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EQUIVALENCE OF DISTRIBUTIONS OF SOME ORNSTEIN-UHLENBECK PROCESSES TAKING VALUES IN HILBERT SPACE

BY

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Abstract. Sufficient conditions for equivalence of distributions in $L^2(0, T; H)$ of two Ornstein–Uhlenbeck processes taking values in a Hilbert space H are given. The Girsanov theorem and some facts in the theory of perturbations of semigroup generators are used.

0. Introduction. Let X and Y be two Ornstein–Uhlenbeck processes on a real separable Hilbert space H. We assume that they are solutions of the following linear stochastic equations:

$$dX = AXdt + dW, \quad X(0) = x \in H, \tag{0.1}$$

$$dY = BYdt + dW, \quad Y(0) = x \in H, \tag{0.2}$$

where W is a cylindrical Wiener process on H, A and B stand for infinitesimal generators of C_0-semigroups from a class to be specified later. By the solution of (0.1) or (0.2) we understand the so-called *mild solution*. Let $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ be the laws (distributions) in $L^2(0, T; H)$ of X and Y. This paper presents sufficient conditions for the equivalence of $\mathscr{L}(X)$ and $\mathscr{L}(Y)$. In the proof of the main result (Theorem 1.1) an approximation technique is used. This technique needs some facts, mentioned in the Appendix, from the theory of perturbations of semigroup generators. The operator B is approximated by a sequence $\{B_n\}$ such that the law equivalence of the mild solutions corresponding to A and B_n follows immediately from the Girsanov theorem and the sequence of densities is relatively weakly compact (more precisely, the sequence of entropies is bounded). The problem of law equivalence of the processes X and Y was considered by Koski and Loges [7] for self-adjoint and commuting generators, by Kozlov [8] and [9] for elliptic generators and by Zabczyk [12] for analytic generators, delay equations and finite dimensional equations. This paper covers a general class of equations for which mild solutions take values in H. The cases of self-adjoint generators and elliptic generators are considered in Section 4 concerning particular cases.

1. Notation and formulation of the main result. Let (ν, H, E) be an abstract Wiener space, i.e., E is a real separable Banach space, H is densely and continuously imbedded in E, and ν is a mean 0 Gaussian measure on E satisfying the condition

$$\int_E (l, z)(k, z)\nu(dz) = \langle l, k\rangle_H, \qquad l, k \in E^* \subseteq H^* = H,$$

where (,) stands for the canonical bilinear form on $E^* \times E$. Let W be a Wiener process in (ν, H, E) (see [10]) defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_t$, $t \geqslant 0$, be the complete sub-σ-fields of $\mathscr{F}$ generated by $\{W(s)\colon 0 \leqslant s \leqslant t\}$.

The space of bounded (or Hilbert–Schmidt) linear operators on H and the operator (or Hilbert–Schmidt) norm will be denoted by $L(H)$ (or $L_2(H)$) and by $\|\ \|$ (or $\|\ \|_2$), respectively. By C_0 we denote the collection of all generators of C_0-semigroups acting on H. In our considerations an important role is played by the class $\mathscr{U}$ of generators $A \in C_0$ such that the semigroup S generated by A satisfies the condition

$$\int_0^1 \|S(t)\|_2^2 dt < \infty.$$

In this and next sections, $A \in \mathscr{U}$ is fixed and S stands for the semigroup generated by A. The following stochastic process, called the *mild solution* of (0.1),

$$X(t) = S(t)x + \int_0^t S(t-s)dW(s), \qquad t \geqslant 0, \tag{1.1}$$

is well defined (see [10]). Moreover, X is an H-valued Gaussian process, $EX(t) = S(t)x$ and

$$E|X(t)|_H^2 = |S(t)x|_H^2 + \int_0^t \|S(s)\|_2^2 ds. \tag{1.2}$$

According to ([4], p. 209), X can be considered as a random element in the space $\mathscr{H}_T = L^2(0, T; H)$. The main result of this paper is

THEOREM 1.1. *Suppose that $A \in \mathscr{U}$ and $B = A + K$, where $(K, D(K))$ is a closed linear operator on H such that*

$$D(K) \supseteq \bigcup_{t>0} \operatorname{Range} S(t) \quad \textit{and} \quad \int_0^1 \|KS(t)\|_2^2 dt < \infty.$$

Then $B \in \mathscr{U}$ and, for all $x \in H$ and $T > 0$, the distributions in $\mathscr{H}_T$ of the solutions of (0.1) *and* (0.2) *are equivalent.*

In the Appendix, the collection of linear operators K satisfying the assumptions of Theorem 1.1 is denoted by $\mathscr{P}_2(A)$. The proof of Theorem 1.1 is postponed to Section 3.

