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HILBERT SPACE VALUED TRACES AND MULTIPLE STRATONOVICH INTEGRALS WITH STATISTICAL APPLICATIONS

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Abstract. Multiple Stratonovich integrals (MSI) with respect to the Wiener process and the Brownian bridge are defined for a class of kernels having k-th order τ-traces which are, in general, different from the traces investigated in earlier work. Asymptotic distributions of V-statistics are derived and the limiting distribution expressed in terms of appropriate MSI. Another application yields an alternative proof of Filippova's theorem on the limiting distribution of von Mises statistical functions.

1. Introduction. The study of Hilbert space valued traces and their connection with multiple Stratonovich integrals (MSI) originated, at least to our knowledge, in a paper of Hu and Meyer [4] in which a new approach to Feynman integrals was presented. In making this approach rigorous, Johnson and Kallianpur [7] introduced several different definitions of traces of which the limiting trace turned out to be the most appropriate one for the proof of the formulae in [4]. The MSI of [7] (the term "Stratonovich integral" was not used in the paper) were defined by using the ideas of lifting. While interesting from the point of view of furnishing formulae for certain types of Feynman integrals, these Stratonovich integrals do not meet the requirements of statistical applications since they are based essentially on Hilbert space techniques and do not take into account the values on the diagonals.

In the present paper, we take a fresh look at the problem. The τ-traces introduced in Section 2 are defined for a subclass (denoted by $\mathscr{S}_p^1$) of the L^2-space of p-th order symmetric kernels. $\mathscr{S}_p^1$ is made into a Hilbert space under a new inner product in such a manner that each of the k-th order τ-traces ($k = 0, 1, \ldots, [p/2]$) is a continuous map from $\mathscr{S}_p^1$ to $L^2[0, 1]^{p-2k}$.

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Results relating MSI to multiple Wiener integrals similar to those obtained in [7] are derived in Section 2. In Section 3, MSI are defined with respect to the Brownian bridge and a Hu–Meyer type formula is proved. The latter result is used in connecting the asymptotic distribution of a U-statistic with that of a V-statistic. Sections 4 and 5 are devoted to statistical applications of our results. In Theorem 4.3 the limiting distribution of a V-statistic is derived in terms of MSI. It is natural that Stratonovich integrals are involved since a V-statistic (in contrast to a U-statistic) allows repeated indices (see Definition 4.2). Hoeffding's pioneering 1948 result [3] is mentioned as a corollary to Theorem 4.3. An application to the asymptotic distribution of von Mises differentiable statistical functionals is made in Section 5. An alternative proof of Filippova's result is given in Theorem 5.3 where the limit is obtained as an MSI with respect to the Wiener process which is shown to be equivalent to the MSI with respect to the Brownian bridge obtained in [2].

2. Hilbert space valued traces and multiple stochastic integrals. In this section we will introduce the multiple Stratonovich integral. This integral, in general, is different from that considered by Johnson and Kallianpur [7], though the two integrals agree for step functions. The Stratonovich integral is closely tied to certain Hilbert space valued traces. In this section we will introduce these traces and discuss their connection with the "limiting traces" of Johnson and Kallianpur.

Definition 2.1 (the class $\mathcal{S}_p$ of step functions). A real valued symmetric function f_p on $[0, 1]^p$ *is in the class* $\mathcal{S}_p$ *of step functions* iff there exists a partition $\{0 = t_1 < \ldots < t_m < t_{m+1} = 1\}$ of $[0, 1]$ and constants $\{a_{i_1,\ldots,i_p};\ i_1, \ldots, i_p = 0, 1, 2, \ldots, m\}$ such that

$$f_p(s_1, \ldots, s_p) = a_{i_1,\ldots,i_p} \quad \text{if } (s_1, \ldots, s_p) \in \Delta_{i_1} \times \ldots \times \Delta_{i_p}, \tag{2.1}$$

where $\Delta_{i_j} = (t_{i_j}, t_{i_j+1}]$ if $1 \leqslant i_j \leqslant m$ and $\Delta_{i_j} = \{0\}$ if $i_j = 0$, $j = 1, 2, \ldots, p$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{W_t; 0 \leqslant t \leqslant 1\}$ be a Wiener process on this space. We first define the multiple Stratonovich integral with respect to the Wiener process for integrands in $\mathcal{S}_p$. This integral turns out to be the same as the multiple Stratonovich integral of Johnson and Kallianpur which will be denoted by $\delta_p^s(\cdot)$.

Definition 2.2 (multiple Stratonovich integral for functions in $\mathcal{S}_p$). Let $f_p \in \mathcal{S}_p$ be given by (2.1). Define the *multiple Stratonovich integral* (MSI) of f_p as

$$\delta_p(f_p) := \sum_{i_1,\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p}\big(W(t_{i_1+1}) - W(t_{i_1})\big)\ldots\big(W(t_{i_p+1}) - W(t_{i_p})\big). \tag{2.2}$$

We will now briefly recall the limiting traces as introduced in [7] and then give a representation for $\delta_p(f_p)$ in terms of those traces.

Definition 2.3 (limiting traces). Let f_p be a symmetric function in $L^2[0, 1]^p$. Fix k, $1 \leqslant k \leqslant [p/2]$. Suppose that for every complete orthonormal system (CONS) $\{\phi_i\}$ for $L^2[0, 1]$

$$(2.3)\qquad \sum_{i_1,\ldots,i_k=1}^{N} \sum_{i_{2k+1},\ldots,i_p=1}^{N} < f_p, \phi_{i_1}\otimes\phi_{i_1}\ldots\phi_{i_k}\otimes\phi_{i_k}\otimes\phi_{i_{2k+1}}\ldots\phi_{i_p} > \phi_{i_{2k+1}}\ldots\phi_{i_p}$$

converges in $L^2[0, 1]^{p-2k}$ to a limit which is independent of the choice of the CONS $\{\phi_i\}$. Then we say that the *k-th limiting trace* for f_p exists, which by definition is the limit of the series in (2.3) and is denoted by $\overrightarrow{\mathrm{Tr}}^k f_p$. $\overrightarrow{\mathrm{Tr}}^0 f_p$ is defined to be the same as f_p.

The following proposition relates the MSI δ_p with the multiple integral of Johnson and Kallianpur, and therefore with multiple Wiener integrals through the Hu–Meyer formula.

Proposition 2.4. *Let $f_p \in \mathscr{S}_p$. Then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for all k, $0 \leqslant k \leqslant [p/2]$, and we have*

$$(2.4)\qquad \delta_p(f_p) = \delta_p^s(f_p) = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p).$$

Furthermore,

$$(2.5)\qquad \mathrm{E}[\delta_p(f_p)]^2 \leqslant C\{ \int_{[0,1]^p} f_p^2(t_1, \ldots, t_p)dt_1\ldots dt_p + \sum_{k=1}^{[p/2]} \int_{[0,1]^{p-k}} f_p^2(t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p)dt_1\ldots dt_k dt_{2k+1}\ldots dt_p\},$$

where

$$C_{p,k} = \frac{p!}{(p-2k)!\,2^k k!},$$

I_j is the j-fold multiple Wiener integral, and C is an appropriate constant.

Proof. Let f_p be given by equation (2.1). Define a CONS $\{e_i\}$ for $L^2[0, 1]$ so that the first m elements of the CONS are given as follows. For $i = 1, 2, \ldots, m$,

$$(2.6)\qquad e_i(t) := \frac{1}{|\Delta_i|^{1/2}} I_{(t_i,t_{i+1}]}(t),$$

where $|\Delta_i| = (t_{i+1} - t_i)$. Then it is clear that

$$(2.7)\qquad f_p = \sum_{i_1,\ldots,i_p=1}^{m} b_{i_1,\ldots,i_p} e_{i_1}\otimes\ldots\otimes e_{i_p},$$

where $b_{i_1,\dots,i_p} = a_{i_1,\dots,i_p}|\Delta_{i_1}|^{1/2}\dots|\Delta_{i_p}|^{1/2}$, $i_j = 1, 2, \dots, m$, $j = 1, 2, \dots, p$. Therefore, by Proposition 3.2 of [7], $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for all $k = 0, 1, \dots, [p/2]$ and, by Theorem 5.1 of [7], $\delta_p^s(f_p)$ exists and is given by the following formulas:

$$\delta_p^s(f_p) = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p), \tag{2.8a}$$

$$\delta_p^s(f_p) = \sum_{i_1,\dots,i_p=1}^{m} b_{i_1,\dots,i_p} I_1(e_{i_1})\dots I_1(e_{i_p}). \tag{2.8b}$$

Hence from (2.6) and (2.7) we have

$$\delta_p^s(f_p) = \sum_{i_1,\dots,i_p=1}^{m} a_{i_1,\dots,i_p}\big(W(t_{i_1+1}) - W(t_{i_1})\big)\dots\big(W(t_{i_p+1}) - W(t_{i_p})\big) = \delta_p(f_p).$$

Therefore (2.4) is proved. By the orthogonality of multiple Wiener integrals of different orders it follows from (2.4) that

$$\mathrm{E}[\delta_p(f_p)]^2 \leqslant C \sum_{k=0}^{[p/2]} \int_{[0,1]^{p-2k}} [\![\overrightarrow{\mathrm{Tr}}^k f_p(t_{2k+1}, \dots, t_p)]\!]^2\, dt_{2k+1}\dots dt_p. \tag{2.9}$$

Finally, we obtain (2.5) from (2.9) and observing that

$$\begin{aligned}\int_{[0,1]^k} f_p(t_1, t_1, \dots, t_k, t_k, \cdot)dt_1\dots dt_k &= \sum_{i_1,\dots,i_p=1}^{m} b_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p} e_{i_{2k+1}} \otimes \dots \otimes e_{i_p}(\cdot) \\ &= \overrightarrow{\mathrm{Tr}}^k f_p(\cdot) \quad \text{for all } k = 1, \dots, [p/2],\end{aligned}$$

the last step following from Theorem 3.1 of [7]. ∎

The inequality in (2.5) will enable us to extend the domain of definition of the integral to a larger class by a denseness argument.

Let us first define the following inner product on $\mathscr{S}_p$:

$$\langle f_p, g_p\rangle_{*,p} := \sum_{k=0}^{[p/2]} \langle {}^k f_p, {}^k g_p\rangle_k, \quad f_p, g_p \in \mathscr{S}_p, \tag{2.10}$$

where $\langle\cdot,\cdot\rangle_k$ is the inner product in $L^2[0, 1]^{p-k}$ and ${}^k f_p$: $[0, 1]^{p-k} \to \boldsymbol{R}$ for $k = 1, 2, \dots, [p/2]$ is defined as

$${}^k f_p(t_1, \dots, t_{p-k}) := f_p(t_1, t_1, \dots, t_k, t_k, t_{k+1}, \dots, t_{p-k}) \tag{2.11}$$

with ${}^0 f_p \equiv f_p$. Define ${}^k g_p$ similarly.

Let $\mathscr{S}_p^*$ be the completion of $\mathscr{S}_p$ in the above inner product. We denote $\langle f_p, f_p\rangle_{*,p}$ by $\|f_p\|_{*,p}^2$. The multiple Stratonovich integral for elements in $\mathscr{S}_p^*$ is defined as follows.

