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STOCHASTIC PROCESSES IN RÉNYI CONDITIONAL PROBABILITY SPACES

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Abstract. In this paper we study stochastic processes in Rényi conditional probability spaces. We prove a conditional analogue of the Kolmogorov fundamental theorem.

0. Introduction. The universally recognized axiomatic foundations of probability theory were given by A. N. Kolmogorov in 1933. The majority of stochastic problems can be considered within this general framework. In some cases, however, Kolmogorov's approach may fail. For instance, we encounter such a situation when we have to deal with unbounded measures. The use of unbounded measures cannot be justified in the theory of Kolmogorov. Hence one cannot speak about the uniform probability distribution on the whole real line or about the random choice of integers with equal non-zero probability.

A new approach to probability theory which overcomes these difficulties was proposed in 1955 by Alfred Rényi [4]. He linked the measure-theoretic concepts of Kolmogorov with the ideas of other mathematicians who regarded the notion of conditional probability as the basic one.

Rényi defines the *conditional probability* as a function $P\colon \Sigma \times \mathscr{B} \to [0, 1]$, where Σ is a σ-algebra of subsets of a given set (space of random events) and $\mathscr{B}$ is a non-empty subfamily of Σ (space of conditions). He assumes that $P(\cdot|B)$ is a probability measure such that $P(B|B) = 1$ for every $B \in \mathscr{B}$ and, moreover, the family of probability measures defined in this way satisfies a natural compatibility condition. In Rényi's theory, unbounded measures are used to construct an important class of conditional probability spaces (see Example 1.3).

In a series of papers (see, e.g., [4]–[6]) and in a book [7] Rényi presented a theory of conditional probability spaces and gave a lot of applications. He claimed the study of stochastic processes as one of the reasons to develop a new theory. However, he defined only random variables and their distributions but not a stochastic process. The present paper in which we study stochastic processes in conditional probability spaces may be treated as a natural expansion of Rényi's ideas. In the sequel, following other authors, we shall call conditional probability spaces *Rényi spaces.*

The main aim of the present paper is to prove a conditional probability analogue of the Kolmogorov fundamental theorem. We shall see that the situation is a bit more complicated in the case of the Rényi spaces. We shall give two versions of this theorem. The idea of the proof is similar to the classical one but several alterations are necessary.

The paper is organized as follows. In Section 1 we introduce the basic notions of the theory. In Section 2 we define random variables and stochastic processes in Rényi spaces. We show that this definition imposes some restrictions on the construction of stochastic processes. We devote Section 3 to the proof of our main result — the Kolmogorov fundamental theorem for Rényi spaces. An essential step of the proof is the theorem on the extension of a conditional probability from an algebra to the σ-algebra generated by it. In Section 4 we prove a version of the Kolmogorov fundamental theorem for the Rényi spaces generated by measures.

1. Preliminaries. We start with the definition of a Rényi space.

DEFINITION 1.1. By a *Rényi space* we mean a system $[\Omega, \Sigma, \mathscr{B}, P]$, where Ω is an arbitrary set (*space of elementary events*), Σ is a σ-algebra of subsets of Ω (*space of random events*), $\mathscr{B}$ is a non-empty subfamily of Σ (*space of conditions*), and P is a non-negative map defined on $\Sigma \times \mathscr{B}$ (*conditional probability*), which fulfills the following conditions:

(I) $P(B|B) = 1$ for every $B \in \mathscr{B}$;

(II) $P(\bigcup_{n=1}^{\infty} A_n|B) = \sum_{n=1}^{\infty} P(A_n|B)$ for every disjoint family $\{A_n\}_{n \in N}$, where $A_n \in \Sigma$, for $n \in N$, and for every $B \in \mathscr{B}$;

(III) $P(A \cap B|C) = P(A|B \cap C) \cdot P(B|C)$ for every $A, B \in \Sigma$ and $C \in \mathscr{B}$ such that $B \cap C \in \mathscr{B}$ (see [4], p. 289).

In the sequel we shall need the following equivalent definition of a Rényi space (see also [1] and [2]).

PROPOSITION 1.2. *Let (Ω, Σ) be a measurable space, let $\mathscr{B}$ be a non-empty subfamily of Σ, and let P be a non-negative map defined on $\Sigma \times \mathscr{B}$. Then $[\Omega, \Sigma, \mathscr{B}, P]$ is a Rényi space if and only if the following conditions are satisfied*:

(I) $P(B|B) = 1$ *for every* $B \in \mathscr{B}$;

(II') $P(\cdot|B)$ *is a probability measure on* (Ω, Σ) *for every* $B \in \mathscr{B}$;

(III') $P(A|C) = P(A|B) \cdot P(B|C)$ *for every* $A \in \Sigma$, $B, C \in \mathscr{B}$ *such that* $A \subset B \subset C$.

The proof is easy and is left to the reader.

The following example of a Rényi space seems to be the most important from the point of view of applications.

EXAMPLE 1.3. If m is a measure on a measurable space (Ω, Σ), then for every non-empty family $\mathscr{B} \subset \{B \in \Sigma\colon\ 0 < m(B) < +\infty\}$ and for the map P defined on $\Sigma \times \mathscr{B}$ by the formula

$$P(A|B) := \frac{m(A \cap B)}{m(B)} \quad \text{for } A \in \Sigma \text{ and } B \in \mathscr{B}$$

the system $\mathscr{R} := [\Omega, \Sigma, \mathscr{B}, P]$ is a Rényi space (see [4], p. 304).

This example motivates the following definition:

Definition 1.4. The Rényi space $\mathscr{R}$ constructed above is called the *Rényi space generated by the measure m*. We shall say that $\mathscr{R}$ is *strictly generated by the measure m* if the equality $\mathscr{B} = \{B \in \Sigma\colon\ 0 < m(B) < +\infty\}$ holds.

Remark 1.5. A necessary and sufficient condition that the measure m generates some Rényi space is that $\{B \in \Sigma\colon\ 0 < m(B) < +\infty\} \neq \emptyset$. It is satisfied if, for example, m is a non-zero and σ-finite measure. It follows from Proposition 1.2 that each usual (Kolmogorov) probability space may be treated as a Rényi space taking m equal to probability measure.

We mention now a special case of Definition 1.4.

Definition 1.6. The *uniform Rényi space on* $\boldsymbol{R}^n$ is a Rényi space generated by the Lebesgue measure m_n on $(\boldsymbol{R}^n, B(\boldsymbol{R}^n))$, where $B(\boldsymbol{R}^n)$ is the Borel σ-algebra on $\boldsymbol{R}^n$.

Definition 1.7. We shall say that the Rényi space $[\Omega, \Sigma, \mathscr{B}^*, P^*]$ is an *extension* of the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$ if $\mathscr{B} \subset \mathscr{B}^*$ and $P^*(A|B) = P(A|B)$ for all $A \in \Sigma$ and $B \in \mathscr{B}$.