2. The case of bounded perturbations. In this section we consider the simple case $B = A + K$, where K is a bounded operator on H. The following infinite dimensional version of the well-known Girsanov theorem whose proof can be found in [9] plays an important role in our considerations.

THEOREM 2.1. *Let ϕ: $[0, T] \times \Omega \to H$ be an $(\mathcal{F}_t)$-nonanticipating process such that*

$$E\exp\Big\{\int_0^T \phi(s)dW(s) - \tfrac{1}{2}\int_0^T |\phi(s)|_H^2 ds\Big\} = 1. \tag{2.1}$$

Then

$$W^*(t) = W(t) - \int_0^t \phi(s)ds, \quad 0 \leqslant t \leqslant T, \tag{2.2}$$

is a Wiener process in (ν, H, E) defined on the probability space $(\Omega, \mathcal{F}, P^*)$, where

$$P^*(d\omega) = \exp\Big\{\int_0^T \phi(s)dW(s) - \tfrac{1}{2}\int_0^T |\phi(s)|_H^2 ds\Big\}P(d\omega).$$

Remark 2.1 (see [5], p. 83). If there exists $\delta > 0$ such that

$$\sup_{0 \leqslant t \leqslant T} E\exp(\delta|\phi(t)|_H^2) < \infty, \tag{2.3}$$

then (2.1) holds.

Let $K \in L(H)$. Then $B = A + K \in C_0$ and the semigroup U generated by B satisfies the following equation:

$$U(t)x = S(t)x + \int_0^t S(t-s)KU(s)xds = S(t)x + \int_0^t U(t-s)KS(s)xds.$$

Moreover, it is easy to see that $A + K \in \mathcal{U}$ and

$$Y(t) = U(t)x + \int_0^t U(t-s)dW(s), \quad t \geqslant 0,$$

is the unique $(\mathcal{F}_t)$-nonanticipating solution of the stochastic integral equation

$$Y(t) = S(t)x + \int_0^t S(t-s)KY(s)ds + \int_0^t S(t-s)dW(s), \quad t \geqslant 0. \tag{2.4}$$

The main result of this section is

THEOREM 2.2. *For arbitrary $T > 0$ and $x \in H$ the distributions $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ of X and Y are equivalent and the Radon–Nikodým derivative*

$$\Psi = \frac{d\mathcal{L}(Y)}{d\mathcal{L}(X)}$$

has the following properties:

$$\Psi(X) = E\Big\{\exp\Big(\int_0^T KX(s)dW(s) - \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds\Big) \;\Big|\; X\Big\}, \tag{2.5}$$

$$E\Psi(X)\log\Psi(X) \leqslant \tfrac{1}{2}\int_0^T |KU(s)x|_H^2 ds + \tfrac{1}{2}\int_0^T\int_0^t \|KU(s)\|_2^2 dsdt. \tag{2.6}$$

Proof. Let $\phi(s) = KX(s)$. The Fernique theorem (see [10]) implies (2.3) (for details see [12], p. 26). From (1.1) and (2.2) we have

$$X(t) = S(t)x + \int_0^t S(t-s)dW(s)$$
$$= S(t)x + \int_0^t S(t-s)KX(s)ds + \int_0^t S(t-s)dW^*(s).$$

Hence X is the nonanticipating solution of (2.4) defined on the probability space $(\Omega, \mathscr{F}, P^*)$. Therefore, for each $\Gamma \in \mathrm{Bor}(\mathscr{H}_T)$ we have

$$\mathscr{L}(Y)(\Gamma) = P(Y \in \Gamma) = P^*(X \in \Gamma)$$
$$= \int_{X \in \Gamma} \exp\{\int_0^T KX(s)dW(s) - \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds\} P(d\omega).$$