Definition 2.5 (multiple Stratonovich integral for elements in $\mathscr{S}_p^*$). Let $f_p \in \mathscr{S}_p^*$ and let $\{f_{p,n}\}$ be a sequence in $\mathscr{S}_p$ such that $\|f_{p,n} - f_p\|_{*,p}$ converges to 0 as $n \to \infty$. From (2.5) we infer that $\delta_p(f_{p,n})$ converges in $L^2(\Omega)$ as $n \to \infty$.

Define the *multiple Stratonovich integral* of f_p, denoted by $\delta_p(f_p)$, as

(2.12) $$\delta_p(f_p) := L^2(\Omega)\text{-}\lim_{n\to\infty} \delta_p(f_{p,n}).$$

There is a useful identification of an element of $\mathscr{S}_p^*$ with an equivalence class of functions which we discuss below.

Let $\mathscr{S}_p^1$ be the class of real-valued measurable symmetric functions f_p on $[0, 1]^p$ such that

(2.13) $$\int_{[0,1]^{p-k}} [{}^k f_p]^2(t_1, \ldots, t_{p-k}) dt_1 \ldots dt_{p-k} < \infty \quad \text{for all } k \ (0 \leqslant k \leqslant [p/2]),$$

where ${}^k f_p$ is defined as in (2.11). We will employ the convention that $\mathscr{S}_0^* = \mathscr{S}_0^1 = \boldsymbol{R}$ and that $\|\cdot\|_{*,0}$ is the usual Euclidean distance in $\boldsymbol{R}$.

We introduce an equivalence relation (denoted by $\sim$) in $\mathscr{S}_p^1$ as follows: For $f_p, g_p \in \mathscr{S}_p^1$, we say that $f_p \sim g_p$ iff

$${}^k f_p = {}^k g_p \text{ a.e. } [0, 1]^{p-k} \quad \text{for all } k \ (0 \leqslant k \leqslant [p/2]).$$

Denote by $[f_p]$ the equivalence class generated by f_p under the above equivalence relation and let $[\mathscr{S}_p^1]$ be defined as

(2.14) $$[\mathscr{S}_p^1] := \{[f_p] : f_p \in \mathscr{S}_p^1\}.$$

We will show now that the class $\mathscr{S}_p^*$ can be identified with $[\mathscr{S}_p^1]$.

Let $g_p \in \mathscr{S}_p^*$ and let $\{f_{p,n}\}$ be a Cauchy sequence in $\mathscr{S}_p$ such that $f_{p,n} \to g_p$ in the norm described above. Since $f_{p,n}$ is Cauchy in the $\|\cdot\|_{p,*}$-norm, ${}^k f_{p,n}$ is Cauchy in the $L^2[0, 1]^{p-2k}$-norm, and hence there exists $h_{p-k} \in L^2[0, 1]^{p-k}$ such that, for $0 \leqslant k \leqslant [p/2]$, ${}^k f_{p,n} \to h_{p-k}$ in $L^2[0, 1]^{p-k}$.

Now define a real-valued function f_p on the set $\{(t_1, \ldots, t_p) : 0 \leqslant t_1 \leqslant \leqslant t_2 \leqslant \ldots \leqslant t_p \leqslant 1\}$ as follows:

(2.15) $$f_p(t_1, \ldots, t_p) := \begin{cases} h_p(t_1, \ldots, t_p) & \text{if } 0 \leqslant t_1 < t_2 < \ldots < t_p \leqslant 1, \\ h_{p-k}(t_1, t_3, \ldots, t_{2k-1}, t_{2k+1}, \ldots, t_p) & \\ \text{if } 0 \leqslant t_1 = t_2 < t_3 = t_4 < \ldots < t_{2k-1} = t_{2k} < t_{2k+1} < \ldots < t_p \leqslant 1, & \\ \qquad 1 \leqslant k \leqslant [p/2], & \\ 0 & \text{otherwise.} \end{cases}$$

Extend f_p to all of $[0, 1]^p$ by symmetry. Then ${}^k f_p = h_{p-k} \in L^2[0, 1]^{p-k}$, so that $f_p \in \mathscr{S}_p^1$. Identify the element $g_p \in \mathscr{S}_p^*$ with $[f_p]$. With this identification, $\mathscr{S}_p^* \subseteq [\mathscr{S}_p^1]$.

Next we show that $\mathscr{S}_p$ is dense in $[\mathscr{S}_p^1]$, and since $\mathscr{S}_p^*$ is the closure of $\mathscr{S}_p$, we have $[\mathscr{S}_p^1] \subseteq \mathscr{S}_p^*$, proving that there is a 1-1 correspondence between $\mathscr{S}_p^*$ and $[\mathscr{S}_p^1]$.

PROPOSITION 2.6. *Let $f_p: [0, 1]^p \to \mathbf{R}$ be in $\mathscr{S}_p^1$. Then there exists a sequence $\{f_{p,n}\}$ in $\mathscr{S}_p$ such that*

$$\|f_{p,n}-f_p\|_{*,p} \to 0 \quad \text{as } n \to \infty.$$

Moreover, if $g_p \sim f_p$, then

$$\|f_{p,n}-g_p\|_{*,p} \to 0.$$

Proof. We need to show that there exists a sequence $\{f_{p,n}\}$ in $\mathscr{S}_p$ such that the following relations hold:

(2.16a) $$\int_{[0,1]^p} [f_{p,n}-f_p]^2 (t_1, \ldots, t_p) dt_1 \ldots dt_p \to 0 \quad \text{as } n \to \infty,$$

(2.16b)

$$\int_{[0,1]^{p-k}} [f_{p,n}-f_p]^2 (t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p) dt_1 \ldots dt_k dt_{2k+1} \ldots dt_p \to 0$$
$$\text{as } n \to \infty \quad \text{for all } k = 1, \ldots, [p/2].$$

To motivate the idea of the proof we consider first the case $p = 2$. Note that we can find a sequence of step functions $\{g_{2,n}\}$ on $L^2[0, 1]^2$ and another sequence of step functions $\{g_{1,n}\}$ on $L^2[0, 1]$ such that, as $n \to \infty$,

$$\int_{[0,1]^2} |g_{2,n}(t, s)-f_2(t, s)|^2 dt ds \to 0 \quad \text{and} \quad \int_{[0,1]} |g_{1,n}(t)-f_2(t, t)|^2 dt \to 0.$$

Define

$$f_{2,n}(t, s) := g_{2,n}(t, s)-\delta(t, s) g_{2,n}(t, s)+\delta(t, s) g_{1,n}(t),$$

where $\delta(t, s)$ is 1 if $t = s$ and is 0 otherwise. Then, clearly, as $n \to \infty$,

$$\int_{[0,1]^2} |f_{2,n}(t, s)-f_2(t, s)|^2 dt ds \to 0 \quad \text{and} \quad \int_{[0,1]} |f_{2,n}(t, t)-f_2(t, t)|^2 dt \to 0.$$

Hence we have the desired sequence, the only problem being that $f_{2,n}$ is not a sequence of step functions. Therefore we need to replace $\delta(t, s)$ by a sequence of step functions converging to it. This is the essential idea of the proof.

Returning to the case of a general p, let us assume for the sake of notational simplicity that p is odd. Note that for each fixed r, $1 \leqslant r \leqslant [p/2]$, we infer, by the denseness of step functions on $[0, 1]^{p-r}$ in $L^2[0, 1]^{p-r}$ that there exists a sequence $\{g_{p-r,n}\}$ of step functions, $g_{p-r,n}: [0, 1]^{p-r} \to \mathbf{R}$, such that

$$\int_{[0,1]^{p-r}} [g_{p-r,n}(t_1, t_2, \ldots, t_r, t_{2r+1}, \ldots, t_p)$$
$$-f_p(t_1, t_1, \ldots, t_r, t_r, t_{2r+1}, \ldots, t_p)]^2 dt_1 \ldots dt_r dt_{2r+1} \ldots dt_p \to 0 \quad \text{as } n \to \infty,$$

(2.17) $$\int_{[0,1]^p} [g_{p,n}(t_1, \ldots, t_p)-f_p(t_1, \ldots, t_p)]^2 dt_1 \ldots dt_p \to 0 \quad \text{as } n \to \infty.$$

Let $\Pi_n = \{0 = \tau_1 < \tau_2 < \ldots < \tau_{l+1} = 1\}$ be a sequence of partitions of $[0, 1]$ (where the dependence of τ_i and l on n is suppressed in the notation) with $|\Pi_n| \to 0$ as $n \to \infty$. We now construct a new set of sequences $\{h_{p,n}^{(r)}\}$ and $\{f_{p,n}^{(r)}\}$ of step functions on $[0, 1]^p$, $0 \leqslant r \leqslant [p/2]$, as follows:

Define $h_{p,n}^{(0)} := 0$. For $1 \leqslant r \leqslant [p/2]$, define

$$h_{p,n}^{(r)}(s_1, \ldots, s_p) := \begin{cases} 1 & \text{if } (s_{2r-1}, s_{2r}) \in \bigcup_{i=1}^{l} (\tau_i, \tau_{i+1}]^2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

The functions $h_{p,n}^{(r)}$ constructed above will play the role of the approximating sequence for the delta function.

Also, define $f_{p,n}^{(0)} := g_{p,n}$. For $r = 1, 2, \ldots, [p/2]$, define

$$\begin{aligned} f_{p,n}^{(r)}(s_1, \ldots, s_p) &= f_{p,n}^{(r-1)}(s_1, \ldots, s_p) - h_{p,n}^{(r)}(s_1, \ldots, s_p) f_{p,n}^{(r-1)}(s_1, \ldots, s_p) \\ &\quad + h_{p,n}^{(r)}(s_1, \ldots, s_p) g_{p-r,n}(s_1, s_3, \ldots, s_{2r-1}, s_{2r+1}, s_{2r+2}, \ldots, s_p). \end{aligned} \tag{2.19}$$

We will show that $f_{p,n} := f_{p,n}^{([p/2])}$ is the required sequence, i.e., it satisfies (2.16a) and (2.16b). In fact, we show something more, namely, for each fixed r ($r = 1, 2, \ldots, [p/2]$), as $n \to \infty$,

$$\begin{gathered} \int_{[0,1]^p} [f_{p,n}^{(r)} - f_p]^2 (t_1, \ldots, t_p) \, dt_1 \ldots dt_p \to 0, \\ \int_{[0,1]^{p-m}} [f_{p,n}^{(r)} - f_p]^2 (t_1, t_1, \ldots, t_m, t_m, t_{2m+1}, \ldots, t_p) dt_1 \ldots dt_m dt_{2m+1} \ldots dt_p \to 0 \\ \text{for } m = 1, 2, \ldots, r. \end{gathered} \tag{2.20}$$

To prove (2.20), we will first show it for $r = 1$, and then assuming that it holds for $r = 1, 2, \ldots, k-1$ ($k-1 < [p/2]$) we will show that it holds for $r = k$. This will prove that (2.20) holds for $r = 0, 1, \ldots, [p/2]$.

