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ON SOME MOMENT CONDITIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

BY

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Abstract. Some moment bounds and conditions for sums of independent random variables are considered. In particular, a method is presented to estimate the absolute moments using the related characteristic functions.

0. Introduction. Estimates for absolute moments of sums of independent random variables (r.v.'s) were studied by many authors (see von Bahr and Esseen [1], Rosenthal [11], Kwapień [7] et al.). If $2 \leqslant p < \infty$, Rosenthal's inequality implies that each sequence of independent identically distributed (i.i.d.) mean zero r.v.'s $\{X_k\}$ generates in L_p the subspace isomorphic to l_2. For $p < 2$ it is not true. In this paper we show that in this case each subspace of such a type is isomorphic to some Orlicz sequence space l_Φ. If $P\{|X_1| \geqslant x\} = x^{-r}h(x)$, where $h(x)$ is a slowly varying function and $0 < p < r < 2$, a calculation of Φ becomes particularly simple.

Esseen and Janson [3] have proved that the condition

$$(0.1) \qquad \Big\|\sum_{k=1}^{n} X_k\Big\|_p \leqslant Cn^{1/r} \quad (n = 1, 2, \ldots),$$

where C is a constant, holds for $r = 2$ iff $\mathrm{E}X_1^2 < \infty$ and $\mathrm{E}X_1 = 0$. If $0 < p < r < 2$, then (0.1) is true iff $P\{|X_1| \geqslant x\} = O(x^{-r})$, $x \to \infty$, $\mathrm{E}X_1 = 0$ for $r > 1$ and

$$\sup\{\mathrm{E}|X_1 I_{\{|X_1| \leqslant a\}}|:\ a > 0\} < \infty$$

for $r = 1$. Braverman [2] have obtained conditions which are equivalent to the two-sided estimate

$$(0.2) \qquad C_1 n^{1/r} \leqslant \Big\|\sum_{k=1}^{n} X_k\Big\|_p \leqslant C_2 n^{1/r} \quad (n = 1, 2, \ldots).$$

Let $a^{(n)} = \{1\}_{k=1}^{n}$ and $b_n = \|a^{(n)}\|_{l_\Psi}$, where Ψ is given. In this paper the analogues of (0.1) and (0.2) are studied, where $n^{1/r}$ is replaced by b_n.

We consider r.v.'s with symmetric distributions only. Using the symmetrization and reasoning as in [2] and [3] one may extend our results to the non-symmetric case. We omit details.

1. The main inequality. All considered r.v.'s are assumed to be defined on the non-atomic probability space (Ω, A, P). It is well known (see [6]) that if $X \in L_p(\Omega)$, $0 < p < 2$, and $f(t)$ is the corresponding characteristic function, then

$$\mathrm{E}|X|^p = C(p) \int_0^\infty \big(1-\mathrm{Re}(f(t))\big)t^{-p-1}\,dt, \tag{1.1}$$

where $C(p)$ depends on p only.

Let $\{Z_k\}_{k=1}^n \subset L_p(\Omega)$ be independent symmetric r.v.'s. Denote by $f_k(t)$ the related characteristic functions and let $\beta = \beta(Z_1, \ldots, Z_n)$ be the solution of the equation

$$\sum_{k=1}^n \int_0^1 \big(1-f_k(t/\beta)\big)t^{-p-1}\,dt = 1. \tag{1.2}$$

It is not difficult to check that the term on the left-hand side is monotonically decreasing with respect to β and it tends to ∞ (respectively, to 0) as $\beta \to 0$ (respectively, $\beta \to \infty$). Therefore, (1.2) has a unique solution.