This proves $\mathscr{L}(Y) \ll \mathscr{L}(X)$ and (2.5). The absolute continuity of $\mathscr{L}(X)$ with respect to $\mathscr{L}(Y)$ can be proved by similar arguments with the replacement of A with $A+K$ and K with $-K$. To prove (2.6) notice that $p(u) = u \log u$ is a convex function. Hence, using Jensen's inequality to (2.5), we have

$$E\Psi(X)\log\Psi(X) \leqslant E\{\exp(\int_0^T KX(s)dW(s) - \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds)$$
$$\times (\int_0^T KX(s)dW(s) - \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds)\}$$
$$= E^{P^*}(\int_0^T KX(s)dW(s) - \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds)$$
$$= E^{P^*}(\int_0^T KX(s)dW^*(s) + \tfrac{1}{2}\int_0^T |KX(s)|_H^2 ds) = \tfrac{1}{2}E^{P^*}\int_0^T |KX(s)|_H^2 ds$$
$$= \tfrac{1}{2}E\int_0^T |KY(s)|_H^2 ds = \tfrac{1}{2}\int_0^T |KU(s)x|_H^2 ds + \tfrac{1}{2}\int_0^T\int_0^t \|KU(s)\|_2^2 ds dt.$$

Thus the proof is complete.

3. Proof of Theorem 1.1. Let $K_n = KS(1/n)$ for $n \in N$. Since K is closed, the operators K_n are bounded. By $\mathscr{L}(Y_n)$ we denote the distribution of the process

$$Y_n(t) = U_n(t)x + \int_0^t U_n(t-s)dW(s),$$

where U_n is the semigroup generated by $B_n = A + K_n$. By Theorem 2.2, $\mathscr{L}(Y_n)$ and $\mathscr{L}(X)$ are equivalent. Let

$$\Psi_n = \frac{d\mathscr{L}(Y_n)}{d\mathscr{L}(X)}, \qquad n \in N.$$

Then, by (2.6),

$$E\Psi_n(X)\log\Psi_n(X) \leqslant \tfrac{1}{2}\int_0^T |K_n U_n(s)x|_H^2 ds + \tfrac{1}{2}\int_0^T\int_0^t \|K_n U_n(s)\|_2^2 ds dt.$$

Hence, by Lemma A.5,

$$\sup_{n\in N} E\Psi_n(X)\log\Psi_n(X) < \infty.$$

According to the De La Vallée–Poussin theorem the sequence $\{\Psi_n(X)\}$ is uniformly integrable and, by the Dunford–Schwartz theorem, it is relatively weakly compact. Let $B = A+K$. By Lemma A.4, $B\in\mathscr{U}$. Let U be the semigroup generated by B and let Y be the solution of (0.2). Then (compare with (1.2))

$$E\int_0^T |Y(t)-Y_n(t)|_H^2 dt = \int_0^T |(U(t)-U_n(t))x|_H^2 dt + \int_0^T \|U(t)-U_n(t)\|_2^2 dt.$$

Hence, by Lemma A.4, the sequence $\{Y_n\}$ converges in $L^2(\Omega;\mathscr{H}_T, dP)$ to Y. Therefore, we have $\mathscr{L}(Y)\ll\mathscr{L}(X)$. Lemma A.6 implies that the absolute continuity of $\mathscr{L}(X)$ with respect to $\mathscr{L}(Y)$ can be proved by the replacement of A with $A+K$ and K with $-K$. This completes the proof.

EXAMPLE 3.1. Let $H = L^2(0,\pi)$, $A = d^2/d\sigma^2$ with the zero boundary conditions. Then $A\in\mathscr{U}$, and for arbitrary $h\in H$ the operator K which multiplies a function f by h (i.e., $Kf = hf$) is closed and satisfies the assumptions of Theorem 1.1.

4. Some special cases. In this section $(A, D(A))$ is a self-adjoint, negative definite linear operator on H. For $\varrho > 0$ the domain $D((-A)^\varrho)$ is treated here as a Hilbert space equipped with the graph norm. The following theorem follows from Theorem 1.1, Lemmas A.8 and A.6 and Corollary A.2.

