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INTRODUCTION TO DISCRIMINANT ANALYSIS IN THE MONOTONE EXPERTS MODEL

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Abstract. A special case of object identification is considered in which it is desirable to classify objects according to values of an unobservable latent variable U. Independent expert opinions concerning U, say Z and Z^*, are supposed to be available. In the paper we give a theoretical basis for solving some identification problems under the assumption that the distribution of (Z, Z^*) is known while the distribution of (U, Z) is not known.

1. Introduction. In practice we are often confronted with two-class discriminant problems concerning some latent trait U, such that the first class consists of objects with values of U smaller than its π-th quantile, say u_π. Suppose that two experts independently give their opinions on U, say Z and Z^*, while our information is reduced to the knowledge of the true distribution of (Z, Z^*) and of the family of distributions of $(U\ Z, Z^*)$ which contains the true one. This refers to situations in which the learning sample consists of objects classified by two experts and their opinions are known but the values of U remain unknown. Thus, one may estimate the distribution of (Z, Z^*) but not that of (U, Z). Such models arise in wide variety of applications, for example in educational testing and psychometrics (see [3]).

Our idea is to replace an identification rule based on Z and concerning U by a respective identification rule based also on Z but concerning Z^*. In other words, we try to replace an identification problem for (U, Z) by an analogous one for (Z^*, Z). The paper gives some theoretical basis for such an approach. In Section 2 we introduce a general model of (U, Z, Z^*) and in this connection consider some monotone orderings of bivariate distributions. We show that the strength of dependence between an expert and a latent trait influences the strength of dependence between experts. In Section 3 a tool for comparing bivariate distributions (called the *divergence curve*) is introduced, and its role in a large family of identification problems is explained. Finally, this tool is used in Section 4 to compare the distributions of (U, Z) and (Z^*, Z).

2. Monotone experts models. A family of distributions of (U, Z, Z^*) will be called the *experts model* if Z and Z^* are conditionally independent given U (symbolically, $Z \perp Z^* \mid U$), i.e., if the distribution function of (Z, Z^*) is defined (in obvious notation) by

$$F_{Z,Z^*}(z, z^*) = \int F_{Z|U=u}(z) F_{Z^*|U=u}(z^*) dF_U(u)$$

for any F_U and any families of conditional distribution functions of $(Z \mid U = u)$ and of $(Z^* \mid U = u)$.

A special attention will be given to experts with the same "mechanism" of delivering opinions so that the distributions of (U, Z) and (U, Z^*) are identical. In this case a triplet (U, Z, Z^*) belongs to the experts model iff there exist i.i.d. random variables ε and ε^* independent of U and a Borel measurable function g such that $Z = g(U, \varepsilon)$ and $Z^* = g(U, \varepsilon^*)$ (see [7]), while for different distributions of (U, Z) and (U, Z^*) the conditional independence $Z \perp Z^* \mid U$ is obviously implied by the relations

$$Z = g(U, \varepsilon) \quad \text{and} \quad Z^* = g^*(U, \varepsilon^*)$$

for some independent random variables ε, ε^*, U and some Borel measurable functions g and g^*.

It is natural to assume some sort of positive dependence between Z and U and between Z^* and U. We shall consider the family of positively quadrant dependent (QD^+) or positively regression dependent (RD^+) pairs of random variables (see [6]):

$(X, Y) \in \mathrm{QD}^+$ iff $P(X \leqslant x, Y \leqslant y) \geqslant P(X \leqslant x) P(Y \leqslant y)$ for all x, y;

$(X, Y) \in \mathrm{RD}^+$ iff $P(X \leqslant x \mid Y = y)$ is a nonincreasing function in y for all x.

It is known [6] that $\mathrm{RD}^+ \subset \mathrm{QD}^+$.