THEOREM 1.1. *Let $0 < p < 2$. There are positive constants $C_j(p)$, $j = 1, 2$, such that for each set of independent symmetric r.v.'s $\{Z_k\}_{k=1}^n \subset L_p(\Omega)$*

$$C_1(p)\beta(Z_1, \ldots, Z_n) \leqslant \Big\| \sum_{k=1}^n Z_k \Big\|_p \leqslant C_2(p)\beta(Z_1, \ldots, Z_n). \tag{1.3}$$

Proof. The upper bound. Put

$$S = \sum_{k=1}^n Z_k. \tag{1.4}$$

The corresponding characteristic function is $f(t) = \prod_{k=1}^n f_k(t)$. Write

$$g_k(t) = \prod_{j=1}^{k-1} f_j(t) \qquad (2 \leqslant k \leqslant n). \tag{1.5}$$

We have

$$1-f(t) = \big(1-f_1(t)\big) + \sum_{k=2}^n g_k(t)\big(1-f_k(t)\big). \tag{1.6}$$

Hence

$$1-f(t) \leqslant \sum_{k=1}^n \big(1-f_k(t)\big).$$

Putting $\beta = \beta(Z_1, \ldots, Z_n)$ and using (1.1), we obtain

$$\mathrm{E}\left|\sum_{k=1}^{n} Z_k\right|^p \leqslant C(p)\Big(\sum_{k=1}^{n} \int_{0}^{\beta^{-1}} \big(1-f_k(t)\big)t^{-p-1}\,dt + 2\int_{\beta^{-1}}^{\infty} t^{-p-1}\,dt\Big).$$

According to (1.2), the first term on the right-hand side is equal to β^p. The desired estimate follows.

The lower bound. We need the following auxiliary result.

PROPOSITION 1.1. *Let V be a weakly compact set of probability distributions on $\mathbf{R}$ and M be the set of the related characteristic functions. Then there is $\delta = \delta(V) > 0$ such that*

$$\sup\{|1-g(t)|:\ g \in M,\ 0 < t < \delta\} < 1/2.$$

Proof. Suppose to the contrary that there are $g_k \in M$ and $t_k \to 0$ such that $|1-g_k(t_k)| \geqslant 1/2$. Using compactness, we may suppose that $g_k(t) \to g(t)$ for all real t and some characteristic function $g(t)$. But then (see [9]) $g_k(t_k) \to g(0) = 1$, which yields a contradiction. ∎

Without loss of generality we may and do assume that

$$\beta(Z_1, \ldots, Z_n) = 1. \tag{1.7}$$

Then, by (1.2), $\beta(Z_1, \ldots, Z_j) \leqslant 1$, so that $\|\sum_{k=1}^{j} Z_k\|_p \leqslant C_2(p)$ $(1 \leqslant j \leqslant n)$. Let V be the set of all r.v.'s X such that $\|X\|_p \leqslant C_2(p)$. Then (1.5) and Proposition 1.1 give us $g_k(t) \geqslant 1/2$ for $0 < t < \delta = \delta(V)$. This and (1.6) imply

$$1-f(t) \geqslant 2^{-1} \sum_{k=1}^{n} \big(1-f_k(t)\big), \qquad 0 < t < \delta.$$

Now we apply the inequality (see [4])

$$1-\mathrm{Re}\big(g(mt)\big) \leqslant m^2\{1-\mathrm{Re}\big(g(t)\big)\}, \tag{1.8}$$

which holds for each characteristic function g and integer m. From the previous considerations we obtain

$$1-f(t) \geqslant (2m^2)^{-1} \sum_{k=1}^{n} \big(1-f_k(mt)\big), \qquad 0 < t < \delta.$$

Putting $m = [1/\delta]+1$ and using (1.1), we obtain

$$\begin{aligned}\mathrm{E}\left|\sum_{k=1}^{n} Z_k\right|^p &\geqslant C(p)\int_{0}^{\delta} \big(1-f(t)\big)t^{-p-1}\,dt \\ &\geqslant 2^{-1}C(p)m^{p-2} \sum_{k=1}^{n} \int_{0}^{m\delta} \big(1-f_k(y)\big)y^{-p-1}\,dy.\end{aligned}$$

Since $m\delta > 1$ and (1.7) holds, the desired estimate follows. ∎

2. Applications. Let us recall some definitions of the theory of sequence spaces. Let $\Psi(x)$ be a non-decreasing function on $[0, \infty)$, $\Psi(0) = 0$, and $\Psi(\infty) = \infty$. We shall suppose that

$$\Psi(2x) \leqslant C\Psi(x) \tag{2.1}$$

for $0 < x < 1$ and some constant C. The Orlicz space l_Ψ consists of all real sequences $a = \{a_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \Psi(|a_k|/t) < \infty \qquad \text{for some } t > 0.$$

It is easy to check that the relation (2.1) implies $\Psi(x+y) \leqslant B(\Psi(x)+\Psi(y))$ for $0 < x,\ y < 1$ and some constant B. Hence, if $a, b \in l_\Psi$, then $a+b \in l_\Psi$.