THEOREM 4.1. *Suppose that there exists $0<\varepsilon_0$ such that $(-A)^{-1+\varepsilon}$ is nuclear for arbitrary $\varepsilon<\varepsilon_0$. Let K_1 and K_2 be closed linear operators on H such that:*

(i) *for some $\gamma < 1/2$, $D(K_1)\cap D(K_2) \supseteq D((-A)^\gamma)$,*

(ii) *for some $\varrho < \varepsilon_0/2$, the operator K_1-K_2 has a bounded extension acting from $D((-A)^\varrho)$ into H.*

Then, for all $x\in H$ and $T>0$, the distributions in $\mathscr{H}_T$ of the solutions of the equations

$$d\tilde{X} = (A+K_1)\tilde{X}dt + dW, \quad \tilde{X}(0) = x\in H, \tag{4.1}$$

$$d\tilde{Y} = (A+K_2)\tilde{Y}dt + dW, \quad \tilde{Y}(0) = x\in H, \tag{4.2}$$

are equivalent.

Now, let A be a self-adjoint, negative definite and uniformly elliptic differential operator of order $2m > d$ on a bounded region G in $\mathbf{R}^d$. We assume that all coefficients of A are in C^∞ and the region G has the restricted cone property (see [1], p. 11). The domain of A is given by

$$D(A) = \{\phi \in H^{2m}(G)\colon \beta_j \phi = 0 \text{ on } \partial G,\ 1 \leqslant j \leqslant m\},$$

where $H^{2m}(G)$ is the Sobolev space of order $2m$ and $\{\beta_j\}$ is a normal system of boundary operators (see [11]). These conditions imply that the operator A has a pure point spectrum $\{-\lambda_k\}$ and $\lambda_k \approx ck^{2m/d}$ as k converges to ∞ (see [1], Section 14, and [3], p. 179). The following theorem is a special case of Theorem 4.1 and, in a little stronger form, it was proved by Kozlov [8] and [9] for elliptic operators on a smooth compact manifold.

THEOREM 4.2. *If K_1 and K_2 are differential operators of orders less than m and with smooth coefficients, then $A+K_i$, $i = 1, 2$, belong to $\mathscr{U}$. If the order of $K_1 - K_2$ is less than $m-d/2$, then the distributions of the solutions of* (4.1) *and* (4.2) *are equivalent.*

APPENDIX

In the Appendix, $(A, D(A))$ is the generator of a C_0-semigroup S acting on H. The class $\mathscr{U}$ is introduced in Section 1. We consider here the class $\mathscr{P}(A)$ of perturbations which was introduced by Hille and Phillips (see [6]). Classes $\mathscr{P}_1(A)$ and $\mathscr{P}_2(A)$ are introduced by the author.

DEFINITION A.1. We say that a *linear operator* $(K, D(K))$ *on* H *belongs to* $\mathscr{P}(A)$ iff K is closed, $D(K) \supseteq \bigcup_{t>0} \operatorname{Range} S(t)$ and

$$\int_0^1 \|KS(t)\|\, dt < \infty.$$

DEFINITION A.2. We say that a *linear operator* $(K, D(K))$ *on* H *belongs to* $\mathscr{P}_1(A)$ iff $K \in \mathscr{P}(A)$ and

$$\int_0^1 \|KS(t)\|^2\, dt < \infty.$$

DEFINITION A.3. We say that a *linear operator* $(K, D(K))$ *on* H *belongs to* $\mathscr{P}_2(A)$ iff $K \in \mathscr{P}(A)$ and

$$\int_0^1 \|KS(t)\|_2^2\, dt < \infty.$$

Remark A.1. If $K \in \mathscr{P}(A)$, then the operators $KS(t)$, $t > 0$, are bounded. Moreover, $D(A) \subseteq D(K)$ and $B = A+K$ is the generator of a C_0-semigroup U (see [2], Chapter 3) such that $D(K) \supseteq \bigcup_{t>0} \operatorname{Range} U(t)$ and

$$U(t) = S(t) + \int_0^t S(t-s) K U(s)\, ds = S(t) + \int_0^t U(t-s) K S(s)\, ds. \tag{A.1}$$

LEMMA A.1. *If* $K \in \mathscr{P}(A)$, *then* $\mathscr{P}(A+K) = \mathscr{P}(A)$.