2. Random variables and stochastic processes. In [4] and [7] Rényi defined random variables on conditional probability spaces and their distributions. Our definition is slightly different.

Proposition 2.1. *Let $[\Omega, \Sigma, \mathscr{B}, P]$ be a Rényi space, X a topological space, $B(X)$ the Borel σ-algebra on X, $\xi\colon \Omega \to X$ a measurable map such that*

$$\mathscr{B}_\xi := \{C \in B(X)\colon\ \xi^{-1}(C) \in \mathscr{B}\} \neq \emptyset \tag{1}$$

and $\Phi_\xi\colon B(X) \times \mathscr{B}_\xi \to \boldsymbol{R}$ a map defined by the formula

$$\Phi_\xi(B|C) := P\big(\xi^{-1}(B)|\xi^{-1}(C)\big)$$

for $B \in B(X)$ and $C \in \mathscr{B}_\xi$. Then $[X, B(X), \mathscr{B}_\xi, \Phi_\xi]$ is a Rényi space (see [4], p. 295).

Definition 2.2. The map ξ defined above will be called a *Rényi random variable*, the map Φ_ξ the *distribution of the random variable* ξ, and $[X, B(X), \mathscr{B}_\xi, \Phi_\xi]$ the *Rényi space generated by the random variable* ξ; if $X = \boldsymbol{R}^n$, then Φ_ξ will be called a *uniform distribution* whenever $[\boldsymbol{R}^n, B(\boldsymbol{R}^n), \mathscr{B}_\xi, \Phi_\xi]$ is uniform (see Definition 1.6).

DEFINITION 2.3. Let $[\Omega, \Sigma, \mathscr{B}, P]$ be a Rényi space, let X be a topological space, and let T be an arbitrary set (usually T is an interval in $\boldsymbol{N}$ or $\boldsymbol{R}$). A map $x: T \times \Omega \to X$ will be called a *Rényi stochastic process* if, for each $t \in T$, the map $x_t := x(t, \cdot): \Omega \to X$ is a Rényi random variable.

PROPOSITION 2.4. *Let $x: T \times \Omega \to X$ be a Rényi stochastic process. Then $(x_{t_1}, \ldots, x_{t_n}): \Omega \to X^n$ is a Rényi random variable.*

Proof. Let $n \in N$ and $t_1, \ldots, t_n \in T$. It is obvious that $(x_{t_1}, \ldots, x_{t_n})$ is measurable, so it remains to prove that $\mathscr{B}_{(x_{t_1}, \ldots, x_{t_n})} \neq \emptyset$. We know that $\mathscr{B}_{x_{t_1}}$ is non-empty as x_{t_1} is a Rényi random variable. Hence there exists $C \in B(X)$ such that $x_{t_1}^{-1}(C) \in \mathscr{B}$. Moreover,

$$(x_{t_1}, \ldots, x_{t_n})^{-1}(C \times X \times \ldots \times X) = x_{t_1}^{-1}(C) \quad \text{and} \quad C \times X \times \ldots \times X \in B(X^n).$$

Thus $\mathscr{B}_{(x_{t_1}, \ldots, x_{t_n})}$ is non-empty. ∎

This leads to the following definition:

DEFINITION 2.5. The distribution of the Rényi random variable $x_{t_1 \ldots t_n} := (x_{t_1}, \ldots, x_{t_n})$ will be called the *n-dimensional joint distribution of the Rényi stochastic process* x. We shall denote it by $\Phi_{x_{t_1} \ldots x_{t_n}}$. The respective space of conditions will be denoted by $\mathscr{B}_{x_{t_1} \ldots x_{t_n}}$.

The next proposition shows that the notion of a Rényi stochastic process is simply a generalization of the usual notion of a stochastic process.

PROPOSITION 2.6. *Let $[\Omega, \Sigma, P]$ be a probability space, X a topological space, T an arbitrary set, and $x: T \times \Omega \to X$ a stochastic process. Then x is a Rényi stochastic process with respect to the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$ strictly generated by the measure P on (Ω, Σ).*

Proof. Let $t \in T$. It is enough to prove that the random variable x_t fulfills condition (1) of Proposition 2.1. We have $x_t^{-1}(X) = \Omega \in \mathscr{B}$. Hence $\mathscr{B} \neq \emptyset$. ∎

The following proposition shows that Definition 2.3 imposes some restrictions on the construction of a stochastic process.

PROPOSITION 2.7. *If $[\Omega, \Sigma, \mathscr{B}, P]$ is a Rényi space, T is an arbitrary set, and $x: T \times \Omega \to X$ is a Rényi stochastic process, then*

(1) *any n-dimensional joint distribution of the process x is not uniform for $n \geqslant 2$;*

(2) *if for some $n \in N$ and $t_1, \ldots, t_n \in T$ the Rényi space generated by the random variable $x_{t_1 \ldots t_n}$ is an extension of the uniform Rényi space on $\boldsymbol{R}^n$, then*

$$\mathscr{B}_{x_{t_1} \ldots x_{t_k}} \cap \{A \in B(\boldsymbol{R}^k): 0 < m_k(A) < +\infty\} = \emptyset$$

for each $k \in N$, $k < n$, where m_k is the Lebesgue measure on $\boldsymbol{R}^k$.

Proof. (1) Let $n \in N$, $n \geqslant 2$, and $t_1, \ldots, t_n \in T$. Choose any set $A \in \mathscr{B}_{x_{t_1}}$. Then $x_{t_1}^{-1}(A) \in \mathscr{B}$. Hence and from the equality $x_{t_1}^{-1}(A) = x_{t_1 \ldots t_n}^{-1}(A \times \boldsymbol{R}^{n-1})$ we deduce that $A \times \boldsymbol{R}^{n-1} \in \mathscr{B}_{x_{t_1} \ldots x_{t_n}}$. On the other hand, $m_n(A \times \boldsymbol{R}^{n-1}) = 0$ or $m_n(A \times \boldsymbol{R}^{n-1}) = +\infty$, and so the Rényi space generated by the random variable $x_{t_1 \ldots t_n}$ cannot be uniform.

(2) We start with the following lemma (see [2], Remark 2.1).