Put for $a \in l_\Psi$

$$\|a\|_\Psi = \inf\{t > 0\colon \sum_{k=1}^{\infty} \Psi(|a_k|/t) \leqslant 1\}. \tag{2.2}$$

This functional is a quasi-norm. If $\Psi(x)$ is convex, then it is a norm and l_Ψ is a Banach space. If $\Psi(x) = x^p$, then $l_\Psi = l_p$.

The following property of Orlicz sequence spaces is well known and may be easily verified.

PROPOSITION 2.1. *The following conditions are equivalent*:
(i) $\Psi_1(x) \leqslant u\Psi_2(x)$ *for* $0 < x < 1$ *and some constant* u;
(ii) $\|a\|_{\Psi_1} \leqslant C\|a\|_{\Psi_2}$ *for all finite sequences* a *and some constant* C.

Let $\{X_k\}_{k=1}^{\infty} \subset L_p(\Omega)$ be a sequence of symmetric i.i.d.r.v.'s and let $f(t)$ be the common characteristic function. Put

$$\Phi_p(x) = \int_0^1 (1-f(tx))t^{-p-1}\,dt. \tag{2.3}$$

The inequality (1.8) implies (2.1) for $\Psi = \Phi_p(x)$. Applying Theorem 1.1, we get the following result.

COROLLARY 2.1. *There are positive constants* $C_j(p)$ $(j = 1, 2)$ *such that for each* $a = \{a_k\}_{k=1}^{n}$

$$C_1(p)\|a\|_{\Phi_p} \leqslant \Big\|\sum_{k=1}^{n} a_k X_k\Big\|_p \leqslant C_2(p)\|a\|_{\Phi_p}.$$

In some cases it is more convenient to use the distribution function instead of the characteristic function. For the $X \in L_p(\Omega)$ we put

$$H_p(z) = \int_z^{\infty} y^{p-1} P\{|X| \geqslant y\}\,dy, \tag{2.4}$$

$$G(z) = \int_0^z yP\{|X| \geqslant y\}\,dy. \tag{2.5}$$

Let

$$U_p(z) = (p/2)z^{-p}H_p(z)+z^{-2}G(z). \tag{2.6}$$

PROPOSITION 2.2. *Let $0 < p < 2$. There are positive constants $a_j = a_j(p)$, $j = 1, 2$, such that for every symmetric r.v. $X \in L_p(\Omega)$*

$$a_1 \Phi_p(x) \leqslant U_p(x^{-1}) \leqslant a_2 \Phi_p(x) \quad (0 < x < 1). \tag{2.7}$$

Proof. Put $F(x) = P\{X < x\}$ and

$$g_p(x) = \int_0^x (1-\cos(u))u^{-p-1}\,du. \tag{2.8}$$

It is not difficult to verify that

$$\Phi_p(x) = 2x^p \int_0^\infty z^p g_p(xz)\,dF(z)$$

and there are positive constants $b_j = b_j(p)$ ($j = 1, 2$) such that $b_1 < g_p(x)x^{p-2} < b_2$ $(0 < x < 1)$ and $b_1 < g_p(x) < b_2$ $(x > 1)$. Hence

$$c_1 \Phi_p(x) \leqslant x^2 \int_0^{x^{-1}} z^2\,dF(z)+x^p \int_{x^{-1}}^\infty z^p\,dF(z) \leqslant c_2 \Phi_p(x), \tag{2.9}$$

where $0 < x < 1$ and $c_1, c_2 > 0$ depend on p only.

Denote the summands in (2.9) by $J_k(x)$, $k = 1, 2$. Integrating by parts, we get

$$J_1(x) = -P\{|X| \geqslant x^{-1}\}/2+x^2 G(x^{-1})/2$$

and

$$J_2(x) = P\{|X| \geqslant x^{-1}\}/2+(p/2)x^p H_p(x^{-1}).$$

Using this, (2.6) and (2.9) the relation (2.7) follows. ∎

The obtained results imply the following assertion.