Proof. Let $L \in \mathscr{P}(A)$. Note that it is enough to show that, for some $0 < \tau$,

$$\int_0^\tau \|LU(t)\|\, dt < \infty .$$

Let $0 < \tau \leqslant 1$ be such that

$$\int_0^\tau \|LS(t)\| dt + \int_0^\tau \|KS(t)\|\, dt \leqslant \tfrac{1}{2}.$$

Then (see [2], pp. 68–71), for each $t \leqslant \tau$,

$$U(t) = S(t) + \int_0^t S(t-s)KS(s)ds + \int_0^t \int_0^s S(t-s)KS(s-s_1)KS(s_1)ds_1\, ds + \ldots$$

Hence

$$\int_0^\tau \|LU(t)\|\, dt \leqslant \int_0^\tau \|LS(t)\|\, dt + \int_0^\tau \int_0^t \|LS(t-s)KS(s)\|\, ds dt + \ldots$$

$$\leqslant \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Therefore, we have $\mathscr{P}(A) \subseteq \mathscr{P}(A+K)$. Since $-K \in \mathscr{P}(A) \subseteq \mathscr{P}(A+K)$, we have $\mathscr{P}(A+K) \subseteq \mathscr{P}(A)$.

Let $K_n = KS(1/n)$ and let U_n be the semigroup generated by $B_n = A + K_n$. The proof of the following lemma is rather routine and is omitted here.

LEMMA A.2. *Let* $K \in \mathscr{P}(A)$. *Then there exist constants* M *and* α *such that, for arbitrary* $n \in N$, U *and* U_n *belong to* $C_0(M, \alpha)$. *Moreover, for each* $v \in D(A) = D(B) = D(B_n)$,

$$\lim_{n\to\infty} B_n v = Bv.$$

According to the Trotter–Kato theorem (see [2], Chapter 3) we have

COROLLARY A.1. *For all* $v \in H$

$$\lim_{n\to\infty} U_n(t)v = U(t)v$$

uniformly with respect to t *on every compact set.*

LEMMA A.3. *Let* $K \in \mathscr{P}_1(A)$. *Then* $\mathscr{P}_1(A+K) = \mathscr{P}_1(A)$.

Proof. Let $L \in \mathscr{P}_1(A)$. We will show that $L \in \mathscr{P}_1(A+K)$. Since $\mathscr{P}(A+K) = \mathscr{P}(A)$, it is enough to verify that, for some $0 < \tau$,

$$\int_0^\tau \|LU(t)\|^2 dt < \infty .$$

Let $0 < \tau \leqslant 1$ be such that

$$\int_0^\tau \|LS(t)\|^2 dt + \int_0^\tau \|KS(t)\|^2 dt \leqslant \tfrac{1}{4}.$$

Using arguments similar to those in the proof of Lemma A.1 we have

$$\Big(\int_0^\tau \|LU(t)\|^2 dt\Big)^{1/2} \leqslant \Big(\int_0^\tau \|LS(t)\|^2 dt\Big)^{1/2}$$
$$+\Big(\int_0^\tau t \int_0^t \|LS(t-s)\|^2 \|KS(s)\|^2 ds dt\Big)^{1/2} + \ldots \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence $\mathscr{P}_1(A) \subseteq \mathscr{P}_1(A+K)$. Since $-K \in \mathscr{P}_1(A)$, we have $\mathscr{P}_1(A+K) \subseteq \mathscr{P}_1(A)$.

LEMMA A.4. *If $A \in \mathscr{U}$ and $K \in \mathscr{P}_1(A)$, then $B = A+K \in \mathscr{U}$ and, for all $\tau > 0$,*

$$\lim_{n\to\infty} \int_0^\tau \|U(t)-U_n(t)\|_2^2 dt = 0. \tag{A.2}$$

Proof. Since (A.1) holds and $K \in \mathscr{P}_1(A+K)$, we have

$$\int_0^\tau \|U(t)\|_2^2 dt \leqslant 2\int_0^\tau \|S(t)\|_2^2 dt + 2\int_0^\tau t \int_0^t \|S(t-s)\|_2^2 \|KU(s)\|^2 ds dt$$
$$\leqslant 2\int_0^\tau \|S(t)\|_2^2 dt + 2\tau \int_0^\tau \|S(t)\|_2^2 dt \int_0^\tau \|KU(t)\|^2 dt < \infty.$$