Let us first consider the case $r = 1$. Note that from (2.19) we obtain

$$\begin{aligned} &\int_{[0,1]^p} [f_{p,n}^{(1)} - f_p]^2 (s_1, \ldots, s_p) ds_1 \ldots ds_p \\ &\leqslant 3 \int_{[0,1]^p} [f_{p,n}^{(0)} - f_p]^2 (s_1, \ldots, s_p) ds_1 \ldots ds_p + 3 \int_{A_n^{(1,0)}} [f_{p,n}^{(0)}]^2 (s_1, \ldots, s_p) ds_1 \ldots ds_p \\ &\quad + 3 \int_{A_n^{(1,0)}} [g_{p-1,n}]^2 (s_1, s_3, s_4, \ldots, s_p) ds_1 ds_2 \ldots ds_p, \end{aligned} \tag{2.21}$$

where

$$A_n^{(k,0)} = \Big((s_1, \ldots, s_p) \in [0, 1]^p \colon (s_{2k-1}, s_{2k}) \in \bigcup_{i=1}^{l} (\tau_i, \tau_{i+1}]^2\Big), \quad k = 1, 2, \ldots, [p/2].$$

Since $\lambda^{\otimes p}(A_n^{(1,0)}) \to 0$ as $n \to \infty$ (λ being the Lebesgue measure on $[0, 1]$), the second and the third terms on the right-hand side of (2.21) converge to 0 as $n \to \infty$.

The first term on the right-hand side of (2.21) converges to 0 as $n \to \infty$ by (2.17) and the fact that $f_{p,n}^{(0)} \equiv g_{p,n}$. Again

$$\int_{[0,1]^{p-1}} [f_{p,n}^{(1)} - f_p]^2 (s_1, s_1, s_3, s_4, \ldots, s_p)\, ds_1 ds_3 ds_4 \ldots ds_p$$

$$= \int_{[0,1]^{p-1}} [(f_{p,n}^{(0)} - 1.f_{p,n}^{(0)} - f_p)(s_1, s_1, s_3, s_4, \ldots, s_p)$$

$$+ 1.g_{p-1,n}(s_1, s_3, s_4, \ldots, s_p)]^2\, ds_1 ds_3 ds_4 \ldots ds_p$$

$$= \int_{[0,1]^{p-1}} [g_{p-1,n}(s_1, s_3, s_4, \ldots, s_p)$$

$$- f_p(s_1, s_1, s_3, s_4, \ldots, s_p)]^2\, ds_1 ds_3 ds_4 \ldots ds_p.$$

The last expression converges to zero as $n \to \infty$ by (2.17). Hence from the above observations we see that (2.20) holds for $r = 1$.

Now suppose that (2.20) holds for $r = 1, 2, \ldots, k-1$, where $k-1 < [p/2]$. We will show that (2.20) holds for $r = k$. Note initially that for m strictly less than k we infer using (2.19) and arguing as in (2.21) that, as $n \to \infty$,

(2.22)

$$\int_{[0,1]^{p-m}} [f_{p,n}^{(k)} - f_p]^2 (t_1, t_1, \ldots, t_m, t_m, t_{2m+1}, \ldots, t_p)\, dt_1 \ldots dt_m dt_{2m+1} \ldots dt_p \to 0.$$

Also it can be similarly seen that

$$\int_{[0,1]^{p-m}} [f_{p,n}^{(k)} - f_p]^2 (t_1, \ldots, t_p)\, dt_1 \ldots dt_p \to 0 \quad \text{as } n \to \infty.$$

Consider now the case $r = k$, $m = k$. Then

$$\int_{[0,1]^{p-k}} [f_{p,n}^{(k)} - f_p]^2 (t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p)\, dt_1 \ldots dt_k dt_{2k+1} \ldots dt_p$$

$$= \int_{[0,1]^{p-k}} [(f_{p,n}^{(k-1)} - 1.f_{p,n}^{(k-1)} - f_p)(t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p)$$

$$+ 1.g_{p-k,n}(t_1, \ldots, t_k, t_{2k+1}, \ldots, t_p)]^2\, dt_1 \ldots dt_k dt_{2k+1} \ldots dt_p$$

$$= \int_{[0,1]^{p-k}} [g_{p-k,n}(t_1, \ldots, t_k, t_{2k+1}, \ldots, t_p)$$

$$- f_p(t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p)]^2\, dt_1 \ldots dt_k dt_{2k+1} \ldots dt_p$$

and the last expression converges to zero as $n \to \infty$ by (2.17) with $r = k$.

From the above observations we see that (2.20) holds for $r = k$. Therefore, as remarked earlier, this proves that (2.20) holds for $r = 1, \ldots, [p/2]$. Substituting $r = [p/2]$ in (2.20) we have, as $n \to \infty$,

$$\int_{[0,1]^p} [f_{p,n}^{([p/2])} - f_p]^2 (t_1, \ldots, t_p)\, dt_1 \ldots dt_p \to 0,$$

$$\int_{[0,1]^{p-m}} [f_{p,n}^{([p/2])} - f_p]^2 (t_1, t_1, \ldots, t_m, t_m, t_{2m+1}, \ldots, t_p)\, dt_1 \ldots dt_m dt_{2m+1} \ldots dt_p \to 0$$

for $m = 1, 2, \ldots, [p/2]$.

Therefore the sequence $f_{p,n} := f_{p,n}^{([p/2])}$ satisfies (2.16a) and (2.16b). ∎

Henceforth, we will not distinguish between the equivalence class $[f_p]$ and the function $f_p \in \mathcal{S}_p^1$ that belongs to it. We will write $\delta_p(f_p)$ for $\delta_p([f_p])$.

The multiple Stratonovich integral introduced in this section can be represented as a sum of multiple Wiener integrals through a formula which is very similar to the Hu–Meyer formula (2.4), though the traces appearing in this formula are in general different from the limiting traces that appear in (2.4). We discuss below the appropriate notion of a trace for this setup.

DEFINITION 2.7. Let $f_p \in \mathcal{S}_p^1$. Fix $1 \leqslant k \leqslant [p/2]$. Then the *k-th τ-trace* of f_p, denoted by $\tau^k f_p$, is an element of $\mathcal{S}_{p-2k}^1$ defined as

(2.23) $$\tau^k f_p(\cdot) := \int_{[0,1]^k} f_p(t_1, t_1, \ldots, t_k, t_k, \cdot)\, dt_1 \ldots dt_k.$$

Define $\tau^0 f_p := f_p$.

Note that if $g_p \sim f_p$, then $\tau^k g_p \sim \tau^k f_p$ for all k, $0 \leqslant k \leqslant [p/2]$.

We state below the following important property of k-th τ-traces as continuous maps.

PROPOSITION 2.8. *If* $\{f_{p,n}\}$, $f_p \in \mathcal{S}_p^1$ *and* $\|f_{p,n} - f_p\|_{*,p} \to 0$ *as* $n \to \infty$, *then*

$$\|\tau^k f_{p,n} - \tau^k f_p\|_{*,p-2k} \to 0 \ \text{as} \ n \to \infty \quad \text{for all} \ k, 0 \leqslant k \leqslant [p/2].$$

In particular, the map $\tau^k \colon \mathcal{S}_p^1 \to L^2[0,1]^{p-2k}$ *is continuous.*

The following proposition is the analogue of the Hu–Meyer formula for δ_p.

PROPOSITION 2.9. *Let* $f_p \in \mathcal{S}_p^1$. *Then*

(2.24) $$\delta_p(f_p) = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(\tau^k(f_p)).$$

Proof. Observe that (2.24) holds for $f_p \in \mathcal{S}_p$ by Proposition 2.4 and the fact that, for such functions, $\overrightarrow{\mathrm{Tr}}^k f_p$ is the same as $\tau^k f_p$. Next let $\{f_{p,n}\}$ be a sequence of functions in $\mathcal{S}_p$ such that $\|f_{p,n} - f_p\|_{*,p} \to 0$. Note that such a sequence exists from Proposition 2.6. It follows from (2.5) that $\delta_p(f_{p,n}) \xrightarrow{L^2} \delta_p(f_p)$. Furthermore, since (2.24) holds for each $f_{p,n}$, we have

(2.25) $$\delta_p(f_{p,n}) = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(\tau^k(f_{p,n})).$$

Proposition 2.8 implies that $\tau^k(f_{p,n}) \overset{L^2[0,1]^{p-2k}}{\to} \tau^k(f_p)$ as $n \to \infty$. Therefore combining the above observations and taking the limit as $n \to \infty$ in (2.25), we have the result. ∎

Johnson and Kallianpur showed that the multiple Wiener integral for integrands which have limiting traces of all orders and are such that these limiting traces are consistent with the second order traces can be expressed as a sum of MSI of Johnson and Kallianpur. We show in the following proposition that a similar result holds for the multiple Stratonovich integrals introduced in this section.

PROPOSITION 2.10. *Let* $f_p \in \mathscr{S}_p^1$. *Then*

$$I_p(f_p) = \sum_{k=0}^{[p/2]} (-1)^k C_{p,k}\, \delta_{p-2k}\big(\tau^k(f_p)\big). \tag{2.26}$$

The proof of the proposition is similar to that of Proposition 2.9, i.e., (2.26) is first shown for elements in $\mathscr{S}_p$ and then, by using Proposition 2.8, (2.5) and the usual denseness arguments, the result is extended to elements in $\mathscr{S}_p^1$.

It is clear from Proposition 2.9 that $\delta_p^s(f_p)$ and $\delta_p(f_p)$ will agree if the limiting traces $\overrightarrow{\mathrm{Tr}}^k(f_p)$ are the same as the traces $\tau^k f_p$. As we remarked earlier, τ-traces are in general not the same as limiting traces, though for functions in $\mathscr{S}_p$ they agree. The following proposition gives another situation in which the two traces are the same.

PROPOSITION 2.11. *Let* $f_p \in L^2[0, 1]^p$ *be symmetric and continuous.*

(i) *Then the Solé and Utzet trace* $T^k f_p$ (see [11] for the definition of $T^k f_p$) *exists and is the same as* $\tau^k f_p$ *for all* $k = 0, 1, \ldots, [p/2]$.

(ii) *Suppose that for some* k $(1 \leqslant k \leqslant [p/2])$ $\overrightarrow{\mathrm{Tr}}^k f_p$ *exists. Then*

$$\overrightarrow{\mathrm{Tr}}^k f_p(\cdot) = \tau^k f_p(\cdot) = \int_{[0,1]^k} f(t_1, t_1, \ldots, t_k, t_k, \cdot)\, dt_1 \ldots dt_k. \tag{2.27}$$

Proof. We will prove (ii). For the proof of (i), the reader may refer to [5]. We will assume that p is odd; the case where p is even can be treated in a similar fashion. The existence of $\overrightarrow{\mathrm{Tr}}^k f_p$ implies that for every CONS $\{e_i\}$ for $L^2[0, 1]$ the following series converges in $L^2[0, 1]^{p-2k}$ as $n \to \infty$:

(2.28)

$$\sum_{i_1, \ldots, i_k, i_{2k+1}, \ldots, i_p = 1}^{n} \langle f_p, e_{i_1} \otimes e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e_{i_k} \otimes e_{i_{2k+1}} \otimes \ldots \otimes e_{i_p} \rangle\, e_{i_{2k+1}} \ldots e_{i_p}(\cdot).$$

To prove (2.27) we will use the CONS introduced by Nualart and Zakai in [8]. Let $\{\Pi_n\}$ be a sequence of partitions of $[0, 1]$ such that $|\Pi_n| \to 0$ as $n \to \infty$. We assume that Π_n is a refinement of Π_{n-1} by only one point. Set $\Pi_1 = \{0, 1\}$. Define $e_1 = 1$. For $m \geqslant 1$, e_{m+1} is defined as follows. Let α be the point of refinement in Π_{m+1}, and assume that this point is in the interval

$(d_1, d_2]$, $d_1, d_2 \in \Pi_m$. Then

$$e_{m+1}(t) := \left(\frac{d_2-\alpha}{(\alpha-d_1)(d_2-d_1)}\right)^{1/2} I_{(d_1,\alpha]}(t) - \left(\frac{\alpha-d_1}{(d_2-\alpha)(d_2-d_1)}\right)^{1/2} I_{(\alpha,d_2]}(t).$$

It can be shown (see [8]) that $\{e_m\}$ is a CONS for $L^2[0, 1]$.