COROLLARY 2.2. *Let $0 < p_j < 2$ ($j = 1, 2$) and $\{X_k^{(j)}\}_{k=1}^\infty \subset L_{p_j}(\Omega)$ be sequences of symmetric i.i.d.r.v.'s. The following conditions are equivalent for some constants C_j ($j = 1, 2, 3$):*

(i) *for all $a_k \in \mathbf{R}$ and $n = 1, 2, \ldots$*

$$\Big\|\sum_{k=1}^n a_k X_k^{(1)}\Big\|_{p_1} \leqslant C_1 \Big\|\sum_{k=1}^n a_k X_k^{(2)}\Big\|_{p_2};$$

(ii) $\Phi_{p_1}^{(1)}(x) \leqslant C_2 \Phi_{p_2}^{(2)}(x)$ $(0 < x < 1)$;

(iii) $U_{p_1}^{(1)}(t) \leqslant C_3 U_{p_2}^{(2)}(t)$ $(t \geqslant 1)$.

Remark. Suppose $X_k^{(1)} = X_k^{(2)} = X_k$ and

$$H_p(z) \leqslant Cz^p P\{|X_1| \geqslant z\}, \tag{2.10}$$

where C is a constant. It is not difficult to verify that (2.10) implies (iii), and therefore (i) for the exponents $p_2 < p = p_1 < 2$. This assertion had been obtained by Kwapień [7] (see also Vakhania et al. [12]). However, it is easy to check that (2.10) does not follow from (iii).

Assume now that the symmetric distribution F is determined by the formula

$$F\{t: |t| \geqslant x\} = x^{-r}h(x), \tag{2.11}$$

where $h(x)$ is a slowly varying function at infinity, i.e., $h(xy)/h(x) \to 1$ $(x \to \infty)$ for all $y > 0$. If $p < r$, then this distribution has a finite absolute p-th moment. In the case $p = r$ the latter is true iff

$$\int_0^\infty h(y)y^{-1}dy < \infty. \tag{2.12}$$

In the sequel we will assume that the condition (2.12) holds.

Let $p < 2$ be fixed. Consider the functions

$$\Psi_r(x) = x^r h(x^{-1}) \quad (0 < p < r < 2), \tag{2.13}$$

$$\Psi_2(x) = x^2 \int_0^{x^{-1}} h(y)y^{-1}dy, \tag{2.14}$$

$$\Psi_p(x) = x^p \int_{x^{-1}}^\infty h(y)y^{-1}dy. \tag{2.15}$$

Let $f(t)$ be the characteristic function of the distribution (2.11). It is well known (see [5], Chapter 2) that $(1-f(t))/\Psi_r(t) \to c > 0$ $(t \to 0)$ for $0 < r \leqslant 2$. Using the Karamata representation for the slowly varying functions, one may prove that there are positive constants $u_j = u_j(p, r, h)$ $(j = 1, 2)$ such that for $0 < p \leqslant r \leqslant 2$ and $0 < x < 1$

$$u_1 \Psi_r(x) \leqslant \Phi_p(x) \leqslant u_2 \Psi_r(x), \tag{2.16}$$

where Φ_p is determined by the formula (2.3).

COROLLARY 2.3. *Suppose $0 < p \leqslant r \leqslant 2$ and $\{X_k\}_{k=1}^\infty$ are i.i.d.r.v.'s with the common distribution function satisfying* (2.11) *and* (2.12) *if $p = r$. Then there are positive constants $c_j = c_j(p, r, h)$ $(j = 1, 2)$ such that, for each $a = \{a_k\}_{k=1}^n$ and $n = 1, 2, \ldots,$*

$$c_1 \|a\|_{\Psi_r} \leqslant \Big\| \sum_{k=1}^n a_k X_k \Big\|_p \leqslant c_2 \|a\|_{\Psi_r}.$$