We will use the term *monotone experts model* for the experts model (U, Z, Z^*) in which (U, Z), (Z, U), (U, Z^*) and (Z^*, U) belong to RD^+. From the considerations of Alam and Wallenius [1] it follows immediately that in the monotone experts models (Z, Z^*) and (Z^*, Z) belong to RD^+. Some other types of dependence between U and Z and between U and Z^*, as well as their influence on dependence between Z and Z^*, were considered in [3].

We will show that in some experts models with positive dependence the strength of dependence between the expert and the latent variable influences the strength of dependence between the experts.

Let us remind that in QD^+ the strength of dependence may be naturally compared by an ordering defined as follows (see [5]):

$(X, Y) \underset{\mathrm{QD}}{\leqslant} (X', Y')$ (i.e. the pair (X', Y') is *stronger dependent than the pair* (X, Y)) if there exist increasing functions ϕ_1, ϕ_2 such that $\phi_1(X')$ is distributed as X and $\phi_2(Y')$ is distributed as Y and

$$P(X \leqslant x, Y \leqslant y) \leqslant P\big(\varphi_1(X') \leqslant x, \varphi_2(Y') \leqslant y\big) \quad \text{for all } (x, y) \in \boldsymbol{R}^2.$$

THEOREM 1. *Let* (U, Z, Z^*) *and* (U, T, T^*) *be such that*

$$Z = g(U, \varepsilon), \quad Z^* = g^*(U, \varepsilon^*), \quad T = g_1(U, \varepsilon_1), \quad T^* = g_1^*(U, \varepsilon_1^*),$$

where g, g^*, g_1, g_1^* *are increasing functions of* u, *and* $\varepsilon, \varepsilon^*, \varepsilon_1, \varepsilon_1^*, U$ *are independent variables. Then*

$$\big((U, Z) \underset{\mathrm{QD}}{\leqslant} (U, T) \wedge (U, Z^*) \underset{\mathrm{QD}}{\leqslant} (U, T^*)\big) \Rightarrow (Z, Z^*) \underset{\mathrm{QD}}{\leqslant} (T, T^*).$$

Proof. The representation of Z, Z^* implies that

$$(U, Z), (U, Z^*), (Z, Z^*) \in \mathrm{QD}^+$$

(cf. [6]), and the same holds for the triplet (U, T, T^*). Tchen [8] proved that

$$(U, Z) \underset{\mathrm{QD}}{\leqslant} (U, T) \Rightarrow \big(r(U, \eta), Z\big) \underset{\mathrm{QD}}{\leqslant} \big(r(U, \eta), T\big)$$

for any function r monotone in u and for any η independent of (U, Z, T). Thus, for $r = g$ and $\eta = \varepsilon$,

$$(U, Z^*) \underset{\mathrm{QD}}{\leqslant} (U, T^*) \Rightarrow (Z, Z^*) \underset{\mathrm{QD}}{\leqslant} (Z, T^*)$$

while for $r = g_1^*$ and $\eta = \varepsilon_1^*$

$$(U, Z) \underset{\mathrm{QD}}{\leqslant} (U, T) \Rightarrow (T^*, Z) \underset{\mathrm{QD}}{\leqslant} (T^*, T).$$

Consequently,

$$(Z, Z^*) \underset{\mathrm{QD}}{\leqslant} (Z, T^*) \underset{\mathrm{QD}}{\leqslant} (T, T^*). \blacksquare$$

3. The divergence curve based on (U, Z). A population of objects is described by the distribution of a pair (U, Z), where U is an unobservable random variable taking on values in a certain set $\boldsymbol{U} \subset \boldsymbol{R}$, and Z is an observable random variable taking values in a set $\boldsymbol{Z} \subset \boldsymbol{R}$.