Let the partition Π_n be $\{0 = \tau_1 < \ldots < \tau_{n+1} = 1\}$, where the dependence of τ_i on n has been suppressed in the notation. Let $\Delta_i \equiv (\tau_i, \tau_{i+1}]$ and $|\Delta_i| := \tau_{i+1} - \tau_i$. Let

$$(2.29)\quad S_{\Pi_n}(u_{2k+1}, \ldots, u_p)$$

$$:= \sum_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_p=1}^{n} \frac{1}{|\Delta_{i_1}|\ldots|\Delta_{i_k}||\Delta_{i_{2k+1}}|\ldots|\Delta_{i_p}|} I_{\Delta_{i_{2k+1}}}(u_{2k+1})\ldots I_{\Delta_{i_p}}(u_p)$$

$$\times \int_{\substack{\Delta_{i_1}\times\Delta_{i_1}\times\ldots\times\Delta_{i_k}\times\Delta_{i_k}\\ \times\Delta_{i_{2k+1}}\times\ldots\times\Delta_{i_p}}} f_p(t_1, t_2, \ldots, t_p)\, dt_1\ldots dt_p.$$

We will show that for all $n \geq 1$

$$(2.30)\quad S_{\Pi_n}(\cdot)$$

$$= \sum_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_p=1}^{n} \langle f_p, e_{i_1}\otimes e_{i_1}\otimes\ldots\otimes e_{i_k}\otimes e_{i_k}\otimes e_{i_{2k+1}}\otimes\ldots\otimes e_{i_p}\rangle e_{i_{2k+1}}\ldots e_{i_p}(\cdot).$$

Observe that to prove (2.30) it suffices to show it for $p = 2$ and $k = 1$, for then the general result can be obtained by iteration and noting that the span of $\{|\Delta_1|^{-1/2} I_{\Delta_1}, \ldots, |\Delta_n|^{-1/2} I_{\Delta_n}\}$ is the same as that of $\{e_1, \ldots, e_n\}$.

We will prove (2.30) for $p = 2$ $(k = 1)$ by induction on n. Note that for $n = 1$ the result holds trivially. Suppose now that the result holds for $n = m$, i.e.,

$$(2.31)\quad \sum_{i=1}^{m} \langle f_2, e_i\otimes e_i\rangle = \sum_{i=1}^{m} |\Delta_i|^{-1} \int_{\Delta_i\times\Delta_i} f_2(t_1, t_2)\, dt_1 dt_2.$$

Then

$$(2.32)\quad S_{\Pi_{m+1}} - S_{\Pi_m} = \frac{1}{d_2-\alpha} \int_{(\alpha,d_2]^2} f(t_1, t_2)\, dt_1 dt_2 + \frac{1}{\alpha-d_1} \int_{(d_1,\alpha]^2} f(t_1, t_2)\, dt_1 dt_2$$

$$- \frac{1}{d_2-d_1} \int_{(d_1,d_2]^2} f(t_1, t_2)\, dt_1 dt_2$$

$$= \frac{\alpha-d_1}{(d_2-\alpha)(d_2-d_1)} \int_{(\alpha,d_2]^2} f(t_1, t_2)\, dt_1 dt_2$$

$$+\frac{d_2-\alpha}{(\alpha-d_1)(d_2-d_1)}\int_{(d_1,\alpha]^2} f(t_1, t_2)\,dt_1\,dt_2$$

$$-\frac{2}{d_2-d_1}\int_{(d_1,\alpha]\times(\alpha,d_2]} f(t_1, t_2)\,dt_1\,dt_2.$$

Again,

$$\int_{[0,1]^2} f_2(t_1, t_2)\,e_{m+1}(t_1)\,e_{m+1}(t_2)\,dt_1\,dt_2 = \frac{d_2-\alpha}{(\alpha-d_1)(d_2-d_1)}\int_{d_1}^{\alpha}\int_{d_1}^{\alpha} f_2(t_1, t_2)\,dt_1\,dt_2$$

$$+\frac{\alpha-d_1}{(d_2-\alpha)(d_2-d_1)}\int_{\alpha}^{d_2}\int_{\alpha}^{d_2} f_2(t_1, t_2)\,dt_1\,dt_2 - 2\frac{1}{d_2-d_1}\int_{d_1}^{\alpha}\int_{\alpha}^{d_2} f_2(t_1, t_2)\,dt_1\,dt_2$$

$$= S_{\Pi_{m+1}} - S_{\Pi_m},$$

the last equality following from (2.32). Therefore, in view of (2.31), we have

$$S_{\Pi_{m+1}} = \sum_{i=1}^{m+1} |\Delta_i|^{-1}\int_{\Delta_i\times\Delta_i} f_2(t_1, t_2)\,dt_1\,dt_2 = \sum_{i=1}^{m+1}\langle f_2, e_i\otimes e_i\rangle.$$

Hence (2.31) holds with m replaced by $m+1$. As remarked before, this proves (2.30) for all $n \geqslant 1$. Now from the definition of limiting trace and (2.30) we have

(2.33) $$S_{\Pi_n} \overset{L^2[0,1]^{p-2k}}{\rightarrow} \overrightarrow{\mathrm{Tr}}^k f_p.$$

We show now that S_{Π_n} converges to $\tau^k f_p$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be such that

$$|f(t_1, \ldots, t_p) - f(s_1, \ldots, s_p)| < \varepsilon \quad \text{whenever } \max_{1\leqslant i\leqslant p} |t_i - s_i| < \delta.$$

Let $N > 0$ be such that $|\Pi_n| < \delta$ for all $n \geqslant N$. Then it can be shown from some straightforward considerations that, for all $\boldsymbol{u}\in[0, 1]^{p-2k}$,

(2.34) $$\Big|S_{\Pi_n}(\boldsymbol{u}) - \int_{[0,1]^k} f(t_1, t_1, \ldots, t_k, t_k, \boldsymbol{u})\,dt_1\ldots dt_k\Big| \leqslant \varepsilon \quad \text{for all } n \geqslant N.$$

As $\varepsilon > 0$ is arbitrary, $S_{\Pi_n}(\cdot)$ converges to $\int_{[0,1]^k} f(t_1, t_1, \ldots, t_k, t_k, \cdot)\,dt_1\ldots dt_k$ uniformly, and therefore in $L^2[0, 1]^{p-2k}$. The above observation combined with (2.33) gives the result. ■

COROLLARY 2.12. *Let $f_p \in L^2[0, 1]^p$ be continuous and symmetric. Suppose that there exists a CONS $\{\phi_i\}$ for $L^2[0, 1]$ such that*

$$f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\,\phi_{i_1}\otimes\ldots\otimes\phi_{i_p} \quad \text{and} \quad \sum_{i_1,\ldots,i_p=1}^{\infty} |a_{i_1,\ldots,i_p}| < \infty.$$

Then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists and we have $\overrightarrow{\mathrm{Tr}}^k f_p = \tau^k f_p$.

Proof. By Theorem 3.1 of [7] we infer that $\vec{\mathrm{Tr}}^k f_p$ exists, and then the result follows from Proposition 2.11. ∎

Except for the special situation considered above, the equality of $\vec{\mathrm{Tr}}^k f_p$ and $\tau^k f_p$ is not clear. In general, the following set of sufficient conditions can be given under which the limiting trace equals the τ-trace.

PROPOSITION 2.13. *Let $f_p \in \mathcal{S}_p^2$ be such that $\vec{\mathrm{Tr}}^k f_p$ exists. Suppose that there exists a CONS $\{\phi_i\}$ for $L^2[0, 1]$ such that:*

(a) *the series $\sum_{i_1,\ldots,i_p=1}^{n} \langle f_p, \phi_{i_1} \otimes \ldots \otimes \phi_{i_p} \rangle \phi_{i_1}(t_1) \ldots \phi_{i_p}(t_p)$ converges pointwise on $[0, 1]^p$ as $n \to \infty$;*

(b) $\sum_{i_1,\ldots,i_p=1}^{n} \langle f_p, \phi_{i_1} \otimes \ldots \otimes \phi_{i_p} \rangle \phi_{i_1} \phi_{i_2}(t_1) \ldots \phi_{i_{2k-1}} \phi_{i_{2k}}(t_k) \phi_{i_{2k+1}}(t_{2k+1}) \ldots \ldots \phi_{i_p}(t_p)$ *converges in $L^1[0, 1]^{p-k}$ as $n \to \infty$.*

Then $\tau^k f_p = \vec{\mathrm{Tr}}^k f_p$.

The proof, being standard, is omitted.

3. Multiple integrals with respect to a Brownian bridge. Let $(\Omega, \mathcal{F}, P)$ and $\{W_t\}$ be as in Section 2. Let $W_t^0 \equiv W_t - tW_1$ be the Brownian bridge. In this section we will discuss multiple integrals with respect to W_t^0. As in the case of the Wiener process, two types of integrals appear naturally. The first integral accounts for the diagonal contribution of the integrand and will be referred to as "the multiple Stratonovich integral (MSI) with respect to the Brownian bridge." It will be denoted by $\delta_p^0(\cdot)$. The second integral – which ignores the contribution of the diagonals like Itô's integral – will be referred to as "the multiple Itô integral (MII) with respect to the Brownian bridge." It will be denoted by $I_p^0(\cdot)$. We like to remark here that $I_p^0(\cdot)$ is not to be interpreted as an iterated integral. The Brownian bridge process does not have orthogonal increments, hence it is not straightforward to define a stochastic integral for adapted or bounded predictable integrands. In this section we will define the multiple integral as in [6] for special elementary functions and then proceed to the general case via a denseness argument. We will also obtain a relation between multiple integrals with respect to the Brownian bridge and those with respect to the Wiener process. This will be useful in connecting our result on asymptotic distribution of von Mises functionals with that of Filippova (see Section 5). Finally we will obtain a formula like the Hu–Meyer formula, connecting $\delta_p^0(\cdot)$ and $I_p^0(\cdot)$.

