{-1}\big) \to 0 \quad \text{as } k \to \infty,$$

because

$$\mathrm{E}\left[\sum_{i=1}^{\infty} \delta_i|G(X_i^*)-bX_i^*|\right] < \infty.$$

Consequently, we get (23). Now, we obtain

$$\bar{F}_n(\lambda) = P\left[\max_{1\leqslant k\leqslant n} |\bar{S}_{n,k}| \leqslant \lambda\right] \leqslant P\left[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| - \max_{1\leqslant k\leqslant n} |\bar{S}_{n,k}-\tilde{S}_{n,k}| \leqslant \lambda\right]$$
$$= P\left[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| - \max_{1\leqslant k\leqslant n} \frac{|\bar{S}_k - b\tilde{S}_k|}{b\sqrt{2\log n}} \leqslant \lambda\right]$$
$$\leqslant P\left[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| - \frac{b_n}{b\sqrt{2\log n}} \max_{1\leqslant k\leqslant n} \frac{|\bar{S}_k - b\tilde{S}_k|}{b_n} \leqslant \lambda\right]$$
$$\leqslant P\left[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| \leqslant \lambda + \frac{O(1)b_n}{b\sqrt{2\log n}}\right],$$

and analogously

$$\bar{F}_n(\lambda) \geqslant P\left[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| \leqslant \lambda - \frac{O(1)b_n}{b\sqrt{2\log n}}\right].$$

By (7) and Lemma 3.7, putting $b_n = \log_2 n$ we obtain (5), and the proof of Theorem 1 is complete.

3. Lemmas. In this section we present without proofs lemmas due to Dehéuvels [3], Höglund [10], Sawyer [15] and Skorokhod [16], we needed in the proof of Theorem 1.

Let $\{\varepsilon(n), n \geqslant 1\}$ be a sequence of positive real numbers strictly decreasing to zero. By $\{\tau_n = \tau(\varepsilon(n)), n \geqslant 1\}$ we denote a sequence of random variables such that

$$\tau_n = \inf\{m\colon \inf(X_1, X_2, \ldots, X_m) \leqslant \varepsilon(n)\}, \tag{3.1}$$

where $\{X_n, n \geqslant 1\}$ is a sequence of i.r.vs.u.d. on $[0, 1]$.

LEMMA 3.1. *The sequence* $\{\tau_n, n \geq 1\}$ *increases with probability one and* $\tau_n \to \infty$ *a.s. as* $n \to \infty$.

LEMMA 3.2. *The random variables* $\tau_{n+1}-\tau_n$, $n \geq 1$, *are independent and if* $\varepsilon(n) = n^{-1}$, *then*

$$\mathrm{E}(\tau_{n+1}-\tau_n) = 1, \quad \sigma^2(\tau_{n+1}-\tau_n) = 2n, \quad n \geq 1, \tag{3.2}$$

$$P[\tau_{n+1}-\tau_n \geq r] = \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^{r-1} \quad \text{for any } r > 0,\ n \geq 1. \tag{3.3}$$

Let us put

$$U_n = \sum_{k=1}^{n-1} (\tau_{k+1}-\tau_k)\frac{1}{k}, \quad U'_n = \sum_{k=1}^{n-1} (\tau_{k+1}-\tau_k)\frac{1}{k+1}. \tag{3.4}$$

Then

$$\mathrm{E}U_n - \log n = O(1), \quad \mathrm{E}U'_n - \log n = O(1), \tag{3.5}$$

$$\sigma^2 U_n - 2\log n = O(1), \quad \sigma^2 U'_n - 2\log n = O(1), \tag{3.6}$$

$$\sum_{k=1}^{n} \mathrm{E}(\tau_{k+1}-\tau_k)^p/k^p \sim \sum_{k=1}^{n} \mathrm{E}(\tau_{k+1}-\tau_k)^p/(k+1)^p \sim p!\log n, \tag{3.7}$$

$$\mathrm{E}(U_n - U'_n) = O(1), \quad \sigma^2(U_n - U'_n) = O(1), \tag{3.8}$$

where $b_n = O(1)$ *means that the sequence* $\{b_n, n \geq 1\}$ *is bounded as* $n \to \infty$.