Given a $\pi \in (0, 1)$, we are interested in classifying objects to one of the two classes which consists, respectively, of objects with values of U not larger than the π-th quantile of U (first class) and larger than the π-th quantile of U (second class). For any π-th quantile of U, say u_π, we find a number a such that $P(U < u_\pi) + aP(U = u_\pi) = \pi$. Let $U_{(\pi)} = 1$ if $U < u_\pi$ or if $U = u_\pi$ and $I < a$, and let $U_{(\pi)} = 2$ if $U > u_\pi$ or if $U = u_\pi$ and $I > a$, where I is a random variable uniformly distributed on $(0, 1)$ and independent of U. Let f denote the density of (U, Z) with respect to a certain measure which is the product measure of a measure λ on $\boldsymbol{U}$ and a measure μ on $\boldsymbol{Z}$. The density of $Z | U_{(\pi)} = 1$ is given by

$$f_1(z) = \int_{u \leqslant u_\pi} f(u, z) d\lambda \Big/ \int_{u \leqslant u_\pi} \int_{\boldsymbol{Z}} f(u, z) d\mu d\lambda.$$

The density f_2 of $Z | U_{(\pi)} = 2$ is defined analogously. In further considerations it is convenient to assume that the supports of $Z|U_{(\pi)} = 1$ and $Z|U_{(\pi)} = 2$ are the same.

A two-class discriminant problem concerning a pair of random variables $(U_{(\pi)}, Z)$, where $U_{(\pi)}$ is the class indicator and Z the observable one, will be called the *two-class discriminant problem based on* (U, Z).

Let δ be a decision rule based on Z, where $\delta(z)$ is the probability that an object with $Z = z$ will be classified to class 1. The result of classification based on δ is a binary random variable – say I_δ – with values 1 and 2. The joint distribution of $(U_{(\pi)}, I_\delta)$ will be denoted by P_δ, where

$$P_\delta(I_\delta = 1) = \int \delta(z) f_Z(z) dz, \quad f_Z(z) = \pi f_1(z) + (1-\pi) f_2(z).$$

Let $a_{ij}(\delta) = P_\delta(I_\delta = j \mid U_{(\pi)} = i)$, $i \neq j$, $i, j = 1, 2$, be the probabilities of misclassification. A natural ordering of identification rules is given by

$$\delta \leqslant \delta' \Leftrightarrow a_{ij}(\delta) \geqslant a_{ij}(\delta') \text{ for } i \neq j,\ i, j = 1, 2. \tag{3.1}$$

According to Theorem 1 in [2], δ is admissible with respect to the ordering (3.1) iff δ is the threshold rule with respect to $h(z) = f_2(z)/f_1(z)$, that is, iff for some $s \in [0, 1]$ and $\varkappa \geqslant 0$

$$\delta(z) = \begin{cases} 1 & \text{if } h(z) < \varkappa, \\ s & \text{if } h(z) = \varkappa, \\ 0 & \text{if } h(z) > \varkappa. \end{cases} \tag{3.2}$$

Let $\Delta_{(U,Z)}$ be the set of all admissible rules of the form (3.2).

LEMMA 1. *For any* $\delta \in \Delta_{(U,Z)}$ *the pair* $(U_{(\pi)}, I_\delta)$ *is positively quadrant dependent.*

Proof. It is enough to show that

$$P_\delta(U_{(\pi)} = 1, I_\delta = 1) \geqslant P_\delta(U_{(\pi)} = 1) P_\delta(I_\delta = 1)$$

or, equivalently,

$$P_\delta(I_\delta = 1 \mid U_{(\pi)} = 1) \geqslant P_\delta(I_\delta = 1), \tag{3.3}$$

i.e.,

$$\int_{h<\varkappa} f_1(z) d\mu + s \int_{h=\varkappa} f_1(z) d\mu \geqslant \int_{h<\varkappa} f_Z(z) d\mu + s \int_{h=\varkappa} f_Z(z) d\mu. \tag{3.4}$$

Because of $f_Z(z) = \pi f_1(z) + (1-\pi) f_2(z)$, (3.4) is equivalent to

$$\int_{h<\varkappa} f_1(z) d\mu + s \int_{h=\varkappa} f_1(z) d\mu \geqslant \int_{h<\varkappa} f_2(z) d\mu + s \int_{h=\varkappa} f_2(z) d\mu. \tag{3.5}$$