LEMMA. *Let $[\Omega, \Sigma, \mathscr{B}, P]$ be a Rényi space, let $\{B_n\}_{n \in N}$ be an increasing sequence of sets from $\mathscr{B}$ such that $P(B_n | B_{n+1}) > 0$ for each $n \in N$, and let $B = \bigcup_{n \in N} B_n \in \mathscr{B}$. Then*

$$\prod_{k=1}^{\infty} P(B_k | B_{k+1}) > 0.$$

Now let $n \in N$, and $t_1, \ldots, t_n \in T$. We shall argue by contradiction. Let us assume that $[\boldsymbol{R}^n, B(\boldsymbol{R}^n), \mathscr{B}_{x_{t_1 \ldots t_n}}, \Phi_{x_{t_1 \ldots t_n}}]$ is an extension of the uniform space on $\boldsymbol{R}^n$ and there exist $k \in N$, $k < n$, and $A \in \mathscr{B}_{x_{t_1 \ldots t_k}}$ such that $0 < m_k(A) < +\infty$. Then, as in the proof of (1), we can show that $B := A \times \boldsymbol{R}^{n-k} \in \mathscr{B}_{x_{t_1 \ldots t_n}}$. Put $B_m := A \times (-m, m)^{n-k}$ for $m \in N$. Since $m_n(B_m) = (2m)^{n-k} \cdot m_k(A)$ we get $0 < m_n(B_m) < +\infty$, and so $B_m \in \mathscr{B}_{x_{t_1 \ldots t_n}}$ for $m \in N$. Then

$$\Phi_{x_{t_1 \ldots t_n}}(B_m | B_{m+1}) = \frac{m_n(B_m)}{m_n(B_{m+1})} = \left\{\frac{m}{m+1}\right\}^{n-k}.$$

Hence

$$\prod_{m=1}^{\infty} \Phi_{x_{t_1 \ldots t_n}}(B_m | B_{m+1}) = \lim_{j \to +\infty} \prod_{m=1}^{j} \left\{\frac{m}{m+1}\right\}^{n-k} = \lim_{j \to +\infty} \left\{\frac{1}{j+1}\right\}^{n-k} = 0.$$

On the other hand, the sequence $\{B_m\}_{m \in N}$ fulfills the hypothesis of the Lemma, which is a contradiction. ∎

The above proposition implies immediately the following corollary.

COROLLARY 2.8. *There is no Rényi stochastic process such that for each $n \in N$ the Rényi space of its n-dimensional joint distribution is an extension of the uniform Rényi space on $\boldsymbol{R}^n$.*

3. Kolmogorov fundamental theorem for Rényi spaces. The main objective of this section is to prove two versions of the Kolmogorov fundamental theorem for Rényi spaces. We shall need the following theorem, which is a generalization of the theorem on the extension of a probability measure from an algebra to the σ-algebra.

THEOREM 3.1 (on the extension of conditional probability). *Assume that Ω is an arbitrary set, $\mathscr{G}$ is an algebra of subsets of Ω, $\mathscr{B}$ is a non-empty subfamily of $\mathscr{G}$, and P is a non-negative map defined on $\mathscr{G} \times \mathscr{B}$ which fulfills the following conditions*:

(1) $P(\Omega | B) = 1$ *for every* $B \in \mathscr{B}$;

Proof. Let $\Omega := \{\omega\colon \omega\colon T \to X\}$. For all $n \in N$, $t = (t_1, \ldots, t_n) \in T^{(n)}$, $A \in B(X^n)$ we define the set

$$C(t, A) = C(t_1, \ldots, t_n, A) := \{\omega \in \Omega\colon (\omega(t_1), \ldots, \omega(t_n)) \in A\}$$

which will be called the *cylindrical set in Ω with the basin A over the coordinates* $t_1, \ldots, t_n$. The class of all cylindrical sets will be denoted by $\mathscr{G}$.

The following lemmas are easy to prove:

LEMMA 1. *Let $C(p_1, \ldots, p_k, A)$, $C(q_1, \ldots, q_v, B) \in \mathscr{G}$, and let*

$$C(p_1, \ldots, p_k, A) = C(q_1, \ldots, q_v, B), \quad \{p_{j_1}, \ldots, p_{j_r}\} = \{p_1, \ldots, p_k\} \cap \{q_1, \ldots, g_v\}.$$

Then there exist $\phi \in S_k$, $\psi \in S_v$, and $C \in B(X^r)$ such that $p_{\phi(i)} = p_{j_i} = q_{\psi(i)}$ for $i = 1, \ldots, r$ and $\bar{\phi}(A) = C \times X^{k-r}$, $\bar{\psi}(B) = C \times X^{v-r}$. Moreover, the form of the permutations ϕ and ψ does not depend on the particular choice of the sets A and B, but only on the sequences $\{p_1, \ldots, p_k\}$, $\{q_1, \ldots, q_v\}$ and $\{p_{j_1}, \ldots, p_{j_r}\}$.

LEMMA 2. *Let $n_i \in N$, $C(t_1^i, \ldots, t_{n_i}^i, A_i\} \in \mathscr{G}$ for $i = 1, \ldots, k$, let $n \in N$, $(p_1, \ldots, p_n) \in T^{(n)}$, and*

$$\bigcup_{i=1}^{k} \{t_1^i, \ldots, t_{n_i}^i\} \subset \{p_1, \ldots, p_n\}.$$

Then there exist sets $B_1, \ldots, B_k \in B(X^n)$ such that

$$C(t_1^i, \ldots, t_{n_i}^i, A_i) = C(p_1, \ldots, p_n, B_i) \quad \text{for } i = 1, \ldots, k.$$

Moreover, for each $i = 1, \ldots, k$, B_i is of the form $B_i = \bar{\pi}_i(A_i \times X^{n-n_i})$, where $\pi_i \in S_n$ and $p_{\pi_i^{-1}(m)} = t_m^i$ for $m = 1, \ldots, n_i$.

LEMMA 3. *Let $C(s_1, \ldots, s_m, B) \in \mathscr{G}$ and let $\{C_n\}_{n \in N}$ be an arbitrary decreasing sequence of the cylindrical sets. Then there exist an injective (but not necessarily infinite) sequence $\{p_n\}_{n \in N}$ of elements from T, an increasing map $k\colon N \to \{m, m+1, \ldots\}$, and, for each $n \in N$, sets $B_n, D_n \in B(X^{k_n})$ such that $\{p_1, \ldots, p_m\} = \{s_1, \ldots, s_m\}$ and*

$$C_n = C(p_1, \ldots, p_{k_n}, D_n), \quad C(s_1, \ldots, s_m, B) = C(p_1, \ldots, p_{k_n}, B) \quad \text{for } n \in N.$$

Moreover, we have $D_v \subset D_n \times X^{k_v - k_n}$, $B_v = B_n \times X^{k_v - k_n} = B \times X^{k_v - m}$ for $n, v \in N$, $n \leqslant v$.

Applying Lemma 2 it is easy to show that the class $\mathscr{G}$ of all cylindrical sets forms an algebra of subsets of Ω.

