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GAUSS MEASURES IN THE SENSE OF BERNSTEIN ON THE HEISENBERG GROUP

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Abstract. Gauss measures in the sense of Bernstein (for a definition thereof which extends the classical one straightforwardly involving non-commutativity) are specified explicitly for the three-dimensional Heisenberg group.

1. Introduction. Based on the famous Bernstein Theorem, a probability measure μ on a locally compact Abelian group G with countable basis of its topology is called *Gaussian in the sense of Bernstein* if there exists a probability space $(\Omega, \mathscr{B}, P)$ and independent G-valued random variables X, Y on it such that $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$ and that the random variables $X \cdot Y$ and $X \cdot Y^{-1}$ are independent (cf. [2], and [3], 5.3.1, Remark 5.3.2). See [3] for properties of such measures, in particular the relationship to other definitions of Gauss measures on groups. In this note the Gauss measures in the sense of Bernstein (for a definition thereof which extends the above one straightforwardly involving non-commutativity) on the (non-Abelian) three-dimensional Heisenberg group $\boldsymbol{H}$ are specified explicitly. $\boldsymbol{H}$ is given as $\boldsymbol{R}^3$ equipped with the multiplication

$$x \cdot y = x+y+\tfrac{1}{2}[x, y], \quad [x, y] = (0, 0, x_1 y_2 - x_2 y_1) \in \boldsymbol{R}^3$$
$$(x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \boldsymbol{R}^3).$$

Clearly, the neutral element e is 0 and $x^{-1} = -x$. It is shown that the Gauss measures in the sense of Bernstein on $\boldsymbol{H}$ are just the Gauss measures on $(\boldsymbol{R}^3, +)$ which are concentrated on a plane containing the x_3-axis. These measures are also Gaussian on $\boldsymbol{H}$ in the sense that they lie in a continuous convolution semigroup which is generated by the sum of a quadratic and a primitive distribution.

2. Gauss measures in the sense of Bernstein.

DEFINITION 1. Let G be a locally compact group admitting a countable basis of its topology. Then a probability measure μ on G is called *Gaussian* *in the sense of Bernstein* if there exists a probability space $(\Omega, \mathscr{B}, P)$ and independent G-valued random variables X, Y on it such that

(i) $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$,

(ii) $(X \cdot Y, Y \cdot X)$ and $(X \cdot Y^{-1}, Y^{-1} \cdot X)$ are independent.

Remark 1. Since $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, it is clear that (ii) is equivalent to the fact that

$$(X \cdot Y, Y \cdot X) \quad \text{and} \quad (X \cdot Y^{-1}, Y^{-1} \cdot X, X^{-1} \cdot Y, Y \cdot X^{-1})$$

are independent.

THEOREM 1. *A probability measure μ on $\boldsymbol{H}$ is Gaussian in the sense of Bernstein iff it is a (one- or two-dimensional) normal distribution on $(\boldsymbol{R}^3, +)$ concentrated on a plane containing the x_3-axis.*

Proof. 1. Assume μ is a normal law on $(\boldsymbol{R}^3, +)$ concentrated on a plane containing the x_3-axis and let X, Y be independent random variables on some probability space $(\Omega, \mathscr{B}, P)$ both distributed according to the law μ. By the Bernstein Theorem for $(\boldsymbol{R}^3, +)$, $X+Y$ and $X-Y$ are independent. However, it follows from the assumption that

$$X \cdot Y = Y \cdot X = X+Y \quad \text{and} \quad X \cdot (-Y) = (-Y) \cdot X = X-Y,$$

so μ is Gaussian in the sense of Bernstein.