3. One-sided estimates. In this section $\{X_k\}_{k=1}^\infty$ is a sequence of symmetric i.i.d.r.v.'s and $f(t)$ is the common characteristic function. Here we consider the upper bound

$$\Big\| \sum_{k=1}^n a_k X_k \Big\|_p \leqslant C \|a\|_\Psi, \tag{3.1}$$

where $a = \{a_k\}_{k=1}^n$, Ψ is a given function and C does not depend on a_k or n. Corollary 2.1 and Proposition 2.1 imply that (3.1) is equivalent to the relation $\Phi_p(x) \leqslant u\Psi(x)$ $(0 < x < 1)$, where u is a constant and Φ_p is determined by the formula (2.3). However, the pointwise conditions on the behaviour $1-f(t)$ $(t \to 0)$ or $P\{|X_1| \geqslant x\}$ $(x \to \infty)$ are more convenient. We obtain them for the case $\Psi = \Psi_r$, where Ψ_r is determined by the formulae (2.13)–(2.15). In the sequel $h(x)$ will be assumed to be continuous.

THEOREM 3.1. *Let $0 < p < r < 2$. The following conditions are equivalent:*
(i) *for the function Ψ_r the estimate* (3.1) *holds;*
(ii) $1-f(t) = O(\Psi_r(t))$ $(t \to 0)$;
(iii) $P\{|X_1| \geqslant x\} = O(\Psi_r(x^{-1}))$ $(x \to \infty)$.

THEOREM 3.2. *Let $0 < p < r = 2$. The following conditions are equivalent:*
(i) *for the function $\Psi = \Psi_2$ the estimate* (3.1) *holds;*
(ii) $1-f(t) = O(\Psi_2(t))$ $(t \to 0)$;
(iii) $P\{|X_1| \geqslant x\} = O(x^{-2}h(x))$ $(x \to \infty)$, *where $h(x)$ is the function given by* (2.14).

Consider the case $p = r < 2$. It is easy to check that the estimate $1-f(t) = O(\Psi_p(t))$ $(t \to 0)$ implies the relation $\Phi_p(t) = O(\Psi_p(t))$ $(t \to 0)$, which is equivalent to (3.1). The author does not know if the converse implication is true. We formulate an analogue of the previous theorems using the function Φ_p.

THEOREM 3.3. *Let $0 < p = r < 2$. Assume* (2.12) *holds and let the function Ψ_p be determined by* (2.15). *Then the following conditions are equivalent:*
(i) *for the function $\Psi = \Psi_p$ the estimate* (3.1) *holds;*
(ii) $\Phi_p(t) = O(\Psi_p(t))$ $(t \to 0)$;
(iii) $\int_x^\infty P\{|X_1| \geqslant y\} y^{p-1} dy = O(\Psi_p(x^{-1}))$ $(x \to \infty)$.

In all theorems of this section the equivalence (ii)$\Leftrightarrow$(iii) is well known and can be easily verified. Using the properties of a slowly varying function (see [5], Chapter 2) one can easily prove that the condition (ii) of Theorems 3.1 and 3.2 implies that $\Phi_p(t) = O(\Psi_r(t))$ $(t \to 0)$. Thus the implication (ii)$\Rightarrow$(i) follows. The equivalence (i)$\Leftrightarrow$(ii) of Theorem 3.3 follows directly from Proposition 2.1 and Corollary 2.1. Therefore, only the implications (i)$\Rightarrow$(ii) in Theorems 3.1 and 3.2 require proofs.

First we establish some auxiliary results. Let us write $a^{(n)} = \{1\}_{k=1}^n$ and $b_n = \|a^{(n)}\|_{\Psi_r}$. From (2.2) we obtain

(3.2) $$n\Psi_r(b_n^{-1}) = 1.$$

Put

(3.3) $$S_n = b_n^{-1} \sum_{k=1}^n X_k.$$

Then

$$f_n(t) = (f(b_n^{-1}t))^n \tag{3.4}$$

is the characteristic function of S_n.