For $\varkappa \leqslant 1$ and for $z \in \{z\colon h(z) \leqslant \varkappa\}$ we have $f_2(z) \leqslant f_1(z)$, so (3.5) is fulfilled. Otherwise, for $\varkappa > 1$ and for $z \in \{z\colon h(z) \geqslant \varkappa\}$, $f_2(z) \geqslant f_1(z)$ and the inequality

$$\int_{h<\varkappa} f_1(z) d\mu + (1-s) \int_{h=\varkappa} f_1(z) d\mu \leqslant \int_{h<\varkappa} f_2(z) d\mu + (1-s) \int_{h=\varkappa} f_2(z) d\mu,$$

which is equivalent to (3.5), is true. ∎

LEMMA 2. *If (U, Z) is positively regression dependent, then in the two-class discriminant problem the set of threshold rules with respect to $z \in Z$ is a minimal admissible class.*

Proof. It follows from [2] that the minimal admissible class with respect to the ordering (3.1) is the class of threshold rules with respect to $h(z)$. Since $(U, Z) \in \mathrm{RD}^+$, we infer that $P(U \leqslant u_\pi | Z = z)$ is a nonincreasing function of z. So, from the equality

$$P(U \leqslant u_\pi | Z = z) = \frac{\pi f_1(z)}{\pi f_1(z) + (1-\pi) f_2(z)} = \frac{\pi}{\pi + (1-\pi) h(z)}$$

it follows that $h(z)$ is a nondecreasing function of z. Thus, it is obvious then that for every threshold rule δ with respect to $h(z)$ there exists a threshold rule δ' with respect to z such that $a_{ij}(\delta) = a_{ij}(\delta')$, $i, j = 1, 2$. ■

A convenient tool to describe the divergence of the distribution of $Z | U_{(\pi)} = 2$ with respect to the distribution of $Z | U_{(\pi)} = 1$ is the divergence curve for the pair $(Z | U_{(\pi)} = 1, Z | U_{(\pi)} = 2)$, denoted by $C_{(U_{(\pi)}, Z)}$ and defined as the set of error rates $a_{12}(\delta)$ and $a_{21}(\delta)$ for all $\delta \in \Delta_{(U,Z)}$:

$$C_{(U_{(\pi)}, Z)} = \{(a_{21}(\delta), a_{12}(\delta));\ \delta \in \Delta_{(U,Z)}\}. \tag{3.6}$$

The divergence curve $C_{(U_{(\pi)}, Z)}$ is a convex continuous nonincreasing curve on the plane joining the points (0, 1) and (1, 0). (The divergence curves in the general case, as well as their properties, are described in [4].)

For any $\pi \in (0, 1)$ and any pairs (U, Z), (U', Z'), the notation

$$C_{(U_{(\pi)}, Z)} \geqslant C_{(U'_{(\pi)}, Z')} \tag{3.7}$$

will mean that the divergence curve $C_{(U_{(\pi)}, Z)}$ lies above the divergence curve for $C_{(U'_{(\pi)}, Z')}$, where $U'_{(\pi)}$ is defined by means of U' analogously as $U_{(\pi)}$ is defined by means of U.

From (3.6) it follows immediately that for any fixed $\pi \in (0, 1)$ we have $C_{(U_{(\pi)}, Z)} \geqslant C_{(U'_{(\pi)}, Z')}$ iff for every decision rule $\delta \in \Delta_{(U,Z)}$ there exists $\delta' \in \Delta_{(U',Z')}$ such that

$$a_{12}(\delta) \geqslant a'_{12}(\delta'), \qquad a_{21}(\delta) \geqslant a'_{12}(\delta').$$

This means that the divergence of the distribution of $Z' | U'_{(\pi)} = 2$ with respect to the distribution of $Z' | U'_{(\pi)} = 1$ is not smaller than the divergence of the distribution of $Z | U_{(\pi)} = 2$ with respect to the distribution of $Z | U_{(\pi)} = 1$.