Now we define a subfamily $\mathscr{B}$ of $\mathscr{G}$, which will be the space of conditions of the Rényi space to be defined. Put

$$\mathscr{B} := \{C(t, B)\colon n \in N, t \in T^{(n)}, B \in \mathscr{B}_t\}.$$

Let now $C_1 \in \mathscr{G}$ and $C_2 \in \mathscr{B}$. Then there exist $n, m \in N$, $t \in T^{(n)}$, $s \in T^{(m)}$, $A \in B(X^n)$, $B \in \mathscr{B}_s$ such that $C_1 = C(t, A)$ and $C_2 = C(s, B)$. Let $k \in N$, $p \in T^{(k)}$ and

$$\{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_m\} \subset \{p_1, \ldots, p_k\}.$$

Then, according to Lemma 2, there exist $\bar{A}, \bar{B} \in B(X^k)$ such that $C_1 = C(p, \bar{A})$ and $C_2 = C(p, \bar{B})$. Let us define $P(C_1 | C_2)$ by the formula

$$P(C_1 | C_2) := \Phi_p(\bar{A} | \bar{B}).$$

We shall prove that P is well defined. For this purpose it is sufficient to show that

(i) $\bar{B} \in \mathscr{B}_p$;

(ii) the definition does not depend on the choice of a sequence p;

(iii) the definition does not depend on the form of C_1 and C_2.

(i) We know that $B \in \mathscr{B}_s$ and $\bar{B}$ is of the form $\bar{B} = \bar{\pi}(B \times X^{k-m})$, where $\pi \in S_k$ is the permutation described in Lemma 2. Hence, and by assumption (2) of Theorem 3.2, we have

$$B \times X^{k-m} \in \mathscr{B}_{s(1)\ldots s(m) p(\pi^{-1}(m+1))\ldots p(\pi^{-1}(k))} = \mathscr{B}_{\pi^{-1}(p)}.$$

According to assumption (1) of Theorem 3.2 we have $\bar{B} = \bar{\pi}(B \times X^{k-m}) \in \mathscr{B}_p$.

(ii) Let $p \in T^{(k)}$ and $q \in T^{(v)}$, where $k, v \in N$, and let

$$\{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_m\} \subset \{p_1, \ldots, p_k\} \cup \{q_1, \ldots, q_v\}.$$

Moreover, let $A_1, B_1 \in B(X^k)$, $A_2, B_2 \in B(X^v)$, $C(t, A) = C(p, A_1) = C(q, A_2)$ and $C(s, B) = C(p, B_1) = C(q, B_2)$. Then, according to (i), $B_1 \in \mathscr{B}_p$ and $B_2 \in \mathscr{B}_q$. Hence, by Lemma 1, there exist $r \in N$, $r \geqslant m$, $\phi \in S_k$, $\psi \in S_v$ such that $p_{\phi(i)} = q_{\psi(i)}$ for $i = 1, \ldots, r$ and $p_{\phi(i)} = s_i$ for $i = 1, \ldots, m$ and there exist $A^*, B^* \in B(X^r)$ such that $\bar{\phi}(A_1) = A^* \times X^{k-r}$, $\bar{\phi}(B_1) = B^* \times X^{k-r}$, $\bar{\psi}(A_2) = A^* \times X^{v-r}$ and $\bar{\psi}(B_2) = B^* \times X^{v-r}$.

Let us note that the following equalities hold:

$$\begin{aligned} C(s_1, \ldots, s_m, p_{\phi(m+1)}, \ldots, p_{\phi(k)}, B \times X^{k-m}) &= C(s, B) = C(p, B_1) \\ = C(\bar{\phi}(p), \bar{\phi}(B_1)) &= C(s_1, \ldots, s_m, p_{\phi(m+1)}, \ldots, p_{\phi(k)}, B^* \times X^{k-r}). \end{aligned}$$

Hence $B^* \times X^{k-r} = B \times X^{k-m}$, and so $B^* = B \times X^{r-m}$. Since $B \in \mathscr{B}_s$, we have

$$B^* \in \mathscr{B}_{s(1)\ldots s(m) p(\phi(m+1))\ldots p(\phi(r))} = \mathscr{B}_{\bar{\phi}(p)}.$$

Then we obtain

$$\begin{aligned} \Phi_p(A_1 | B_1) &= \Phi_p(\bar{\phi}^{-1}(A^* \times X^{k-r}) | \bar{\phi}^{-1}(B^* \times X^{k-r})) \\ &= \Phi_{\bar{\phi}(p)}(A^* \times X^{k-r} | B^* \times X^{k-r}) = \Phi_{p(\phi(1))\ldots p(\phi(r))}(A^* | B^*) \end{aligned}$$

$$= \Phi_{q(\psi(1))\dots q(\psi(r))}(A^*|B^*) = \Phi_{\bar{\psi}(q)}(A^*\times X^{v-r}|B^*\times X^{v-r})$$
$$= \Phi_q(\bar{\psi}^{-1}(A^*\times X^{v-r})|\bar{\psi}^{-1}(B^*\times X^{v-r})) = \Phi_q(A_2|B_2),$$

as required.

(iii) Let now $n, k, m, v\in N$, $t\in T^{(n)}$, $w\in T^{(k)}$, $s\in T^{(m)}$, $u\in T^{(v)}$, $A\in B(X^n)$, $D\in B(X^k)$, $B\in\mathscr{B}_s$ and $E\in\mathscr{B}_u$. Let us assume that $C(t, A) = C(w, D)$ and $C(s, B) = C(u, E)$.

According to Lemma 2 there exist $r\in N$, $p\in T^{(r)}$ and $A_1, B_1, D_1, E_1\in B(X^r)$ such that $C(t, A) = C(p, A_1)$, $C(s, B) = C(p, B_1)$, $C(w, D) = C(p, D_1)$ and $C(u, E) = C(p, E_1)$. Hence $A_1 = D_1$ and $B_1 = E_1$. Then $\Phi_p(A_1|B_1) = \Phi_p(D_1|E_1)$. This, according to (ii), completes the proof of the consistency of the definition.

We now prove that the map P defined above fulfills assumptions (1)–(5) of Theorem 3.1.

(1) Let $m\in N$, $s\in T^{(m)}$, $B\in\mathscr{B}_s$. Since $\Omega = C(s, X^m)$, we obtain

$$P(\Omega|C(s, B)) = P(C(s, X^m)|C(s, B)) = \Phi_s(X^m|B) = 1.$$

(2) Let $n, v, m\in N$, $t\in T^{(n)}$, $r\in T^{(v)}$, $s\in T^{(m)}$, $A\in B(X^n)$, $D\in B(X^v)$, $B\in\mathscr{B}_s$ and $C(t, A)\cap C(r, D) = \emptyset$. According to Lemma 2 there exist $k\in N$, $p\in T^{(k)}$ such that

$$\{t_1, \dots, t_n\}\cup\{r_1, \dots, r_v\}\cup\{s_1, \dots, s_m\}\subset\{p_1, \dots, p_k\}$$

and $\bar{A}, \bar{D}\in B(X^k)$, $\bar{B}\in\mathscr{B}_p$ such that

$$C(t, A) = C(p, \bar{A}),\quad C(r, D) = C(p, \bar{D}),\quad C(s, B) = C(p, \bar{B}).$$

As $C(p, \bar{A})\cap C(p, \bar{D}) = \emptyset$, we have $\bar{A}\cap\bar{D} = \emptyset$. Thus

$$\begin{aligned}P(C(t, A)\cup C(r, D)|C(s, B)) &= P(C(p, \bar{A}\cup\bar{D})|C(p, \bar{B}))\\ &= \Phi_p(\bar{A}\cup\bar{D}|\bar{B}) = \Phi_p(\bar{A}|\bar{B})+\Phi_p(\bar{D}|\bar{B})\\ &= P(C(t, A)|C(s, B))+P(C(r, D)|C(s, B)),\end{aligned}$$

as required.