Note that it is enough to show that (A.2) holds for some $\tau > 0$. Since

$$\int_0^t \|K_n S(s)\|^2 ds = \int_{1/n}^{t+1/n} \|KS(s)\|^2 ds,$$

we may choose $\tau \in (0, 1]$ and $M_0 \in N$ such that

$$\int_0^\tau \|K_n S(s)\|^2 ds \leqslant \tfrac{1}{4} \quad \text{for } n \geqslant M_0.$$

If $n \geqslant M_0$, then

$$\int_0^\tau \|U(t)-U_n(t)\|_2^2 dt \leqslant 2\int_0^\tau t \int_0^t \|\big(U(t-s)-U_n(t-s)\big)K_n S(s)\|_2^2 ds dt$$
$$+2\int_0^\tau t \int_0^t \|U(t-s)(K-K_n)S(s)\|_2^2 ds dt$$
$$\leqslant \tfrac{1}{2}\int_0^\tau \|U(t)-U_n(t)\|_2^2 dt + 2\int_0^\tau \|U(t)\|_2^2 dt \int_0^\tau \|(K-K_n)S(t)\|^2 dt.$$

Hence

$$\int_0^\tau \|U(t)-U_n(t)\|_2^2 dt \leqslant 4\int_0^\tau \|U(t)\|_2^2 dt \int_0^\tau \|(K-K_n)S(t)\|^2 dt. \tag{A.3}$$

Since S is continuous in the operator norm on the interval $(0, \infty)$, the right-hand side of (A.3) converges to 0 and we get (A.2). Thus the proof is complete.

LEMMA A.5. *Let $K \in \mathscr{P}_2(A)$. Then, for $\tau > 0$,*

$$\sup_{n\in N} \int_0^\tau \|K_n U_n(t)\|_2^2 dt < \infty. \tag{A.4}$$

Proof. Note that, as in the proofs of the previous lemmas, it is enough to verify (A.4) for some $\tau > 0$. Let $\tau \in (0, 1]$ and $M_1 \in N$ be such that

$$\int_0^\tau \|K_n S(s)\|_2^2 ds \leqslant \tfrac{1}{4} \quad \text{for } n \geqslant M_1.$$

By (A.1) we have

$$K_n U_n(t) = K_n S(t) + \int_0^\tau K_n U_n(t-s) K_n S(s) ds.$$

Hence, for $n \geqslant M_1$, we obtain

$$\begin{aligned}\int_0^\tau \|K_n U_n(t)\|_2^2 dt &\leqslant 2 \int_0^\tau \|K_n S(t)\|_2^2 dt + 2 \int_0^\tau t \int_0^t \|K_n U_n(t-s) K_n S(s)\|_2^2 ds dt \\ &\leqslant \tfrac{1}{2} + 2 \int_0^\tau \|K_n S(t)\|_2^2 dt \int_0^\tau \|K_n U_n(t)\|_2^2 dt \\ &\leqslant \tfrac{1}{2} + \tfrac{1}{2} \int_0^\tau \|K_n U_n(t)\|_2^2 dt\end{aligned}$$

and, consequently,

$$\int_0^\tau \|K_n U_n(t)\|_2^2 dt \leqslant 1 \quad \text{for } n \geqslant M_1.$$

LEMMA A.6. *If $A \in \mathscr{U}$ and $K \in \mathscr{P}_1(A)$, then $\mathscr{P}_2(A) = \mathscr{P}_2(A+K)$.*

Proof. Let $L \in \mathscr{P}_2(A)$. By (A.1) we have

$$\begin{aligned}\int_0^1 \|LU(t)\|_2^2 dt &\leqslant 2 \int_0^1 \|LS(t)\|_2^2 dt + 2 \int_0^1 t \int_0^t \|LS(t-s) KU(s)\|_2^2 ds dt \\ &\leqslant 2 \int_0^1 \|LS(t)\|_2^2 dt + 2 \int_0^1 \|LS(t)\|_2^2 dt \int_0^1 \|KU(t)\|^2 dt < \infty.\end{aligned}$$

Hence $\mathscr{P}_2(A) \subseteq \mathscr{P}_2(A+K)$. The inclusion $\mathscr{P}_2(A+K) \subseteq \mathscr{P}_2(A)$ can be proved in the same way.