According to (3.1) we get

$$\|S_n\|_p \leqslant C \quad (n = 1, 2, \ldots). \tag{3.5}$$

Hence the sequence $\{S_n\}_{n=1}^{\infty}$ is weakly compact. Let N be the set of all characteristic functions corresponding to the limit distributions of $\{S_n\}_{n=1}^{\infty}$ and $M = N \cup \{f_n\}_{n=1}^{\infty}$. It follows from Proposition 1.1 that there is $\delta = \delta(M) > 0$ such that $g(t) > 1/2$ for all $g \in M$ and $0 < t < \delta$. Therefore, on $(0, \delta]$ the function $\phi_g(t) = -(\log(g(t)))/\Psi_r(t)$ is defined and

$$g(t) = \exp(-\phi_g(t)\Psi_r(t)) \quad (0 < t < \delta). \tag{3.6}$$

The condition (ii) of Theorems 3.1 and 3.2 is equivalent to the boundedness of $\phi_f(t)$ in the neighbourhood of zero.

The following proposition is well known.

Proposition 3.1. *If $v(x)$ is a slowly varying function, then $v(xy)/v(x) \to 1$ $(x \to \infty)$ uniformly on each segment $0 < \mu \leqslant y \leqslant v$.*

Proposition 3.2. *Let $g \in M$ and $f_{n(k)}(t) \to g(t)$ for every real t. Then $\phi_f(b_{n(k)}^{-1}t) \to \phi_g(t)$ uniformly on every segment $0 < \mu \leqslant t \leqslant \delta = \delta(M)$.*

Proof. Put for $r = 2$

$$H(x) = \int_0^x h(y)y^{-1}\,dy. \tag{3.7}$$

Let $v(x) = h(x)$ if $r < 2$ and $v(x) = H(x)$ if $r = 2$. From (2.13) and (2.14) we obtain $\Psi_r(x) = x^r v(x^{-1})$. Therefore, (3.4) and (3.6) give us

$$f_n(t) = \exp(-nb_n^{-r}t^r v(b_n t^{-1})\phi_f(b_n^{-1}t)). \tag{3.8}$$

The formula (3.2) implies

$$\phi_f(b_n^{-1}t) = -(\log(f_n(t)))\frac{v(b_n)}{v(b_n t^{-1})}t^{-r}, \tag{3.9}$$

where $0 < t < \delta$. Hence

$$|\phi_f(b_n^{-1}t) - \phi_f(b_m^{-1}t)| \leqslant |\log(f_n(t)) - \log(f_m(t))|\frac{v(b_n)}{v(b_n t^{-1})}t^{-r}$$

$$+ |\log(f_m(t))|\left|\frac{v(b_n)}{v(b_n t^{-1})} - \frac{v(b_m)}{v(b_m t^{-1})}\right|t^{-r}.$$

We have $f_n(t) \geqslant 1/2$ for $0 < t < \delta$ and all $n = 1, 2, \dots$ Hence

$$|\log(f_n(t)) - \log(f_m(t))| \leqslant 2|f_n(t) - f_m(t)|$$

and $|\log(f_n(t))| \leqslant 2$. It is well known (see [9]) that the convergence of characteristic functions is uniform on each finite segment. The function (3.7) is slowly varying and, by (3.2), $b_n \to \infty$. Combining the above estimates and Proposition 3.1 completes the proof. ∎

PROPOSITION 3.3. *The sequence $\{\phi_f(b_n^{-1}t)\}_{n=1}^{\infty}$ is compact in the sense of the uniform convergence on every segment $0 < \mu \leqslant t \leqslant \delta$.*

This proposition follows directly from the previous one and the weak compactness of $\{S_n\}_{n=1}^{\infty}$.

Proof of (i)$\Rightarrow$(ii) in Theorems 3.1 and 3.2. Suppose (i) holds. Using the formula (3.2) we get

$$\lim_{n\to\infty} b_{n+1}/b_n = 1. \tag{3.10}$$

Put $b_0 = 1$. Since $b_n \to \infty$, we have

$$1 < d = \sup_{n\geqslant 0} b_{n+1}/b_n < \infty. \tag{3.11}$$

According to Proposition 3.3, we obtain

$$B = \sup\{\phi_f(b_n^{-1}t)\colon\ \delta/d \leqslant t \leqslant \delta,\ n = 1, 2, \dots\} < \infty.$$

It is easy to check up that, for every $t \in (0, \delta)$, there is an integer $n \geqslant 0$ such that $\delta/d \leqslant b_n t = s \leqslant \delta$. Therefore, $0 < \phi_f(t) = \phi_f(b_n^{-1}(b_n t)) = \phi_f(b_n^{-1}s) \leqslant B$. This estimate and (3.6) imply (ii). ∎

Remark. The above proof shows that (ii) follows from the condition (3.5), which is weaker than (i). Putting $h(x) \equiv 1$, we obtain the Esseen and Janson result [3], mentioned in the Introduction.