DEFINITION 3.1 (multiple Stratonovich integral with respect to a Brownian bridge). Let $f_p \in \mathcal{S}_p$ be given by equation (2.1). Define the *multiple Stratonovich integral of f_p with respect to the Brownian bridge $\{W_t^0\}$*, denoted by $\delta_p^0(f_p)$, as follows:

$$(3.1) \quad \delta_p^0(f_p) := \sum_{i_1,\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p} \big(W^0(t_{i_1+1}) - W^0(t_{i_1})\big) \ldots \big(W^0(t_{i_p+1}) - W^0(t_{i_p})\big).$$

To extend the definition of the integral to a larger class we use the following moment bound, due to Filippova [2], for $\delta_p^0(f_p)$ $(f_p \in \mathscr{S}_p)$:

$$\text{(3.2)} \quad \mathrm{E}[\delta_p^0(f_p)]^2 \leqslant C^* \Big\{ \int_{[0,1]^p} f_p^2(t_1, \ldots, t_p)\, dt_1 \ldots dt_p$$

$$+ \sum_{k=1}^{[p/2]} \int_{[0,1]^{p-k}} f_p^2(t_1, t_1, \ldots, t_k, t_k, t_{2k+1}, \ldots, t_p)\, dt_1 \ldots dt_k dt_{2k+1} \ldots dt_p \Big\},$$

where C^* is an appropriate constant. The moment bound (3.2) (which is the same as the bound given in (2.5) excepting a constant) enables us to extend the definition of the integral to $\mathscr{S}_p^1$, just as in the case of the Wiener process.

Definition 3.2 (multiple Stratonovich integral with respect to the Brownian bridge for integrands in $\mathscr{S}_p^1$). Let f_p: $[0, 1]^p \to \boldsymbol{R}$ be in $\mathscr{S}_p^1$ and $f_{p,n} \in \mathscr{S}_p$ be such that $\|f_{p,n} - f_p\|_{p,*} \to 0$ as $n \to \infty$. Define the *multiple Stratonovich integral of f_p with respect to the Brownian bridge* $\{W_t^0\}$, denoted by $\delta_p^0(f_p)$, as the limit in $L^2(\Omega)$ of $\delta_p^0(f_{p,n})$ (which exists by (3.2)).

Note that (3.2) holds also for $f_p \in \mathscr{S}_p^1$.

The following simple relation between $\delta_p(f_p)$ and $\delta_p^0(f_p)$ will be useful in Section 5.

Proposition 3.3. *Let $f_p \in \mathscr{S}_p^1$. Then*

$$\text{(3.3)} \qquad \delta_p(f_p) = W_1^p f_p^{(p)} + \sum_{k=1}^{p} \binom{p}{k} W_1^{p-k} \delta_k^0 (f_p^{(p-k)})$$

and

$$\text{(3.4)} \qquad \delta_p^0(f_p) = (-1)^p W_1^p f_p^{(p)} + \sum_{k=1}^{p} \binom{p}{k} (-1)^{p-k} W_1^{p-k} \delta_k (f_p^{(p-k)}),$$

where for $k = 1, \ldots, p-1$

$$f_p^{(p-k)}(t_1, \ldots, t_k) := \int_{[0,1]^{p-k}} f_p(t_1, \ldots, t_k, \boldsymbol{t})\, d\boldsymbol{t}, \qquad \boldsymbol{t} \in [0, 1]^{p-k},$$

$$f_p^{(0)} := f_p \quad \textit{and} \quad f_p^{(p)} := \int_{[0,1]^p} f_p(t_1, \ldots, t_p)\, d\boldsymbol{t}.$$

Proof. Initially, let $f_p \in \mathscr{S}_p$ be as in (2.1). Then

$$\text{(3.5)} \quad \delta_p(f_p) = \sum_{i_1, \ldots, i_p = 1}^{m} a_{i_1, \ldots, i_p} \big(W(t_{i_1+1}) - W(t_{i_1})\big) \ldots \big(W(t_{i_p+1}) - W(t_{i_p})\big)$$

$$= \sum_{i_1, \ldots, i_p = 1}^{m} a_{i_1, \ldots, i_p} \big(W^0(t_{i_1+1}) - W^0(t_{i_1}) + (t_{i_1+1} - t_{i_1}) W_1\big) \ldots$$

$$\ldots \big(W^0(t_{i_p+1}) - W^0(t_{i_p}) + (t_{i_p+1} - t_{i_p}) W_1\big)$$

$$= W_1^p \sum_{i_1,\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p}(t_{i_1+1}-t_{i_1})\ldots(t_{i_p+1}-t_{i_p})$$

$$+\sum_{k=1}^{p}\binom{p}{k} W_1^{p-k} \sum_{i_1,\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p}(W^0(t_{i_1+1})-W^0(t_{i_1}))\ldots$$

$$\ldots(W^0(t_{i_k+1})-W^0(t_{i_k}))(t_{i_{k+1}+1}-t_{i_k})\ldots(t_{i_p+1}-t_{i_p}).$$

Now, since

$$f_p^{(p-k)}(t_1,\ldots,t_k)$$
$$= \sum_{i_1,\ldots,i_k=1}^{m}\Big\{\sum_{i_{k+1},\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p}(t_{i_{k+1}+1}-t_{i_k})\ldots(t_{i_{p+1}}-t_{i_p})\Big\} I_{\Delta_{i_1}}(t_1)\ldots I_{\Delta_{i_k}}(t_k),$$

we infer from (3.5) and Definition 3.1 that

$$\delta_p(f_p) = W_1^p f_p^{(p)} + \sum_{k=1}^{p}\binom{p}{k} W_1^{p-k}\delta_k^0(f_p^{(p-k)}).$$

Let now $f_p \in \mathscr{S}_p^1$, and let $\{f_{p,n}\}$ be a sequence in $\mathscr{S}_p$ such that $\|f_{p,n}-f_p\|_{p,*}\to 0$ as $n\to\infty$. It is a simple observation that

$$\|f_{p,n}^{(p-k)}-f_p^{(p-k)}\|_{k,*}\to 0 \text{ as } n\to\infty \qquad \text{for all } k=0,1,\ldots,p.$$

From (2.5) and (3.2) we infer that $\delta_p(f_{p,n})\to\delta_p(f_p)$ and $\delta_k^0(f_{p,n}^{(p-k)})\to\delta_k^0(f_p^{(p-k)})$ in $L^2(\Omega)$ as $n\to\infty$ for all $k=1,\ldots,p$. Hence (3.3) follows for f_p on noting that it holds for $f_{p,n}$ for all n and then taking the limit as $n\to\infty$. Equation (3.4) is proved similarly. ∎

We now introduce the multiple integral with respect to the Brownian bridge that does not account for the diagonals. To construct this integral we proceed as in [6].

DEFINITION 3.4 (the class of special elementary functions ($\mathscr{S}_p^s$)). A real-valued symmetric function f_p on $[0,1]^p$ *is in* $\mathscr{S}_p^s$ iff there exist Borel measurable sets A_i $(1\leqslant i\leqslant m)$ with

$$\bigcup_{i=1}^{m} A_i = [0,1] \quad \text{and} \quad A_i\cap A_j=\emptyset \text{ if } i\neq j,$$

and real numbers $a_{i_1,\ldots,i_p}$ $(1\leqslant i_j\leqslant m,\ j=1,2,\ldots,p)$ such that $a_{i_1,\ldots,i_p}=0$ unless $i_1,i_2,\ldots,i_p$ are all distinct, and

$$f_p(s_1,\ldots,s_p)=a_{i_1,\ldots,i_p}$$
$$\text{if } (s_1,\ldots,s_p)\in A_{i_1}\times\ldots\times A_{i_p},\ i_j=1,2,\ldots,m,\ j=1,2,\ldots,p.$$

DEFINITION 3.5 (multiple Itô integral with respect to the Brownian bridge). Let $f_p\in\mathscr{S}_p^s$ be as in Definition 3.4. Define the *multiple Itô integral of* f_p *with* *respect to the Brownian bridge*, denoted by $I_p^0(f_p)$, as

$$I_p^0(f_p) := \sum_{i_1,\ldots,i_p=1}^{m} a_{i_1,\ldots,i_p} W^0(A_{i_1})\ldots W^0(A_{i_p}), \tag{3.6}$$

where $W^0(A) := W(A) - \lambda(A) W_1$ for a Borel set A in $[0, 1]$, and λ is the Lebesgue measure.

It is clear from the definition of δ_p^0 that if the sets $\{A_{i_j}\}$ above are intervals, then I_p^0 and δ_p^0 agree. The following proposition shows that in fact the two integrals are equal for a general special elementary function.

PROPOSITION 3.6. *Let $f_p \in \mathcal{S}_p^s$ be as in Definition 3.2. Then $I_p^0(f_p) = \delta_p^0(f_p)$ a.e.*

The proof is straightforward and is omitted.

Proposition 3.6 and (3.2) show that

$$\mathrm{E}[I_p^0(f_p)]^2 \leqslant \int_{[0,1]^p} f_p^2(t_1, \ldots, t_p)\, dt_1 \ldots dt_p \quad \text{for } f_p \in \mathcal{S}_p^s,$$

and therefore the definition of I_p^0 can be extended to $L^2[0, 1]^p$ by the usual denseness arguments. MII for elements in $L^2[0, 1]^p$ is denoted once more by $I_p^0(\cdot)$.

PROPOSITION 3.7. *Let $f_p \in L^2[0, 1]^p$. Then*

$$I_p(f_p) = W_1^p f_p^{(p)} + \sum_{k=1}^{p} \binom{p}{k} W_1^{p-k} I_k^0(f_p^{(p-k)}) \tag{3.7}$$

and

$$I_p^0(f_p) = (-1)^p W_1^p f_p^{(p)} + \sum_{k=1}^{p} \binom{p}{k} (-1)^{p-k} W_1^{p-k} I_k(f_p^{(p-k)}). \tag{3.8}$$

The proof follows by first proving the result for f_p in $\mathcal{S}_p^s$ and then extending in the usual manner.

We close this section with a Hu and Meyer type formula for Brownian bridge with integrands in $\mathcal{S}_p^1$. This formula will be useful in connecting the asymptotic distributions of U-statistics with that of V-statistics.

PROPOSITION 3.8. *Let $f_p \in \mathcal{S}_p^1$. Then*

$$\delta_p^0(f_p) = \sum_{r=0}^{[p/2]} C_{p,r} I_{p-2r}^0(\tau^r f_p) \tag{3.9}$$

and

$$I_p^0(f_p) = \sum_{r=0}^{[p/2]} C_{p,r} (-1)^r \delta_{p-2r}^0(\tau^r f_p). \tag{3.10}$$

The proof is omitted.

4. Asymptotic distributions of U- and V-statistics and stochastic integrals. In this section we will discuss the asymptotic distribution of a V-statistic under certain assumptions of square integrability of the kernel over the diagonal. The limit is a multiple Stratonovich integral with respect to the Wiener process. We also show in this section that a certain linear combination of related U-statistics has the same asymptotic limit as the original V-statistic.

We begin by giving some basic definitions. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{W_t;\ 0 \leqslant t \leqslant 1\}$ be a Wiener process on this space. Let $W_t^0 \equiv \equiv W_t - tW_1$ be the Brownian bridge. The integrals $I_p(\cdot)$, $\delta_p(\cdot)$, $\delta_p^s(\cdot)$, $I_p^0(\cdot)$, $\delta_p^0(\cdot)$ are to be understood as in Sections 2 and 3. Let $(\Omega_1, \mathscr{F}_1, P_1)$ be another probability space.