Let the notation $X \sim Y$ mean that random variables X, Y have the same distributions.

THEOREM 2. *For any fixed $\pi \in (0, 1)$*

$$C_{(U_{(\pi)}, Z)} \geqslant C_{(U'_{(\pi)}, Z')} \tag{3.8}$$

iff for all $\delta \in \Delta_{(U,Z)}$, $\delta' \in \Delta_{(U',Z')}$ such that $I'_{\delta'} \sim I_\delta$

$$(U_{(\pi)}, I_\delta) \underset{\mathrm{QD}}{\leqslant} (U'_{(\pi)}, I'_{\delta'}). \tag{3.9}$$

Proof. Sufficiency. Let $\delta \in \Delta_{(U,Z)}$, $\delta' \in \Delta_{(U',Z')}$ be such that

$$P'_{\delta'}(I'_{\delta'} = 1) = P_{\delta}(I_{\delta} = 1).$$

Then (3.9) is equivalent to $a'_{ij}(\delta) \leqslant a_{ij}(\delta)$, $i \neq j$, $i, j = 1, 2$, which implies (3.8).

Necessity. Let $\delta_0 \in \Delta_{(U,Z)}$, $\delta'_0 \in \Delta_{(U',Z')}$ be such that

$$P'_{\delta'_0}(I'_{\delta'_0} = 1) = P_{\delta_0}(I_{\delta_0} = 1) = p_0. \tag{3.10}$$

For rules $\delta' \in \Delta_{(U',Z')}$ the inequalities $a'_{ij}(\delta') \leqslant a_{ij}(\delta_0)$, $i \neq j$, $i, j = 1, 2$, are fulfilled iff

$$k[a'_{12}(\delta') - a_{12}(\delta_0)] = a'_{21}(\delta') - a_{21}(\delta_0) \qquad \text{for some } k \geqslant 0. \tag{3.11}$$

Since (3.10) is equivalent to the equalities

$$-\pi a_{12}(\delta_0) + (1-\pi) a_{21}(\delta_0) = p_0 - \pi, \qquad -\pi a'_{12}(\delta'_0) + (1-\pi) a'_{21}(\delta'_0) = p_0 - \pi,$$

the equation (3.11) is fulfilled for $\delta' = \delta'_0$ if $k = \pi/(1-\pi)$. By the conditions $I'_{\delta'_0} \sim I_{\delta_0}$, the inequalities $a'_{ij}(\delta'_0) \leqslant a_{ij}(\delta_0)$, $i \neq j$, $i, j = 1, 2$, are true iff

$$(U_{(\pi)}, I_{\delta_0}) \underset{\text{QD}}{\leqslant} (U'_{(\pi)}, I'_{\delta'_0}). \ ■$$

COROLLARY 1. *If* (U, Z), $(U', Z') \in \mathrm{RD}^+$ *and* $U \sim U'$, $Z \sim Z'$, *then*

$$(U, Z) \underset{\text{QD}}{\leqslant} (U', Z') \Leftrightarrow \big(\forall \pi \in (0, 1)\big)\ C_{(U_{(\pi)}, Z)} \geqslant C_{(U'_{(\pi)}, Z')}.$$

Proof. It follows from Lemma 2 that $\Delta_{(U,Z)}$ and $\Delta_{(U',Z')}$ are the sets of threshold rules with respect to $z \in Z$. So

$$(U, Z) \underset{\text{QD}}{\leqslant} (U', Z') \Leftrightarrow \big(\forall \pi \in (0, 1)\big)\ (U_{(\pi)}, I_{\delta}) \underset{\text{QD}}{\leqslant} (U'_{(\pi)}, I'_{\delta'})$$

for all $\delta \in \Delta_{(U,Z)}$, $\delta' \in \Delta_{(U',Z')}$ such that $I'_{\delta'} \sim I_{\delta}$. Thus, the corollary follows now from Theorem 2.