(4) Let $m\in N$, $s\in T^{(m)}$ and $B\in\mathscr{B}_s$. Then

$$P(C(s, B)|C(s, B)) = \Phi_s(B|B) = 1.$$

(5) Let $t\in T^{(n)}$, $s\in T^{(m)}$, $r\in T^{(v)}$, $A\in B(X^n)$, $B\in\mathscr{B}_s$, $D\in\mathscr{B}_r$ and $C(t, A)\subset C(s, B)\subset C(r, D)$. By Lemma 2 there exist $k\in N$, $p\in T^{(k)}$ such that

$$\{t_1, \dots, t_n\}\cup\{s_1, \dots, s_m\}\cup\{r_1, \dots, r_v\}\subset\{p_1, \dots, p_k\}$$

and there exist $\bar{A}\in B(X^k)$ and $\bar{B}, \bar{D}\in\mathscr{B}_p$ such that

$$C(t, A) = C(p, \bar{A}),\quad C(s, B) = C(p, \bar{B}),\quad C(r, D) = C(p, \bar{D}).$$

As $C(p, \bar{A}) \subset C(p, \bar{B}) \subset C(p, \bar{D})$, we have $\bar{A} \subset \bar{B} \subset \bar{D}$. Thus

$$P\big(C(t, A) \mid C(r, D)\big) = \Phi_p(\bar{A}|\bar{D}) = \Phi_p(\bar{A}|\bar{B}) \cdot \Phi_p(\bar{B}|\bar{D})$$
$$= P\big(C(t, A) \mid C(s, B)\big) \cdot P\big(C(s, B) \mid C(r, D)\big),$$

as required.

Thus, it remains to show that P fulfills assumption (3) of Theorem 3.1.

Let $m \in N$, $s \in T^{(m)}$, $B \in \mathscr{B}_s$ and let $\{C_n\}_{n \in N}$ be an arbitrary decreasing sequence of cylindrical sets. It is enough to show that

$$\lim_{n \to +\infty} P\big(C_n \mid C(s, B)\big) = 0.$$

Let us note that the sequence $P\big(C_n \mid C(s, B)\big)$ is decreasing and bounded from below. Hence it has the limit. We shall argue by contradiction. Let us assume that this limit is greater than 0. Then there exists $\varepsilon > 0$ such that $P\big(C_n | C(s, B)\big) > \varepsilon$ for every $n \in N$.

Let us take an injective sequence $\{p_n\}_{n \in N}$ of elements from T, an increasing map k: $N \to \{m, m+1, \ldots\}$ and, for each $n \in N$, sets $B_n, D_n \in B(X^{k_n})$ fulfilling the conclusion of Lemma 3. Then

$$P\big(C_n | C(s, B)\big) = \Phi_{p(1)\ldots p(k_n)}(D_n|B_n) \quad \text{for every } n \in N.$$

Since $\Phi_{p(1)\ldots p(k_n)}(\cdot|B_n)$ is a probability measure on $\big(X^{k_n}, B(X^{k_n})\big)$, for every $n \in N$ there exists a non-empty compact set $K_n \subset D_n$ such that

$$\Phi_{p(1)\ldots p(k_n)}(D_n \setminus K_n | B_n) < \varepsilon \cdot 2^{-(n+1)}$$

(see [3, Theorems 19.16 and 19.18]).

Let us set

$$K_n^* := (K_1 \times X^{k_n - k_1}) \cap \ldots \cap (K_{n-1} \times X^{k_n - k_{n-1}}) \cap K_n \quad \text{for } n \in N.$$

It is easy to see that the sets K_n^* are compact and

$$K_{n+1}^* = (K_n^* \times X^{k_{n+1} - k_n}) \cap K_{n+1}.$$

Let us put $\Lambda_n := C(p_1, \ldots, p_{k_n}, K_n^*)$ for $n \in N$. Then $\{\Lambda_n\}_{n \in N}$ is a decreasing sequence of cylindrical sets. Moreover,

$$\begin{aligned}
P\big(C_n \setminus \Lambda_n \mid C(s, B)\big) &= P\big(C(p_1, \ldots, p_{k_n}, D_n \setminus K_n^*) \mid C(p_1, \ldots, p_{k_n}, B_n)\big) \\
&= \Phi_{p(1)\ldots p(k_n)}(D_n \setminus K_n^* | B_n) \\
&\leqslant \Phi_{p(1)\ldots p(k_n)}\big(D_n \setminus (K_1 \times X^{k_n - k_1}) | B_n\big) + \ldots \\
&\quad + \Phi_{p(1)\ldots p(k_n)}(D_n \setminus K_n | B_n) \\
&\leqslant \Phi_{p(1)\ldots p(k_n)}\big((D_1 \setminus K_1) \times X^{k_n - k_1} \mid B_1 \times X^{k_n - k_1}\big) + \ldots \\
&\quad + \Phi_{p(1)\ldots p(k_n)}(D_n \setminus K_n | B_n)
\end{aligned}$$

$$= \Phi_{p(1)\dots p(k_1)}(D_1 \backslash K_1 | B_1) + \dots$$
$$+ \Phi_{p(1)\dots p(k_n)}(D_n \backslash K_n | B_n) < \varepsilon/4 + \dots + \varepsilon \cdot 2^{-(n+1)} < \varepsilon/2.$$

Hence $P(A_n | C(s, B)) > \varepsilon/2$, and so $K_n^* \neq \emptyset$ for $n \in N$.

Let us define

$$\Gamma_n := K_n^* \times \mathop{\times}_{i=n+1}^{\infty} (\mathrm{proj}_{k_{i-1}+1} K_1 \times \dots \times \mathrm{proj}_{k_i} K_i),$$

where $\mathrm{proj}_j(x) := x_j$ for $x \in X^{k_i}$, $j = k_{i-1}+1, \dots, k_i$, $i \in N$.

From what has already been proved it follows that $\{\Gamma_n\}_{n\in N}$ is a decreasing sequence of non-empty subsets of X^N. By the Tikhonov product theorem, all Γ_n are compact. Hence $\{\Gamma_n\}_{n \in N}$ form a centred system of closed subsets of the compact set Γ_1. This implies that

$$\emptyset \neq \bigcap \{\Gamma_m \colon m \in N\} \subset \Gamma_n \subset C(p_1, \dots, p_{k_n}, K_n)$$
$$\subset C(p_1, \dots, p_{k_n}, D_n) = C_n \quad \text{for each } n \in N.$$

Thus $\bigcap \{C_n \colon n \in N\} \neq \emptyset$, which is a contradiction.