DEFINITION 4.1. Let $X_1, X_2, \ldots$ be i.i.d. random variables defined on $(\Omega_1, \mathscr{F}_1)$, with the common continuous distribution function F. Let $f_p\colon \boldsymbol{R}^p \to \boldsymbol{R}$ be a symmetric function. The *U-statistic* $(U_n(f_p))$ corresponding to the kernel f_p is defined as

$$(4.1)\qquad U_n(f_p) \equiv U_n(X_1, \ldots, X_n) := \binom{n}{p}^{-1} \sum_{\mathscr{C}} f_p(X_{i_1}, \ldots, X_{i_p}),$$

where $\mathscr{C} := \{(i_1, \ldots, i_p) \in \{1, 2, \ldots, n\}^p;\ i_l \neq i_m \text{ if } l \neq m;\ l, m = 1, 2, \ldots, p\}$.

DEFINITION 4.2. Let $X_1, X_2, \ldots$ and f_p be as in Definition 4.1. The *V-statistic* $(V_n(f_p))$ corresponding to the kernel f_p is defined as

$$(4.2)\qquad V_n(f_p) \equiv V_n(X_1, \ldots, X_n) := n^{-p} \sum_{i_1, \ldots, i_p = 1}^{n} f_p(X_{i_1}, \ldots, X_{i_p}).$$

From now on, we will assume that $\{X_i\}$ is an i.i.d. sequence of $U[0, 1]$ variates. We note that all the results in this section with obvious modifications hold for a general i.i.d. sequence (i.e. not necessarily $U[0, 1]$ variables) with a continuous distribution function.

The central theorem of this section is Theorem 4.3 stated below. To prove the theorem we will need several lemmas which we give following the statement of the theorem. Many of the arguments in Lemmas 4.8–4.12 are contained in the work of Rubin and Vitale [9] but for the sake of completeness we have reproduced those arguments here. After proving the lemmas, we then give the proof of the theorem. Corollary 4.13 is a well-known result due to Hoeffding [3], which is a direct consequence of Theorem 4.3. Finally, Proposition 4.14 shows that the asymptotic distribution of an appropriately centered and standardized V-statistic is the same as that of a certain linear combination of U-statistics.

Rubin and Vitale [9] derive the asymptotic distribution of a U-statistic by using the Hoeffding decomposition for the kernel, i.e. expressing the U-statistic as a finite sum of lower order U-statistics with centered kernels, each of which – appropriately standardized – has a multiple Wiener integral as the asymptotic distribution (see also [1]). We describe now the notion of the centering of a kernel, which will also be useful in deriving the asymptotic distribution of a V-statistic.

Let $f_p: [0, 1]^p \to \boldsymbol{R}$ be a symmetric integrable function. We define the "centering" of f_p, denoted by $f_{p,c}: [0, 1]^p \to \boldsymbol{R}$, as follows:

$$(4.3)\quad f_{p,c}(x_1, \ldots, x_p) := f_p(x_1, \ldots, x_p) + \sum_{\pi} \sum_{m=1}^{p-1} \frac{(-1)^m}{m!(p-m)!} \int_{[0,1]^m} f_p(x_{\pi(1)}, \ldots, x_{\pi(p-m)}, y_m)\, dy_m + (-1)^p \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\, dy_1 \ldots dy_p,$$

where $y_m \in [0, 1]^m$ and the outer summation in (4.3) is over all permutations π of the p symbols $\{1, 2, \ldots, p\}$. Note that when $p = 1$,

$$f_{p,c}(x_1) = f_p(x_1) - \int_0^1 f_p(y)\, dy,$$

and when $p = 2$,

$$f_{p,c}(x_1, x_2) = f_p(x_1, x_2) - \int_0^1 f_p(x_1, y)\, dy - \int_0^1 f_p(x_2, y)\, dy + \int_{[0,1]^2} f_p(y_1, y_2)\, dy_1 dy_2.$$

Also, for $r = 1, 2, \ldots, p-1$ define the function $f_p^{(r)}: [0, 1]^{p-r} \to \boldsymbol{R}$ as follows:

$$(4.4)\quad f_p^{(r)}(x_1, \ldots, x_{p-r}) = \int_{[0,1]^r} f_p(x_1, \ldots, x_{p-r}, y)\, dy.$$

Denote the centering of the function $f_p^{(r)}$ by $f_{p,c}^{(r)}$, i.e.,

$$(4.5)\quad f_{p,c}^{(r)}(x_1, \ldots, x_{p-r}) = f_p^{(r)}(x_1, \ldots, x_{p-r}) + (-1)^{p-r} \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\, dy_1 \ldots dy_p + \sum_{\pi_*} \sum_{s=1}^{p-r-1} \frac{(-1)^s}{s!(p-r-s)!} \int_{[0,1]^s} f_p^{(r)}(x_{\pi_*(1)}, \ldots, x_{\pi_*(p-r-s)}, z_s)\, dz_s,$$

where π_* runs over all permutations of the $p-r$ symbols $\{1, 2, \ldots, p-r\}$, and define

$$f_{p,c}^{(0)} := f_{p,c}.$$

In Lemma 4.4 we represent an integrable kernel f_p as a linear combination of $f_{p,c}^{(r)}$'s which is precisely the Hoeffding decomposition for the kernel f_p. We now state our main theorem.

THEOREM 4.3. *Let $f_p: [0, 1]^p \to \boldsymbol{R}$ be a measurable, symmetric function satisfying*

$$(4.6)\quad \int_{[0,1]^r} f_p^2(x_1, x_1, \ldots, x_r, x_r, \ldots)\, dx_1 \ldots dx_r < \infty,$$

where $x_1, x_2, \ldots, x_r$ *appear* $s_1, s_2, \ldots, s_r$ *times, respectively, for all* $(s_1, \ldots, s_r) \in \in Z_+^r$ *such that* $s_1+\ldots+s_r = p$ *and for all* $r = 1, 2, \ldots, p$. *Then*

$$[\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] = \sum_{r=0}^{p-1} \binom{p}{r} V_n(f_{p,c}^{(r)}) + o_p(1)$$

and

$$n^{1/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \xrightarrow{\mathscr{L}} p\delta_1(f_{p,c}^{(p-1)}); \tag{4.7}$$

moreover, if for fixed $0 < k < p$ *we have* $f_{p,c}^{(p-j)} = 0$ *a.e. in* $[0, 1]^j$ *for all* $j = 1, 2, \ldots, k$, *then*

$$n^{(k+1)/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \xrightarrow{\mathscr{L}} \binom{p}{k+1} \delta_{k+1}(f_{p,c}^{(p-k-1)}). \tag{4.8}$$

LEMMA 4.4. *Let* f_p *be as in Theorem* 4.3. *Then*

$$f_p(x_1, \ldots, x_p) - \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p = \sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!(p-r)!} f_{p,c}^{(r)}(x_{\pi(1)}, \ldots, x_{\pi(p-r)}), \tag{4.9}$$

where π *runs over all the permutations of the* p *symbols* $\{1, 2, \ldots, p\}$.

Proof. By substituting the expression for $f_{p,c}^{(r)}$ from equation (4.5) on the right-hand side of (4.9), we have

$$\sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!(p-r)!} f_{p,c}^{(r)}(x_{\pi(1)}, \ldots, x_{\pi(p-r)}) \tag{4.10}$$

$$= \sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!(p-r)!} \Big(f_p^{(r)}(x_{\pi(1)}, \ldots, x_{\pi(p-r)}) + (-1)^{p-r} \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p$$

$$+ \sum_{\pi_*} \sum_{s=1}^{p-r-1} \frac{(-1)^s}{s!(p-r-s)!} \int_{[0,1]^s} f_p^{(r)}(x_{\pi_*(\pi(1))}, \ldots, x_{\pi_*(\pi(p-r-s))}, z_s)\,dz_s\Big)$$

$$= \sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!(p-r)!} \Big(f_p^{(r)}(x_{\pi(1)}, \ldots, x_{\pi(p-r)}) + (-1)^{p-r} \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p$$

$$+ (p-r)! \sum_{s=1}^{p-r-1} \frac{(-1)^s}{s!(p-r-s)!} \int_{[0,1]^s} f_p^{(r)}(x_{\pi(1)}, \ldots, x_{\pi(p-r-s)}, z_s)\,dz_s\Big)$$

$$= \sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!} \Big(\sum_{s=0}^{p-r-1} \frac{(-1)^s}{s!(p-r-s)!} f_p^{(r+s)}(x_{\pi(1)}, \ldots, x_{(\pi(p-r-s))})\Big)$$

$$- \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p.$$

By substituting $r+s=k$ and rearranging the summations in (4.10), we have

$$\sum_{\pi}\sum_{r=0}^{p-1}\frac{1}{r!(p-r)!}f_{p,c}^{(r)}(x_{\pi(1)},\dots,x_{\pi(p-r)})$$

$$=\sum_{\pi}\left[\sum_{k=0}^{p-1}\left(\sum_{s=0}^{k}\frac{1}{(k-s)!}\frac{(-1)^s}{s!(p-k)!}\right)f_p^{(k)}(x_{(\pi(1))},\dots,x_{(\pi(p-k))})\right]$$

$$-\int_{[0,1]^p}f_p(y_1,\dots,y_p)\,dy_1\dots dy_p$$

$$=\frac{1}{p!}\sum_{\pi}f_p^{(0)}(x_{(\pi(1))},\dots,x_{(\pi(p))})-\int_{[0,1]^p}f_p(y_1,\dots,y_p)\,dy_1\dots dy_p$$

$$=f_p(x_1,\dots,x_p)-\int_{[0,1]^p}f_p(y_1,\dots,y_p)\,dy_1\dots dy_p. \blacksquare$$

LEMMA 4.5. *Let $\{\phi_i\}_1^\infty$ be a CONS in $L^2[0,1]$ such that $\phi_1\equiv 1$. Let M be a fixed positive integer and let $g_1,\dots,g_r$ be real-valued continuous functions defined on $\mathbb{R}^M$. Let $(Y_{1n},\dots,Y_{rn})$ be a vector sequence of random variables on $(\Omega_1,\mathscr{F}_1,P_1)$ such that each component Y_{jn} of the vector converges in probability to a constant c_j. Then*

$$\sum_{j=1}^{r}Y_{jn}g_j\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_2(X_i),\dots,\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_M(X_i)\right)\xrightarrow{\mathscr{L}}\sum_{j=1}^{r}c_jg_j(I_1(\phi_2),\dots,I_1(\phi_M)).$$

The proof is straightforward and is omitted.

LEMMA 4.6 (Filippova [2]). *Let ξ_{nm} be a double array of square integrable random variables and ξ_n be a sequence of square integrable random variables defined on $(\Omega_1,\mathscr{F}_1,P_1)$ so that $\xi_{nm}\to\xi_n$ in L^2 as $m\to\infty$, uniformly in n. Then $L(\xi_{nm},\xi_n)\to 0$ as $m\to\infty$, uniformly in n, where $L(X,Y)$ is the Lévy distance between the laws of X and Y.*

For the proof see [2].

LEMMA 4.7. *Let ξ_{nm},ξ_n be square integrable random variables on $(\Omega_1,\mathscr{F}_1,P_1)$ for $n,m\geqslant 1$ such that $\xi_{nm}\to\xi_n$ in L^2 as $m\to\infty$, uniformly in n. Let η_m,η be square integrable random variables on $(\Omega,\mathscr{F},P)$ such that $\xi_{nm}\to\eta_m$ in distribution as $n\to\infty$ for each fixed m and $\eta_m\to\eta$ in L^2 as $m\to\infty$. Then $\xi_n\to\eta$ in distribution as $n\to\infty$.*

The proof being standard is omitted.

The following lemma is essentially contained in the work of Rubin and Vitale [9].