It is worth noting that instead of assuming that $U \sim U'$ and $Z \sim Z'$ it is enough to assume that there exist increasing functions ϕ and ψ such that $U \sim \phi(U')$ and $Z \sim \psi(Z')$.

4. Some identification rules based on expert opinions. Let (U, Z, Z^*) be the monotone experts model. For simplicity we assume that U is a continuous random variable, but this assumption is not essential. For a given π, we want to construct an identification rule for $(U_{(\pi)}, Z)$, keeping in mind that the distribution of (Z^*, Z) is all we know. So, we "replace" the considered two-class discriminant problem based on (U, Z) by a suitable two-class discriminant problem based on (Z^*, Z).

Let us introduce $Z^*_{(\pi)}$ (valued 1 or 2) as an analogue of $U_{(\pi)}$: for any π-th quantile of Z^*, say z^*_{π}, we find a number a such that

$$P(Z^* < z^*_{\pi}) + aP(Z^* = z^*_{\pi}) = \pi,$$

and put $Z^*_{(\pi)} = 1$ if $Z^* < z^*_\pi$ or if $Z^* = z^*_\pi$ and $I < a$, where I is a random variable uniformly distributed on (0, 1) and independent of Z^*. So we consider $(Z^*_{(\pi)}, Z)$ instead of $(U_{(\pi)}, Z)$.

THEOREM 3. *In any monotone experts model* (U, Z, Z^*) *for any fixed* $\pi \in (0, 1)$,

$$C_{(Z^*_{(\pi)}, Z)} \geqslant C_{(U_{(\pi)}, Z)}.$$

Proof. By Theorem 2 it is enough to prove that, for all $\delta \in \Delta_{(U,Z)}$, $\delta' \in \Delta_{(Z^*,Z)}$ such that $I'_{\delta'} \sim I_\delta$, we have

$$(Z^*_{(\pi)}, I'_{\delta'}) \underset{\mathrm{QD}}{\leqslant} (U_{(\pi)}, I_\delta),$$

which is equivalent to

$$\Pr(U_{(\pi)} = 1, I_\delta = 1) \geqslant \Pr(Z^*_{(\pi)} = 1, I'_{\delta'} = 1). \tag{4.1}$$

We have to show that, for any chosen π, $q \in (0, 1)$,

$$\begin{aligned} P(U < u_\pi, Z < z_q) + bP(U < u_\pi, Z = z_q) \\ \geqslant P(Z^* < z^*_\pi, Z < z_q) + aP(Z^* = z^*_\pi, Z < z_q) + bP(Z^* < z^*_\pi, Z = z_q) \\ + abP(Z^* = z^*_\pi, Z = z_q), \end{aligned} \tag{4.2}$$

where a and b are defined by

$$P(Z^* < z^*_\pi) + aP(Z^* = z^*_\pi) = \pi, \quad P(Z < z_q) + bP(Z = z_q) = q.$$

Since both sides of (4.2) are linear functions of b, it is enough to prove (4.2) for b equal to 0 and 1. For $b = 0$, we will show that for any z

$$P(U < u_\pi, Z < z) \geqslant P(Z^* < z^*_\pi, Z < z) + aP(Z^* = z^*_\pi, Z < z). \tag{4.3}$$

We have

$$P(U < u_\pi, Z < z) = \int_{u < u_\pi} F_{Z|U=u}(z) dF_U(u)$$

and the right-hand side of (4.3) is equal to

$$\begin{aligned} \int_{u < u_\pi} F_{Z|U=u}(z)[F_{Z^*|U=u}(z^*_\pi) + aP(Z^* = z^*_\pi \mid U = u)] dF_U(u) \\ + \int_{u \geqslant u_\pi} F_{Z|U=u}(z)[F_{Z^*|U=u}(z^*_\pi) + aP(Z^* = z^*_\pi \mid U = u)] dF_U(u). \end{aligned}$$