According to Theorem 3.1 there exists a map $\bar{P}\colon \sigma(\mathscr{G}) \times \mathscr{B} \to [0, 1]$ such that $[\Omega, \sigma(\mathscr{G}), \mathscr{B}, \bar{P}]$ is a Rényi space and for every $C_1 \in \mathscr{G}$ and $C_2 \in \mathscr{B}$ the equality $\bar{P}(C_1|C_2) = P(C_1|C_2)$ holds.

Let us define a map $x\colon T \times \Omega \to X$ by the formula

$$x(t, \omega) = \omega(t) \quad \text{for } t \in T,\ \omega \in \Omega.$$

Let $n \in N$, $\boldsymbol{t} = (t_1, \dots, t_n) \in T^{(n)}$. It is easy to show that $x_{t_1\dots t_n}\colon \Omega \to X^n$ is a measurable map. Moreover, if $B \in \mathscr{B}_{\boldsymbol{t}}$, then $x_{t_1\dots t_n}^{-1}(B) = C(\boldsymbol{t}, B) \in \mathscr{B}$. Therefore $\mathscr{B}_{\boldsymbol{t}} \subset \mathscr{B}_{x_{t_1\dots t_n}}$. Hence $\mathscr{B}_{x_{t_1\dots t_n}} \neq \emptyset$, and so $x\colon T \times \Omega \to X$ is a Rényi stochastic process.

It remains to prove that the Rényi space generated by the random variable $x_{t_1\dots t_n}$ is an extension of the Rényi space $[X^n, B(X^n), \mathscr{B}_{\boldsymbol{t}}, \Phi_{\boldsymbol{t}}]$. Since we have proved above that $\mathscr{B}_{\boldsymbol{t}} \subset \mathscr{B}_{x_{t_1\dots t_n}}$, it suffices to show that for $A \in B(X^n)$ and $B \in \mathscr{B}_{\boldsymbol{t}}$ the equality $\Phi_{x_{t_1\dots t_n}}(A|B) = \Phi_{\boldsymbol{t}}(A|B)$ holds. For this purpose let us notice that

$$\Phi_{x_{t_1\dots t_n}}(A|B) = P(x_{t_1\dots t_n}^{-1}(A) \,|\, x_{t_1\dots t_n}^{-1}(B)) = P(C(\boldsymbol{t}, A) | C(\boldsymbol{t}, B)) = \Phi_{\boldsymbol{t}}(A|B).$$

The question naturally arises of when there exists a Rényi stochastic process whose n-dimensional joint distributions are equal to given conditional probabilities on the space $(X^n, B(X^n))$. The following theorem answers this question:

THEOREM 3.3 (Kolmogorov fundamental theorem for Rényi spaces — variant II). *If T is an arbitrary non-empty set, X is a Polish space and $\{[X^n, B(X^n), \mathscr{B}_{\boldsymbol{t}}, \Phi_{\boldsymbol{t}}]\colon n \in N, \boldsymbol{t} \in T^{(n)}\}$ is afamily of Rényi spaces, then the following two conditions are equivalent*:

(I) *The family* $\{[X^n, B(X^n), \mathscr{B}_t, \Phi_t]: n \in N, t \in T^{(n)}\}$ *fulfills compatibility conditions* (1), (3), *and* (4) *from the hypothesis of Theorem* 3.2 *and the condition*

(2′) $B \in \mathscr{B}_{t_1 \ldots t_n}$ *if and only if* $B \times X^{m-n} \in \mathscr{B}_{t_1 \ldots t_m}$ *for all* $n, m \in N$, $n \leqslant m$, $(t_1, \ldots, t_m) \in T^{(m)}$.

(II) *There exist a Rényi space* $[\Omega, \Sigma, \mathscr{B}, P]$ *and a Rényi stochastic process* $x\colon T \times \Omega \to X$ *such that for all* $n \in N$ *and* $t = (t_1, \ldots, t_n) \in T^{(n)}$ *the Rényi space generated by the random variable* $x_{t_1 \ldots t_n}$ *is equal to the Rényi space* $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$.

Proof. (I) $\Rightarrow$ (II). We shall construct the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$ and the Rényi stochastic process $x\colon T \times \Omega \to X$ as in the proof of Theorem 3.2. It is now sufficient to prove that $\mathscr{B}_{x_{t_1 \ldots t_n}} \subset \mathscr{B}_t$ for all $n \in N$, $t = (t_1, \ldots, t_n) \in T^{(n)}$.

Let $B \in \mathscr{B}_{x_{t_1 \ldots t_n}}$. Hence $C(t, B) = x^{-1}_{t_1 \ldots t_n}(B) \in \mathscr{B}$. This means that there exist $m \in N$, $s \in T^{(m)}$, and $A \in \mathscr{B}_s$ such that $C(t, B) = C(s, A)$. By Lemma 1 there exist $r \in N$, $C \in B(X^r)$, and permutations $\phi = S_n$ and $\psi \in S_m$ such that $t_{\phi(i)} = s_{\psi(i)}$ for $i = 1, \ldots, r$ and $\bar{\phi}(B) = C \times X^{n-r}$ and $\bar{\psi}(A) = C \times X^{m-r}$. As $C \times X^{m-r} = \bar{\psi}(A) \in \mathscr{B}_{\bar{\psi}(s)}$, applying condition (2′) we get $C \in \mathscr{B}_{s(\psi(1)) \ldots s(\psi(r))}$. Hence

$$\bar{\phi}(B) = C \times X^{n-r} \in \mathscr{B}_{t(\phi(1)) \ldots t(\phi(n))}.$$

Thus $B \in \mathscr{B}_t$, as required.

(II) $\Rightarrow$ (I). It is obvious that the family of the Rényi spaces generated by the Rényi stochastic process $x\colon T \times \Omega \to X$ fulfills the compatibility conditions (1), (2′), (3) and (4). ∎

4. Kolmogorov fundamental theorem for Rényi spaces generated by measure. At the beginning of this section we quote a theorem, which gives us the necessary and sufficient conditions for a Rényi space to be generated by a measure.

THEOREM 4.1 (see [1, théorème (8.9)]). *If* $[\Omega, \Sigma, \mathscr{B}, P]$ *is a Rényi space, then the following two conditions are equivalent*:

(I) $[\Omega, \Sigma, \mathscr{B}, P]$ *is a Rényi space generated by some measure on* (Ω, Σ).

(II) *There exists a Rényi space* $[\Omega, \Sigma, \mathscr{B}^*, P^*]$, *which is an extension of the Rényi space* $[\Omega, \Sigma, \mathscr{B}, P]$ *and such that for all* $B_1, B_2 \in \mathscr{B}^*$ *there exists* $B \in \mathscr{B}^*$ *with the properties* $B_1 \cup B_2 \subset B$, $P^*(B_1|B) > 0$, *and* $P^*(B_2|B) > 0$.