Now, let $(A, D(A))$ be a self-adjoint, negative definite linear operator on H. Note that $A \in \mathscr{U}$ iff A^{-1} is nuclear. Therefore, we assume that A^{-1} is a nuclear operator on H.

LEMMA A.7. *If* $\varrho < 1/2$, *then* $K = (-A)^{\varrho} \in \mathscr{P}_1(A)$.

Proof. Let $\{-\lambda_k, e_k\}$ be the sequence of all eigenvalues and the corresponding normalized eigenvectors of A. The C_0-semigroup S generated by A has the form

$$S(t)v = \sum_{k=1}^{\infty} \exp(-\lambda_k t)\langle v, e_k\rangle_H e_k.$$

Moreover,

$$Kv = \sum_{k=1}^{\infty} \lambda_k^{\varrho}\langle v, e_k\rangle_H e_k \quad \text{for } v \in D(K).$$

Hence, for $v \in H$,

$$|KS(t)v|_H^2 = \sum_{k=1}^{\infty} \lambda_k^{2\varrho} \exp(-2\lambda_k t)\langle v, e_k\rangle_H^2$$

$$\leqslant |v|_H^2 \sup_{k \in N} \lambda_k^{2\varrho} \exp(-2\lambda_k t) \leqslant |v|_H^2 \varrho^{2\varrho} t^{-2\varrho} \exp(-2\varrho)$$

and, consequently,

$$\int_0^1 \|KS(t)\|^2 dt \leqslant \varrho^{2\varrho} \exp(-2\varrho) \int_0^1 t^{-2\varrho} dt < \infty.$$

LEMMA A.8. *If* $(K, D(K))$ *is a closed operator such that* $D((-A)^{\varrho}) \subseteq D(K)$ *for some* $\varrho < 1/2$, *then* $K \in \mathscr{P}_1(A)$.

Proof. Since $K(-A)^{-\varrho}$ is bounded, we have

$$\int_0^1 \|KS(t)\|^2 dt \leqslant \int_0^1 \|K(-A)^{-\varrho}\|^2 \, \|(-A)^{\varrho} S(t)\|^2 dt < \infty.$$

Using similar arguments we can easily obtain

LEMMA A.9. *Suppose that there exists* $0 < \varepsilon_0$ *such that* $(-A)^{-1+\varepsilon}$ *is nuclear for arbitrary* $\varepsilon < \varepsilon_0$. *Then, for each* $\varrho < \varepsilon_0/2$, $(-A)^{\varrho} \in \mathscr{P}_2(A)$.

COROLLARY A.2. *If* $(K, D(K))$ *is a closed operator such that* $D((-A)^{\varrho}) \subseteq D(K)$ *for some* $\varrho < \varepsilon_0/2$, *then* $K \in \mathscr{P}_2(A)$.

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REFERENCES

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton 1965.

[2] E. B. Davies, *One-Parameter Semigroups*, London Math. Soc. Monographs 15, Academic Press, London 1980.

[3] T. Funaki, *Random motion of strings and related stochastic evolution equations*, Nagoya Math. J. 89 (1983), pp. 129–193.

[4] I. I. Gikhman and A. V. Skorokhod, *Theory of Random Processes* (in Russian), Nauka, Moscow 1965.

[5] — *Stochastic Differential Equations* (in Russian), Naukova Dumka, Kijev 1968.

[6] E. Hille and R. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Providence, R. I., 1957.

[7] T. Koski and W. Loges, *Asymptotic statistical inference for a stochastic heat flow problem*, Statist. Probab. Lett. 3 (1985), pp. 185–189.

[8] S. M. Kozlov, *Equivalence of measures in Itô's linear partial differential equations* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1977) (4), pp. 147–152.

[9] — *Some questions of stochastic partial differential equations* (in Russian), Trudy Sem. Petrovsk. 4 (1978), pp. 147–172.

[10] H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Math. 463, Springer-Verlag, Berlin 1975.

[11] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, *I*, *II*, Springer-Verlag, New York 1972.

[12] J. Zabczyk, *Law equivalence of Ornstein–Uhlenbeck processes and control equivalence of linear systems*, Preprint 457, Inst. of Math. Polish Acad. of Sci., June 1989.

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