4. Two-sided estimates. In this section we consider the two-sided estimates

$$c_1\|a\|_\Psi \leqslant \Big\|\sum_{k=1}^{n} a_k X_k\Big\|_p \leqslant c_2\|a\|_\Psi, \tag{4.1}$$

where, as before, $\{X_k\}_{k=1}^{\infty}$ is a sequence of symmetric i.i.d.r.v.'s, Ψ is one of the functions (2.13)$-$(2.15) and c_1, c_2 are positive constants.

THEOREM 4.1. *Let $0 < p < r < 2$. The following conditions are equivalent:*

(i) *for the function Ψ_r the estimate* (4.1) *holds;*

(ii) *there are positive constants u, v, w such that*

$$u\Psi_r(t) \leqslant 1 - f(t) \leqslant v\Psi_r(t) \qquad (0 < t < w);$$

(iii) *there are positive constants A, B, C such that*

$$A\Psi_r(x^{-1}) \leqslant P\{|X_1| \geqslant x\} \leqslant B\Psi_r(x^{-1}) \qquad (x \geqslant C).$$

THEOREM 4.2. *Let $0 < p < r = 2$. The following conditions are equivalent:*
(i) *for the function Ψ_2 the estimate* (4.1) *holds;*
(ii) *there are positive constants u, v, w such that*

$$u\Psi_2(t) \leqslant 1-f(t) \leqslant v\Psi_2(t) \quad (0 < t < w);$$

(iii) *there are positive constants A, B, C such that*

$$Ax^{-2}h(x) \leqslant P\{|X_1| \geqslant x\} \leqslant Bx^{-2}h(x) \quad (x \geqslant C),$$

where $h(x)$ is a function from (2.14).

THEOREM 4.3. *Let $0 < p = r < 2$ and let* (2.12) *be fulfilled. Then the following conditions are equivalent:*
(i) *for the function Ψ_p the estimate* (4.1) *holds;*
(ii) *there are positive constants u, v, w such that*

$$u\Psi_p(t) \leqslant \Phi_p(t) \leqslant v\Psi_p(t) \quad (0 < t < w);$$

(iii) *there are positive constants A, B, C such that*

$$A\Psi_p(x^{-1}) \leqslant \int_x^\infty P\{|X_1| \geqslant y\} y^{p-1} dy \leqslant B\Psi_p(x^{-1})$$

for $x > C$.

As above, we only need to prove the implication (i) $\Rightarrow$ (ii) of Theorems 4.1 and 4.2. So, assume that (i) holds. Theorems 3.1 and 3.2 give us the upper estimate in (ii). Using these theorems once more (the implication (ii) $\Rightarrow$ (i) for $q \in (p, r)$), we get

$$C(q) \equiv \sup_n \|S_n\|_q < \infty,$$

where S_n is determined by the formula (3.3).

Denote by Y_g an r.v. with the characteristic function $g(t)$. Let M be the set determined in Section 3. We have $\|Y_g\|_q \leqslant C(q)$ for every $g \in M$. Using the theorem on moments convergence (see [9]) we obtain the next assertion.