LEMMA 4.8. *Let f_p be as in Theorem 4.3, and let $\{\phi_i\}$ be a CONS for $L^2[0,1]$ with $\phi_1\equiv 1$. Then, as $M\to\infty$,*

$$(4.11)\qquad \sum_{i_1,\ldots,i_k=1}^{M}\ \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{M} b^{(k)}_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} \times\Big(n^{-(p-r)/2}\sum_{\mathscr{S}}\phi_{i_1}(X_{j_1})\ldots\phi_{i_k}(X_{j_k})\,\phi_{i_{2k+1}}(X_{j_{2k+1}})\,\phi_{i_{p-r}}(X_{j_{p-r}})\Big)$$

$$\overset{L^2(\Omega)}{\to} \sum_{i_1,\ldots,i_k=1}^{\infty}\ \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{\infty} b^{(k)}_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} \times\Big(n^{-(p-r)/2}\sum_{\mathscr{S}}\phi_{i_1}(X_{j_1})\ldots\phi_{i_k}(X_{j_k})\,\phi_{i_{2k+1}}(X_{j_{2k+1}})\,\phi_{i_{p-r}}(X_{j_{p-r}})\Big)$$

uniformly in n for all $r = 1, 2, \ldots, p$, *and* $k \leqslant [p-r]/2$, *where*

$$\mathscr{S} = \{(j_1, \ldots, j_k, j_{2k+1}, \ldots, j_{p-r}) \in \{1, \ldots, n\}^{p-r-k};\quad j_l \neq j_m \text{ if } l \neq m;\ l, m = 1, \ldots, k, 2k+1, \ldots, p-r)\}$$

and

$$(4.12)\qquad b^{(k)}_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} = \int_{[0,1]^{p-r-k}} \big(f^{(r)}_{p,c}(x_1, x_1, \ldots, x_k, x_k, x_{2k+1}, \ldots, x_{p-r}) \times\phi_{i_1}(x_1)\ldots\phi_{i_k}(x_k)\,\phi_{i_{2k+1}}(x_{2k+1})\ldots\phi_{i_{p-r}}(x_{p-r})\big)\,dx_1\ldots dx_k dx_{2k+1}\ldots dx_{p-r}.$$

The dependence of the coefficients $b^{(k)}_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}$ *on r has been suppressed in the notation for simplicity.*

Furthermore,

$$(4.13)\qquad \sum_{i_1,\ldots,i_{p-r}=2}^{M} b^{(0)}_{i_1,\ldots,i_{p-r}}\Big(n^{-(p-r)/2}\sum_{\substack{j_1,\ldots,j_{p-r}=1\\ j_l\neq j_q}}^{n}\phi_{i_1}(X_{j_1})\ldots\phi_{i_{p-r}}(X_{j_{p-r}})\Big) \overset{\mathscr{L}}{\to} \sum_{i_1,\ldots,i_{p-r}=2}^{M} b^{(0)}_{i_1,\ldots,i_{p-r}}\,n^{-(p-r)/2}\sum_{\substack{j_1,\ldots,j_{p-r}=1\\ j_l\neq j_q}}^{n}\phi_{i_1}(X_{j_1})\ldots\phi_{i_{p-r}}(X_{j_{p-r}})$$

uniformly in n for all $r = 1, 2, \ldots, p$, *where*

$$(4.14)\quad b^{(0)}_{i_1,\ldots,i_{p-r}} = \int_{[0,1]^{p-r}} \big(f^{(r)}_{p,c}(x_1, \ldots, x_{p-r})\,\phi_{i_1}(x_1)\ldots\phi_{i_{p-r}}(x_{p-r})\big)\,dx_1\ldots dx_{p-r}.$$

Proof. We will only prove (4.11). The proof of (4.13) is similar. Note initially that since $\phi_1 \equiv 1$, we have

$$\sum_{i_1,\ldots,i_k=1}^{M}\ \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{M} b^{(k)}_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}\,n^{-(p-r)/2}\sum_{\mathscr{S}}\phi_{i_1}(X_{j_1})\ldots \ldots\phi_{i_k}(X_{j_k})\,\phi_{i_{2k+1}}(X_{j_{2k+1}})\,\phi_{i_{p-r}}(X_{j_{p-r}})$$

$$= \sum_{s=1}^{k}\ \sum_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}=2}^{M} b^{(k)}_{1,\ldots,1,i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}\,n^{-(p-r)/2}\sum_{\mathscr{S}}\phi_1(X_{j_1})\ldots \ldots\phi_1(X_{j_{s-1}})\,\phi_{i_1}(X_{j_s})\ldots\phi_{i_k}(X_{j_k})\,\phi_{i_{2k+1}}(X_{j_{2k+1}})\ldots\phi_{i_{p-r}}(X_{j_{p-r}})$$

$$= \sum_{s=1}^{k} \sum_{i_s,\ldots,i_k,i_{2k+1},\ldots i_{p-r}=2}^{M} \beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}$$

$$\times n^{-(p-r)/2} \sum_{\mathcal{S}} \phi_{i_s}(X_{j_s}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}}),$$

where

$$\beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} = b^{(k)}_{\underbrace{1,\ldots,1}_{s-1 \text{ times}},\ldots,i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}$$

$$= \int_{[0,1]^{p-r-k}} f^{(r)}_{p,c}(x_1, x_1, \ldots, x_k, x_k, x_{2k+1}, \ldots, x_{p-r})$$

$$\times \phi_{i_s}(x_s) \ldots \phi_{i_k}(x_k) \phi_{i_{2k+1}}(x_{2k+1}) \ldots \phi_{i_{p-r}}(x_{p-r})\, dx_1 \ldots dx_k\, dx_{2k+1} \ldots dx_{p-r}.$$

Hence we need only to show that

$$S_M := \sum_{i_s,\ldots,i_k,i_{2k+1},\ldots i_r=2}^{M} \beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}$$

$$\times n^{-(p-r)/2} \sum_{\mathcal{S}} \phi_{i_s}(X_{j_s}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})$$

converges in L^2, uniformly in n as $M \to \infty$.

Again, if $\mathscr{C}$ is a finite subset of the set $\{(i_s, \ldots, i_k, i_{2k+1}, \ldots, i_{p-r}) \in \in \{2, 3, \ldots\}^{p-r-k-s+1}\}$, then

$$\mathrm{E}\Big[\sum_{\mathscr{C}} \beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} n^{-(p-r)/2} \sum_{\mathcal{S}} \phi_{i_s}(X_{j_s}) \ldots$$

$$\ldots \times \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})\Big]^2$$

$$\leqslant (p-r-k-s+1)! \sum_{\pi} \mathrm{E}\Big[\sum_{\mathscr{C}} \beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}} n^{-(p-r)/2}$$

$$\times \sum_{\substack{1 \leqslant j_{\pi(s)} < \ldots < j_{\pi(k)} \\ < j_{\pi(2k+1)} < \ldots < j_{\pi(p-r)} \leqslant n}} \phi_{i_s}(X_{j_s}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})\Big]^2,$$

where π runs over all the permutations of the $p-r-k-s+1$ symbols $\{s, s+1, \ldots, k, 2k+1, \ldots, p-r\}$.

Again, since $\mathrm{E}(\phi_i(X_u)\phi_j(X_v)) = 0$ whenever $i, j > 1$ and $(i, u) \neq (j, v)$, the right-hand side of the above inequality equals

$$(p-r-k-s+1)! \sum_{\pi} \sum_{\mathscr{C}} [\beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}]^2 n^{-(p-r)}$$

$$\times \sum_{\substack{1 \leqslant j_{\pi(s)} < \ldots < j_{\pi(k)} \\ < j_{\pi(2k+1)} < \ldots < j_{\pi(p-r)} \leqslant n}} \mathrm{E}[\phi_{i_s}(X_{j_s}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})]^2$$

$$= (p-r-k-s+1)! \sum_{\mathscr{C}} (\beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}})^2 n^{-(p-r)} \frac{n!}{(n-p+r+k+s-1)!}$$

$$< (p-r-k-s+1)! \sum_{\mathscr{C}} (\beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}})^2 .$$

Now, since

$$\sum_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}=1}^{\infty} (\beta^{(k)}_{i_s,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}})^2 < \infty,$$

the above inequality shows that S_M is a Cauchy sequence in L^2, uniformly in n, and therefore S_M converges in L^2, uniformly in n. The lemma now follows from earlier discussions. ■

LEMMA 4.9. *Let $\{\phi_i\}$ be a CONS for $L^2[0, 1]$. Then for $i_j \geqslant 1$ $(j = 1, 2, \ldots, u)$*

$$\frac{1}{n^{u/2}} \sum_{j=1}^{n} \phi_{i_1}(X_j)\ldots\phi_{i_u}(X_j) \xrightarrow{P} 0 \quad as\ n\to\infty \qquad whenever\ u > 2.$$

The proof is straightforward and is omitted.

LEMMA 4.10. *Let $\{\phi_i\}$ be a CONS for $L^2[0, 1]$ with $\phi_1 \equiv 1$. Then:*
(a) *For all $u \geqslant 3$, $i_1, \ldots, i_u > 1$ and $r = 1, 2, \ldots, u-2$,*

$$(4.15)\qquad n^{-u/2} \sum_{\substack{j_1,\ldots,j_u=1\\ j_l\neq j_m}}^{n} \phi_{i_1}\otimes\ldots\otimes\phi_{i_u}(X^{(s_1)}_{j_1}, \ldots, X^{(s_r)}_{j_r}) \xrightarrow{P} 0 \ \ as\ n\to\infty \quad whenever\ \max(s_1, \ldots, s_r) > 2,$$

where $X^{(s_l)}_{j_l} = (X_{j_l}, X_{j_l}, \ldots)$ is an s_l-dimensional vector, $l = 1, 2, \ldots, r$, and $s_1, \ldots, s_r$ are positive integers with $\sum_{i=1}^{r} s_i = u$.
(b) *For all $u \geqslant 2$ and $i_1, \ldots, i_u > 1$,*

$$(4.16)\qquad n^{-u/2} \sum_{\substack{j_1,\ldots,j_u=1\\ j_l\neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_u}(X_{j_u}) \xrightarrow{\mathscr{L}} I_u(\phi_{i_1}\otimes\ldots\otimes\phi_{i_u})$$

and

$$(4.17)\qquad n^{-u/2} \sum_{\substack{j_1,\ldots,j_b,j_{2b+1},\ldots,j_u=1\\ j_l\neq j_m}}^{n} \phi_{i_1}\otimes\ldots\otimes\phi_{i_u}(X_{j_1}, X_{j_1}, X_{j_2}, X_{j_2}, \ldots, X_{j_b}, X_{j_b}, X_{j_{2b+1}}, \ldots, X_{j_u}) \xrightarrow{\mathscr{L}} \delta_{i_1,i_2}\ldots\delta_{i_{2b-1},i_{2b}} I_{u-2b}(\phi_{i_{2b+1}}\otimes\ldots\otimes\phi_{i_u}) \quad as\ n\to\infty$$

for all $b = 1, 2, \ldots, [u/2]$,

where $\delta_{i,j}$ is the Kronecker delta function, i.e., $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$, and $I_0(\phi_{i_{u+1}}\otimes\ldots\otimes\phi_{i_u})$ is understood to be 1.