LEMMA 3.3. *Let* U_n, U'_n *be given by* (3.4). *Then*

$$-2+U'_n \leq \tilde{S}_{\tau_n} - \tilde{S}_{\tau_1} \leq U_n \ \text{a.s.}, \quad n \geq 2, \tag{3.9}$$

$$\tilde{S}_{\tau_{n-1}} \leq \tilde{S}_m \leq \tilde{S}_{\tau_n} \quad \text{for } m \in \langle \tau_{n-1}, \tau_n), \tag{3.10}$$

where

$$\tilde{S}_n = \sum_{k=1}^{n} X_k^*, \quad \tilde{S}_{\tau_n} = \sum_{k=1}^{\tau_n} X_k, \quad n \geq 1,\ X_k^* = \inf(X_1, \ldots, X_k),\ k \geq 1.$$

LEMMA 3.4. *We have*

$$\limsup_{n\to\infty} \tau_n/n\log_2 n = 1 \ \text{a.s.}, \tag{3.11}$$

$$\limsup_{n\to\infty} [\tau_{n+1}-\tau_n-1]/n\log_2 n = 1 \ \text{a.s.} \tag{3.12}$$

LEMMA 3.5. *For all* $A > 0$,

$$[n\log n\log_2 n \ldots (\log_p n)^{1+A}]^{-1} \leq X_n \leq [\log_2 n + \log_3 n + \ldots + (1+A)\log_p n]/n \ \text{a.s.}$$

for sufficiently large n.

LEMMA 3.6. *Under the assumptions of Theorem* 1,

$$\frac{\sum_{m=1}^{n} \delta_m \big(G(X_m^*)-bX_m^*\big)}{b\sqrt{2\log n}} + \frac{\sum_{m=1}^{n} (1-\delta_m)\big(G(X_m^*)-bX_m^*\big)}{b\sqrt{2\log n}} \xrightarrow{P} 0$$

as $n \to \infty$, *and*

$$\mathrm{E}\Big[\sum_{m=1}^{\infty} \delta_m |G(X_m^*)-bX_m^*|\Big] < \infty,$$

where

$$\delta_m = \begin{cases} 1 & \text{if } X_m \leqslant \delta, \\ 0 & \text{if } X_m > \delta, \end{cases} \quad 0 < \delta < 1,$$

$$G(t) = \inf\{x \geqslant 0\colon\ F(x) \geqslant t\}.$$

LEMMA 3.7. *For any pair of reals* $0 \leqslant a < b < \infty$ *we have*

$$T(b)-T(a) \leqslant \sqrt{8/\pi}\,(b-a)e^{-a^2/2},$$

where

$$T(x) = P\big[\sup_{t\in\langle 0,1\rangle} |W(t)| \leqslant x\big],$$

and $\{W(t),\, t\in\langle 0, 1\rangle\}$ *is a standard Brownian motion* (see [15]).

LEMMA 3.8 (the Skorokhod representation theorem; see Theorem A.1 in [5] and Theorem 4 in [14]). *Let* $Y_1, Y_2, \ldots, Y_n$ *be mutually independent random variables with zero means and* $\sigma^2 Y_i = \sigma_i^2$, $1 \leqslant i \leqslant n$. *Then there exists a sequence of nonnegative, mutually independent random variables* $z_1, z_2, \ldots, z_n$ *with the following properties*:

The joint distributions of the random variables $Y_1, Y_2, \ldots, Y_n$ *are identical to the joint distributions of the random variables* $W(z_1)$, $W(z_1+z_2)-W(z_1), \ldots, W(z_1+\ldots+z_n)-W(z_1+\ldots+z_{n-1})$, $\mathrm{E}z_i = \sigma_i^2$, *and* $\mathrm{E}|z_i|^k \leqslant C_k \mathrm{E}(Y_i)^{2k}$, $k \geqslant 1$, *where* $C_k = 2(8/\pi^2)^{k-1}\Gamma(k+1)$.

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Received on 17.5.1989