PROPOSITION 4.1. *Suppose $g_k \in M$ and $g_k(t) \to g(t)$ for all $t \in \boldsymbol{R}$, where g is a characteristic function. Then $\|Y_{g_k}\|_p \to \|Y_g\|_p$.*

The condition (i) implies

$$c_1 \leqslant \|S_n\|_p \leqslant c_2 \quad (n = 1, 2, \ldots). \tag{4.2}$$

PROPOSITION 4.2. *Let $\sup\{f_n(t)\colon \mu \leqslant t \leqslant \nu, n = 1, 2, \ldots\} = 1$ for some $0 < \mu < \nu < \infty$. Then there exist a non-degenerate $g \in M$ and $t_0 \in [\mu, \nu]$ such that $g(t_0) = 1$.*

Proof. There exist $t_k \in [\mu, \nu]$ and integers $n(k) \uparrow \infty$ such that $f_{n(k)}(t_k) \to 1$. Since the set M is weakly compact, we may assume that $t_k \to t_0 \in [\mu, \nu]$ and $f_{n(k)}(t) \to g(t)$ for all $t \in \boldsymbol{R}$ and some $g \in M$. Hence $f_{n(k)}(t_k) \to g(t_0)$, i.e., $g(t_0) = 1$.

Since the r.v.'s X_k are symmetric, the degeneration of g implies $Y_g = 0$. But from Proposition 4.1 we obtain $\|S_{n(k)}\|_p \to \|Y_g\|_p = 0$, which contradicts (4.2). ∎

PROPOSITION 4.3. *There is an integer m such that*

$$\sup\{f_n(t)\colon 1/(db_m) \leqslant t \leqslant 1/b_m,\ n = 1, 2, \ldots\} = \gamma < 1, \tag{4.3}$$

where d is determined by the formula (3.11).

Proof. If this assertion does not hold, then, for every integer m, there are non-degenerate $g_m \in M$ and $t_m \in [1/(db_m), 1/b_m]$ such that $g(t_m) = 1$. Hence g_m corresponds to a symmetric lattice distribution. If a_m is the maximal step of this distribution, then $a_m \geqslant 2\pi/t_m \geqslant 2\pi b_m$.

Taking into account the weak compactness of M, one can choose integers $m(k)$ such that $g_{m(k)}(t) \to g(t)$ for all real t and some $g \in M$. It can be easily verified that $a_m \to \infty$ implies $g(t) \equiv 1$, i.e., $Y_g = 0$. But from (4.2) and Proposition 4.1 we obtain $\|Y_g\|_p \geqslant c_1 > 0$ for all $g \in M$. This contradiction proves the proposition. ∎

The following assertion follows easily from (3.10).

PROPOSITION 4.4. *For each $0 \leqslant \mu \leqslant \nu$ and an integer n_0 there is a $\delta > 0$ such that*

$$(0, \delta) \subset \bigcup_{n \geqslant n_0} [\mu/b_n, \nu/b_n].$$

Proof of (i) ⇒ (ii) in Theorems 4.1 and 4.2. We use the notation of the previous section. We need to show that there are constants $\varepsilon, \delta > 0$ such that $\phi_f(t) > \varepsilon$ if $0 < t < \delta$.

Put $I = [1/(db_m), 1/b_m]$ such that (4.3) holds for the integer m. Choose $\delta_1 > 0$ for which (3.6) is fulfilled. There is an n_1 such that $t/b_n < \delta_1$ for all $t \in I$ and $n \geqslant n_1$. Combining this, (4.3) and (3.9) we get

$$\phi_f(b_n^{-1} t) \geqslant (-\log(\gamma)) \frac{v(b_n)}{v(b_n t^{-1})} t^{-r}.$$

Since $t \in I$, we obtain $t^{-r} \geqslant b_m^r$. Using Proposition 3.1, we conclude that there is an integer n_2 such that

$$\inf\{v(b_n)/v(b_n t^{-1})\colon t \in I, n \geqslant n_2\} = \alpha > 0.$$

Hence, for $t \in I$ and $n \geqslant n_0 = \max\{n_1, n_2\}$,

$$\phi_f(b_n^{-1} t) \geqslant (-\log(\gamma)) b_m^r \alpha = \varepsilon > 0. \tag{4.4}$$

Now apply Proposition 4.4, where $\mu = 1/(db_m)$ and $\nu = 1/b_m$. If $t \in (0, \delta)$, then $s = tb_n \in I$ for some $n \geqslant n_0$. From (4.4) we obtain

$$\phi_f(t) = \phi_f(b_n^{-1}(tb_n)) = \phi_f(b_n^{-1} s) \geqslant \varepsilon > 0.$$

This and (3.6) give us the lower estimate in (ii). ∎

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