2. Assume the probability measure μ on $\boldsymbol{H}$ is Gaussian in the sense of Bernstein, and let X and Y be independent $\boldsymbol{H}$-valued random variables on $(\Omega, \mathscr{B}, P)$ for which (i) and (ii) of Definition 1 hold. It follows from (ii) that

$$X \cdot Y + Y \cdot X = X + Y + \tfrac{1}{2}[X, Y] + Y + X + \tfrac{1}{2}[Y, X] = 2(X+Y)$$

and

$$X \cdot (-Y) + (-Y) \cdot X = X - Y - \tfrac{1}{2}[X, Y] - Y + X - \tfrac{1}{2}[Y, X] = 2(X-Y)$$

are independent, so by the Bernstein Theorem on $(\boldsymbol{R}^3, +)$ the probability measure μ is Gaussian on $(\boldsymbol{R}^3, +)$. On the other hand, by Remark 1,

$$2(X \cdot Y) - (X \cdot Y + Y \cdot X) = 2(X+Y) + [X, Y] - 2(X+Y) = [X, Y]$$

and

$$-\big(X \cdot (-Y) + (-X) \cdot Y\big) = -(X - Y - \tfrac{1}{2}[X, Y] - X + Y - \tfrac{1}{2}[X, Y]) = [X, Y]$$

have to be independent, so $[X, Y] \overset{\text{a.s.}}{=} c \in \boldsymbol{H}$. But since X, Y are i.i.d., we have $\mathscr{L}([X, Y]) = \mathscr{L}([Y, X]) = \mathscr{L}(-[X, Y])$, so $c = 0$, i.e.

$$[X, Y] = \left(0, 0, \det\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\right) \overset{\text{a.s.}}{=} 0$$

$$(X = (X_1, X_2, X_3),\ Y = (Y_1, Y_2, Y_3) \in \boldsymbol{R}^3).$$

So (X_1, X_2), (Y_1, Y_2) have to be linearly dependent a.s., which yields that μ is concentrated on a plane containing the x_3-axis. ∎

Let G be a Lie group, and $M^1(G)$ the set of probability measures on G. A continuous convolution semigroup (c.c.s.) $\{\mu_t\}_{t \geqslant 0}$ on G is a map

$$[0, \infty[\ni t \mapsto \mu_t \in M^1(G)$$

which is continuous with respect to the weak topology on $M^1(G)$ and which satisfies

$$\mu_{t+s} = \mu_t * \mu_s \qquad (t, s \geqslant 0)$$

and $\mu_0 = \delta_e$ (the Dirac probability measure at e). The *generating distribution* of $\{\mu_t\}_{t \geqslant 0}$ is given by

$$\mathscr{A}(f) = \lim_{t \to 0+} \frac{1}{t} \int_G [f(x) - f(e)]\, \mu_t(dx)$$

for C^∞-functions f on G with compact support (cf. [4], Satz 1). See [4] also for further details on generating distributions.

Remark 2. Let μ be as in Theorem 1 and let E be a plane which contains the x_3-axis and supports μ. There is a c.c.s. $\{\mu_t\}_{t \geqslant 0}$ on $(E, +)$ such that $\mu_1 = \mu$ and whose generating distribution has the form

$$\mathscr{A}(f) = \sum_{i=1}^{3} b_i \frac{\partial}{\partial x_i} f(0) + \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(0), \tag{1}$$

where the matrix $(a_{ij})_{1 \leqslant i,j \leqslant 3}$ is symmetric and positive semidefinite. But, by the position of E, $\{\mu_t\}_{t \geqslant 0}$ is also a c.c.s. on $\boldsymbol{H}$. So any probability measure on $\boldsymbol{H}$ which is Gaussian in the sense of Bernstein is Gaussian in the sense that it is embeddable into a c.c.s. on $\boldsymbol{H}$ with generating distribution of the form (1).

Remark 3. Clearly, if μ is a probability measure in the sense of Bernstein on $\boldsymbol{H}$, then we have $\mu^{*n}(B) = \mu(B/\sqrt{n})$ for $n \in N$ and every Borel subset $B \subset \boldsymbol{H}$. We do not know about the converse.

Remark 4. Unfortunately, our reasoning does not seem to work for Heisenberg groups of higher dimension, since the Lie bracket has to be interpreted geometrically as a determinant.

Remark 5. Since it turns out that with the above definition the Gauss measures in the sense of Bernstein are uninteresting from the point of view of the group structure, the question remains open if there are "better" extensions of the classical Bernstein property to the non-commutative case. The situation is somewhat similar to that in [1], Proposition 3, where it is shown that those Gauss measures which are in some sense the most "natural" ones as far as the group structure is concerned are not stable in the sense of Tortrat.

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