The next two theorems answer the question, as to whether the measure which generates a Rényi space is uniquely determined up to the multiplicative factor.

PROPOSITION 4.2. *Assume that* $[\Omega, \Sigma, \mathscr{B}, P]$ *is a Rényi space generated by both a measure* m *and a measure* M *on* (Ω, Σ), *and* $\mathscr{B}$ *fulfills the following condition*:

(a) *for all* $B_1, B_2 \in \mathscr{B}$ *there exists* $B \in \mathscr{B}$ *such that* $B_1 \cup B_2 \subset B$.
Then there exists $\lambda > 0$ *such that* $M(B) = \lambda \cdot m(B)$ *for every* $B \in \mathscr{B}$.

Proof. Let $C \in \mathscr{B}$. Set $\lambda := M(C)/m(C)$. It is obvious that $\lambda > 0$. We shall show that for such λ the conclusion of the theorem is true.

Take an arbitrary $B \in \mathscr{B}$. Then there exists $D \in \mathscr{B}$ such that $B \cup C \subset D$. We have

$$M(B)/M(D) = M(B \cap D)/M(D) = P(B|D)$$
$$= m(B \cap D)/m(D) = m(B)/m(D).$$

Analogously we obtain $M(C)/M(D) = m(C)/m(D)$. Hence

$$M(B)/m(B) = M(D)/m(D) = M(C)/m(C) = \lambda. \blacksquare$$

Remark 4.3. It is clear that if $\mathscr{B}$ is an additive class, then it fulfills condition (a) from the above proposition.

THEOREM 4.4. *If $[\Omega, \Sigma, \mathscr{B}, P]$ is a Rényi space strictly generated by a σ-finite measure m on (Ω, Σ) and M is an arbitrary measure on (Ω, Σ) which generates the space $[\Omega, \Sigma, \mathscr{B}, P]$, then there exists $\lambda > 0$ such that $M(A) = \lambda \cdot m(A)$ for each $A \in \Sigma$.*

Proof. First note that

$$\mathscr{B} = \{B \in \Sigma\colon\ 0 < m(B) < +\infty\} \subset \{B \in \Sigma\colon\ 0 < M(B) < +\infty\}.$$

Hence $\mathscr{B}$ is an additive class and, according to Remark 4.3, the hypotheses of Proposition 4.2 are satisfied. Thus, there exists $\lambda > 0$ such that $M(B) = \lambda \cdot m(B)$ for $B \in \mathscr{B}$. Now it is enough to show that the above equality holds for $A \in \Sigma \backslash \mathscr{B}$. We shall consider the two cases $m(A) = 0$ and $m(A) = +\infty$.

Let $m(A) = 0$. Take some $B \in \mathscr{B}$. Then $0 < m(B \backslash A) = m(B) = m(A \cup B) < +\infty$. Hence it follows that $B \backslash A$, $A \cup B \in \mathscr{B}$. Let us note that $M(A \cup B) = \lambda \cdot m(A \cup B) = \lambda \cdot m(B \backslash A) = M(B \backslash A)$ and $0 < M(A \cup B) = M(B \backslash A) < +\infty$. Consequently, $M(A) = M(A \cup B) - M(B \backslash A) = 0$. Thus $M(A) = \lambda \cdot m(A)$ for every $A \in \Sigma$ such that $m(A) < +\infty$.

Now let $m(A) = +\infty$. As m is σ-finite, there exists a disjoint family $\{B_n\}_{n \in N}$ such that $B_n \in \Sigma$, $m(B_n) < +\infty$, and $\Omega = \bigcup \{B_n\colon\ n \in N\}$. Then

$$M(A) = \sum_{n=1}^{\infty} M(A \cap B_n) = \lambda \cdot \sum_{n=1}^{\infty} m(A \cap B_n) = \lambda \cdot m(A) = +\infty,$$

which concludes the proof. $\blacksquare$

THEOREM 4.5 (Kolmogorov fundamental theorem for Rényi spaces generated by a measure). *Assume that T is an arbitrary non-empty set, X is a Polish space and $\{[X^n, B(X^n), \mathscr{B}_t, \Phi_t]\colon\ n \in N,\ t \in T^{(n)}\}$ is a family of Rényi spaces which satisfies the following conditions:*

(1) *for all $n \in N$, $t \in T^{(n)}$ there exists a σ-finite measure m_t on $(X^n, B(X^n))$ such that the Rényi space $\{[X^n, B(X^n), \mathscr{B}_t, \Phi_t]\colon\ n \in N,\ t \in T^{(n)}\}$ is strictly generated by this measure;*

(2) $m_t(A) = m_{\bar{\pi}(t)}(\bar{\pi}(A))$ *for all* $n \in N$, $t \in T^{(n)}$, $\pi \in S_n$, $A \in B(X^n)$;

(3) $m_{t(1)\dots t(n)}(A) = m_{t(1)\dots t(m)}(A \times X^{m-n})$ *for all* $m, n \in N$, $n \leqslant m$, $(t_1, \dots, t_m) \in T^{(m)}$, $A \in B(X^n)$.

Then there exist a Rényi space $[\Omega, \Sigma, \mathscr{B}^*, P^*]$ *strictly generated by some* σ*-finite measure* M *on* (Ω, Σ) *and a Rényi stochastic process* x: $T \times \Omega \to X$ *such that for all* $n \in N$, $t \in T^{(n)}$ *the Rényi space generated by the random variable* $x_{t_1\dots t_n}$ *is equal to the Rényi space* $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$. *Moreover, there exists* $\lambda > 0$ *such that* $M(x_{t_1\dots t_n}^{-1}(A)) = \lambda \cdot m_t(A)$ *for* $n \in N$, $t = (t_1, \dots, t_n) \in T^{(n)}$, *and* $A \in B(X^n)$.

Proof. It is easy to show that the family $\{[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$: $n \in N$, $t \in T^{(n)}\}$ fulfills conditions (1), (2′), (3) and (4) in Theorems 3.2 and 3.3. Let us define the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$ and the Rényi stochastic process x: $T \times \Omega \to X$ in such a way as in the proof of Theorem 3.2. Then for all $n \in N$, $t \in (t_1, \dots, t_n) \in T^{(n)}$ we have

$$\mathscr{B}_{x_{t_1\dots t_n}} = \mathscr{B}_t = \{B \in B(X^n)\colon\ 0 < m_t(B) < +\infty\}$$

and

$$\Phi_{x_{t_1\dots t_n}}(A|B) = \Phi_t(A|B) = m_t(A \cap B)/m_t(B)$$

for $A \in B(X^n)$ and $B \in \mathscr{B}_t$. Moreover, $x_{t_1\dots t_n}^{-1}(B) = C(t, B) \in \mathscr{B}$ if and only if $B \in \mathscr{B}_t$.