Thus, (4.3) is equivalent to

$$\begin{aligned} \int_{u < u_\pi} F_{Z|U=u}(z)[1 - F_{Z^*|U=u}(z^*_\pi) - aP(Z^* = z^*_\pi \mid U = u)] dF_U(u) \\ - \int_{u \geqslant u_\pi} F_{Z|U=u}(z)[F_{Z^*|U=u}(z^*_\pi) + aP(Z^* = z^*_\pi \mid U = u)] dF_U(u) \geqslant 0. \end{aligned} \tag{4.4}$$

For $(Z, U) \in \mathrm{RD}^+$, $F_{Z|U=u}(z)$ is a nonincreasing function of u; therefore, replacing $F_{Z|U=u}(z)$ by $F_{Z|U=u_\pi}(z)$ on the left-hand side of (4.4), we get the expression which is less than or equal to the left-hand side of (4.4), and finally we have

$$F_{Z|U=u_\pi}(z)(\pi-\pi) = 0.$$

For $b = 1$ we proceed analogously after replacing $Z < z_q$ by $Z \leqslant z_q$. ■

Note that if $U \sim Z^*$, then Theorem 3 follows from Corollary 1.

Example. Let us consider a normal experts model (U, Z, Z^*), i.e. the model with

$$Z = \varrho U + (1-\varrho^2)^{1/2}\varepsilon,\ \varrho \in (0, 1), \qquad Z^* = \varrho^* U + (1-\varrho^{*2})^{1/2}\varepsilon^*,\ \varrho^* \in (0, 1),$$

where U, ε, ε^* are i.i.d. $N(0, 1)$ random variables. It is easy to check that the normal experts model is a monotone experts model and that

$$(U, Z) \sim N_2(0, 0, 1, 1, \varrho), \qquad (Z, Z^*) \sim N_2(0, 0, 1, 1, \varrho\varrho^*).$$

It is known that in the family of bivariate normal distributions the ordering $\underset{\mathrm{QD}}{\leqslant}$ is concordant with the ordering based on the correlation coefficients. From Corollary 1 it follows that if

$$(U, Z) \sim N_2(0, 0, 1, 1, \varrho) \quad \text{and} \quad (U', Z') \sim N_2(0, 0, 1, 1, \varrho'),$$

then

$$\varrho \leqslant \varrho' \Leftrightarrow (\forall \pi \in (0, 1))\, C_{(U_{(\pi)}, Z)} \geqslant C_{(U'_{(\pi)}, Z')}.$$

In Fig. 1, an example of divergence curves $C_{(U_{(\pi)}, Z)}$ and $C_{(Z^*_{(\pi)}, Z)}$ in normal experts model is presented.

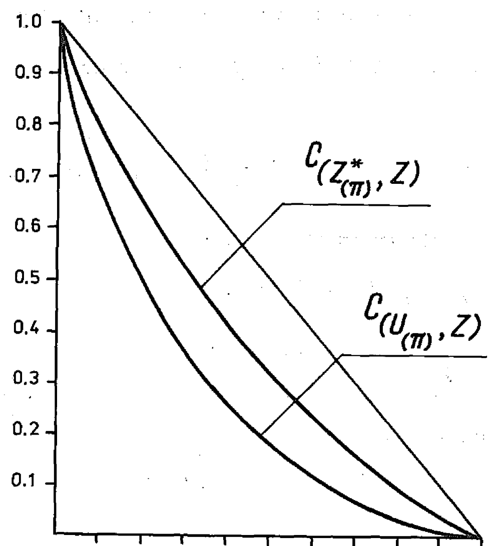

Fig. 1. Divergence curves $C_{(U_{(\pi)}, Z)}$ and $C_{(Z^*_{(\pi)}, Z)}$ in normal experts model for $\varrho = \varrho^* = .5$ and $n = .25$

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