We shall show now that the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$ is generated by some measure M on (Ω, Σ). For this purpose it is enough to check whether condition (II) from Theorem 4.1 is fulfilled.

Let $B_1, B_2 \in \mathscr{B}$. Then there exist $n, m \in N$, $t \in T^{(n)}$, $s \in T^{(m)}$, $C_1 \in \mathscr{B}_t$, and $C_2 \in \mathscr{B}_s$ such that $B_1 = C(t, C_1)$ and $B_2 = C(s, C_2)$. According to Lemma 2 there exist $k \in N$, $p \in T^{(k)}$, $D_1, D_2 \in B(X^k)$ such that $B_1 = C(p, D_1)$ and $B_2 = C(p, D_2)$. Then $D_1, D_2 \in \mathscr{B}_p$, which implies $0 < m_p(D_i) < +\infty$ for $i = 1, 2$. Hence $0 < m_p(D_1 \cup D_2) < +\infty$, and so $D_1 \cup D_2 \in \mathscr{B}_p$. Putting $B := B_1 \cup B_2$ we get

$$B = C(p, D_1 \cup D_2) \in \mathscr{B}.$$

Moreover,

$$\begin{aligned} P(B_i|B) &= P(C(p, D_i)\,|\,C(p, D_1 \cup D_2)) = \Phi_p(D_i|D_1 \cup D_2) \\ &= m_p(D_i)/m_p(D_1 \cup D_2) > 0 \quad \text{for } i = 1, 2, \end{aligned}$$

as required.

Hence, by Theorem 4.1, there exists a measure M on (Ω, Σ), which generates the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$.

Next we prove the last statement of the theorem.

Let $n \in N$, $t \in T^{(n)}$. Let us note that the Rényi space $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$, which is strictly generated by the σ-finite measure m_t, is also generated by the measure M_t defined by the formula $M_t(A) := M(C(t, A))$ for $A \in B(X^n)$.

It is so, because

$$\mathscr{B}_t = \{B \in B(X^n)\colon\ C(t, B) \in \mathscr{B}\} \subset \{B \in B(X^n)\colon\ 0 < M(C(t, B)) < +\infty\}$$

and we have

$$\Phi_t(A|B) = P(C(t, A) \mid C(t, B)) = M(C(t, A \cap B))/M(C(t, B))$$

for all $A \in B(X^n)$, $B \in \mathscr{B}_t$.

Then, according to Theorem 4.4, for all $n \in N$, $t \in T^{(n)}$ there exists $\lambda_t > 0$ such that $M(C(t, A)) = M_t(A) = \lambda_t \cdot m_t(A)$ for each $A \in B(X^n)$. It is now sufficient to prove that $\lambda_t = \lambda_s$ for all $n, m \in N$, $t \in T^{(n)}$, and $s \in T^{(m)}$.

Let $B_1 \in \mathscr{B}_t$, $B_2 \in \mathscr{B}_s$. Then, by Lemma 2, there exist $k \in N$, $p \in T^{(k)}$, $C_1, C_2 \in B(X^k)$ such that $C(t, B_1) = C(p, C_1)$ and $C(s, B_2) = C(p, C_2)$. According to Lemma 1 and by hypotheses (2), (3) we obtain $m_p(C_1) = m_t(B_1)$. Then

$$\lambda_t \cdot m_t(B_1) = M(C(t, B_1)) = M(C(p, C_1)) = \lambda_p \cdot m_p(C_1) = \lambda_p \cdot m_t(B_1).$$

As $0 < m_t(B_1) < +\infty$, we have $\lambda_t = \lambda_p$. Analogously we show that $\lambda_s = \lambda_p$. Hence $\lambda_t = \lambda_s$, which completes the proof of the last part of the theorem.

Next we show that the measure M is σ-finite. Let $t \in T$. Then m_t is a σ-finite measure on $(X, B(X))$. Hence there exists a family of sets $\{A_k\}_{k \in N}$ such that $A_k \in B(X)$, $m_t(A_k) < +\infty$ for $k \in N$, and $X = \bigcup \{A_k\colon\ k \in N\}$. Set $B_k := C(t, A_k)$ for $k \in N$. Then

$$\bigcup \{B_k\colon\ k \in N\} = C(t, X) = \Omega$$

$$\text{and} \qquad M(B_k) = M(C(t, A_k)) = \lambda \cdot m_t(A_k) < +\infty,$$

as required.

Let now $[\Omega, \Sigma, \mathscr{B}^*, P^*]$ be the Rényi space strictly generated by the measure M. It remains to prove that the Rényi space generated by the random variable $x_{t_1 \ldots t_n}$, with respect to the Rényi space $[\Omega, \Sigma, \mathscr{B}^*, P^*]$, is equal to the Rényi space $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$ for all $n \in N$, $t \in T^{(n)}$.

By Theorem 3.3 the Rényi space $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$ is equal to the Rényi space generated by the random variable $x_{t_1 \ldots t_n}$ with respect to the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$. Since the Rényi space $[\Omega, \Sigma, \mathscr{B}^*, P^*]$ is an extension of the Rényi space $[\Omega, \Sigma, \mathscr{B}, P]$, it is sufficient to prove that

$$\{B \in B(X^n)\colon\ x^{-1}_{t_1 \ldots t_n}(B) \in \mathscr{B}^*\} \subset \mathscr{B}_t.$$

Let $C(t, B) = x^{-1}_{t_1 \ldots t_n}(B) \in \mathscr{B}^*$. Then we obtain $0 < M(C(t, B)) < +\infty$. Hence $0 < m(C(t, B)) < +\infty$. Thus $B \in \mathscr{B}_t$, which completes the proof. ∎

Remark 4.6. It follows from the proof above that the words "σ-finite" and "strictly" can be cancelled in condition (1) and in the assertion of Theorem 4.5.

Remark 4.7. Condition (1) in Theorem 4.5 can be replaced by the following condition:

(1′) *For all* $n \in N$, $t \in T^{(n)}$ *there exists a σ-finite measure* m_t *on* $(X^n, B(X^n))$ *such that the Rényi space* $[X^n, B(X^n), \mathscr{B}_t, \Phi_t]$ *is generated by this measure.*

Then the assertion of Theorem 4.5 *remains true, only the words "is equal to" should be replaced by "is an extension of".*

Proof. Let $[X^n, B(X^n), \bar{\mathscr{B}}_t, \bar{\Phi}_t]$ be the Rényi space strictly generated by m_t for $n \in N$, $t \in T^{(n)}$. Then the family of the Rényi spaces $\{[X^n, B(X^n), \bar{\mathscr{B}}_t, \bar{\Phi}_t]: n \in N, t \in T^{(n)}\}$ fulfills the hypotheses of Theorem 4.5. For all $n \in N$, $t \in T^{(n)}$ we have $\mathscr{B}_t \subset \bar{\mathscr{B}}_t = \mathscr{B}_{x_{t_1 \ldots t_n}}$. Now the result follows from Theorem 4.5. ■

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