Proof. We will prove the lemma by induction on u. We observe first that (a) holds for $u = 3$, since $\max(s_1, \ldots, s_r) > 2$ implies in this case that $r = 1$

and $s_1 = 3$, and therefore (4.15) follows from Lemma 4.9. We also note that for $u = 2$ the left-hand side of (4.17) is equal to $n^{-1}\sum_{j_1=1}^{n}\phi_{i_1}(X_{j_1})\phi_{i_2}(X_{j_1})$, which clearly converges in distribution to δ_{i_1,i_2} by the law of large numbers. And for $u = 2$ the left-hand side of (4.16) is equal to

$$n^{-1}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}\phi_{i_1}(X_{j_1})\phi_{i_2}(X_{j_2}),$$

which is the same as

$$n^{-1}\Big(\sum_{j_1=1}^{n}\phi_{i_1}(X_{j_1})\Big)\Big(\sum_{j_2=1}^{n}\phi_{i_2}(X_{j_2})\Big)-n^{-1}\sum_{j_1=1}^{n}\phi_{i_1}(X_{j_1})\phi_{i_2}(X_{j_1}).$$

The last expression converges in distribution by the central limit theorem and the law of large numbers to $I_1(\phi_{i_1})I_1(\phi_{i_2})-\delta_{i_1,i_2}$, which is precisely $I_2(\phi_{i_1}\otimes\phi_{i_2})$. Hence (4.16) and (4.17) hold for $u = 2$.

INDUCTION HYPOTHESES. Suppose that $p > 3$ is such that (a) holds for $3 \leqslant u \leqslant p$ and the relations (4.16), (4.17) hold for $2 \leqslant u \leqslant p-1$.

We will now show that (a) holds for $u = p+1$ and the relations (4.16), (4.17) hold for $u = p$.

Let us first consider (a). For the proof of the relation (4.15) for $u = p+1$ and $r = 1, 2, \ldots, p-1$ we will show that (4.15) holds for $r = 1$ and under the assumption that it holds for $r = 1, 2, \ldots, m-1$ and $m < p-1$ we prove that it holds for $r = m$. Note that from Lemma 4.9 it follows that (4.15) holds for $r = 1$. Suppose now that, for fixed $m < p-1$, (4.15) holds for $r = 1, 2, \ldots, m-1$. Consider now $r = m$ and assume without loss of generality that $s_1 > 2$. Then

$$(4.18)\quad n^{-(p+1)/2}\sum_{\substack{j_1,\ldots,j_m=1\\ j_l\neq j_q}}^{n}\phi_{i_1}\otimes\ldots\otimes\phi_{i_{p+1}}(X_{j_1}, X_{j_1}, \ldots, X_{j_m}, X_{j_m}, \ldots)$$

$$= n^{-s_1/2}\sum_{j_1=1}^{n}\phi_{i_1}\otimes\ldots\otimes\phi_{i_{s_1}}(X_{j_1}, X_{j_1}, \ldots)\times$$

$$\times n^{-(p+1-s_1)/2}\sum_{\substack{j_2,\ldots,j_m=1\\ j_l\neq j_q}}^{n}\phi_{i_{s_1}}(X_{j_1}, X_{j_1}, \ldots)-\sum_{a=2}^{m}n^{-(p+1)/2}\sum_{\substack{j_2,\ldots,j_m=1\\ j_l\neq j_q}}^{n}\phi_{i_1}\otimes\ldots$$

$$\ldots\otimes\phi_{i_{p+1}}(\underbrace{X_{j_a}, X_{j_a}, \ldots,}_{s_1\text{ times}} X_{j_2}, X_{j_2}, \ldots, X_{j_m}, X_{j_m}),$$

where $X_{j_1}, X_{j_2}, \ldots, X_{j_m}$ appear $s_1, s_2, \ldots, s_m$ times, respectively. Note that the last term in (4.18) converges to zero in probability, since (4.15) holds for $r = m-1$ and $u = p+1$ by assumption.

Now we will show that the first term in (4.18) converges to zero in probability. To see this, observe initially by Lemma 4.9 that

$$n^{-s_1/2}\sum_{j_1=1}^{n}\phi_{i_1}\otimes\ldots\otimes\phi_{i_{s_1}}(X_{j_1}, X_{j_1}, \ldots)\xrightarrow{P}0.$$

Hence we only need to show that

$$n^{-(p+1-s_1)/2} \sum_{\substack{j_2,\dots,j_m=1 \\ j_l \neq j_q}}^{n} \phi_{i_{s_1+1}} \otimes \dots \otimes \phi_{i_{p+1}} (X_{j_2}, X_{j_2}, \dots, X_{j_m}, X_{j_m})$$

($X_{j_2}, X_{j_3}, \dots, X_{j_m}$ appear $s_2, s_3, \dots, s_m$ times, respectively) converges in distribution to a finite random variable as $n \to \infty$.

Consider first the case $\max(s_2, \dots, s_m) > 2$. Then, since (a) holds for $u \leqslant p$ by the induction hypothesis on (a), we have

$$n^{-(p+1-s_1)/2} \sum_{\substack{j_2,\dots,j_m=1 \\ j_l \neq j_q}}^{n} \phi_{i_{s_1+1}} \otimes \dots \otimes \phi_{i_{p+1}} (X_{j_2}, X_{j_2}, \dots, X_{j_m}, X_{j_m}) \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

where $X_{j_2}, X_{j_3}, \dots, X_{j_m}$ appear $s_2, s_3, \dots, s_m$ times, respectively.

Finally, consider the case $\max(s_2, \dots, s_m) \leqslant 2$. Then, since (4.16) and (4.17) hold for $u \leqslant p-1$ by the induction hypothesis, we see that

$$n^{-(p+1-s_1)/2} \sum_{\substack{j_2,\dots,j_{p+1}=1 \\ j_l \neq j_m}}^{n} \phi_{i_{s_1+1}} \otimes \dots \otimes \phi_{i_{p+1}} (X_{j_2}, X_{j_2}, \dots, X_{j_m}, X_{j_m})$$

converges in distribution as $n \to \infty$, where $X_{j_2}, X_{j_3}, \dots, X_{j_m}$ appear $s_2, s_3, \dots, s_m$ times, respectively.

Therefore, the above observations imply that

$$n^{-s_1/2} \sum_{j_1=1}^{n} \phi_{i_1} \otimes \dots \otimes \phi_{i_{s_1}} (X_{j_1}, X_{j_1}, \dots)$$

and

$$n^{-(p+1-s_1)/2} \sum_{\substack{j_2,\dots,j_m=1 \\ j_l \neq j_q}}^{n} \phi_{i_{s_1+1}} \otimes \dots \otimes \phi_{i_{p+1}} (X_{j_2}, X_{j_2}, \dots, X_{j_m}, X_{j_m})$$

converge to 0 in probability as $n \to \infty$, where $X_{j_1}, X_{j_2}, \dots, X_{j_m}$ appear $s_1, s_2, \dots, s_m$ times, respectively. As remarked earlier this shows that

$$n^{-(p+1)/2} \sum_{\substack{j_1,\dots,j_m=1 \\ j_l \neq j_q}}^{n} \phi_{i_1} \otimes \dots \otimes \phi_{i_{p+1}} (X_{j_1}, X_{j_1}, \dots, X_{j_m}, X_{j_m}) \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

where $X_{j_1}, X_{j_2}, \dots, X_{j_m}$ appear $s_1, s_2, \dots, s_m$ times, respectively. Hence (4.15) holds for $r = m$. Therefore we have proved (a) for $u = p+1$.

Now we will prove (4.16) for $u = p$. For the sake of simplicity we assume that p is odd, the proof in the case when p is even is similar and is omitted. We observe that

$$(4.19) \quad n^{-p/2} \sum_{\substack{j_1,\dots,j_p=1 \\ j_l \neq j_q}}^{n} \phi_{i_1}(X_{j_1}) \dots \phi_{i_p}(X_{j_p})$$

$$= n^{-p/2} \sum_{k=1}^{[p/2]} C_{p,k} (-1)^k \frac{1}{p!} \sum_{\pi} \Big(\sum_{j=1}^{n} \phi_{i_{\pi(1)}}(X_j) \phi_{i_{\pi(2)}}(X_j) \Big) \dots$$

$$\ldots \times \left(\sum_{j=1}^{n} \phi_{i_{\pi(2k-1)}}(X_j)\,\phi_{i_{\pi(2k)}}(X_j)\right)\left(\sum_{j=1}^{n} \phi_{i_{\pi(2k+1)}}(X_j)\right)\ldots\left(\sum_{j=1}^{n} \phi_{i_{\pi(p)}}(X_j)\right)$$

$$+n^{-p/2} \sum_{j_1,\ldots,j_p=1}^{n} \phi_{i_1}(X_{j_1})\,\phi_{i_2}(X_{j_2})\ldots\phi_{i_p}(X_{j_p})+o_p(1).$$

The proof of this observation uses the hypothesis that (4.15) holds for $3 \leqslant u \leqslant p$ and that (4.16), (4.17) hold for $u \leqslant p-1$. The details are left to the reader.

Observe next that the right-hand side of (4.19) equals

$$\sum_{k=1}^{[p/2]} C_{p,k}(-1)^k \frac{1}{p!}\sum_{\pi}\frac{1}{n}\left(\sum_{j=1}^{n} \phi_{i_{\pi(1)}}(X_j)\,\phi_{i_{\pi(2)}}(X_j)\right)\ldots$$

$$\ldots \times \frac{1}{n}\left(\sum_{j=1}^{n} \phi_{i_{\pi(2k-1)}}(X_j)\,\phi_{i_{\pi(2k)}}(X_j)\right)\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} \phi_{i_{\pi(2k+1)}}(X_j)\right)\ldots\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} \phi_{i_{\pi(p)}}(X_j)\right)$$

$$+\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} \phi_{i_1}(X_j)\right)\ldots\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} \phi_{i_p}(X_j)\right)+o_p(1).$$

By Lemma 4.5 and the law of large numbers the above expression converges in distribution to

$$\sum_{k=1}^{[p/2]} C_{p,k}(-1)^k \frac{1}{p!}\sum_{\pi}\delta_{i_{\pi(1)},i_{\pi(2)}}\ldots\delta_{i_{\pi(2k-1)},i_{\pi(2k)}} I_1(\phi_{i_{\pi(2k+1)}})\ldots I_1(\phi_{i_{\pi(p)}})$$
$$+I_1(\phi_{i_1})\ldots I_1(\phi_{i_p}).$$

Moreover, from Theorem 6.1 of [7] we obtain

$$\text{(4.20a)}\qquad \frac{1}{p!}\sum_{\pi}\delta_{i_{\pi(1)},i_{\pi(2)}}\ldots\delta_{i_{\pi(2k-1)},i_{\pi(2k)}} I_1(\phi_{i_{\pi(2k+1)}})\ldots I_1(\phi_{i_{\pi(p)}})$$
$$= \delta_p^s(\overrightarrow{\mathrm{Tr}}^k(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}))$$

and

$$\text{(4.20b)}\qquad I_1(\phi_{i_1})\ldots I_1(\phi_{i_p}) = \delta_p^s(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}).$$

We remind the reader that the δ appearing on the left-hand side of (4.20a) is the Kronecker delta function, while δ_p^s on the right-hand side of (4.20) refers to the MSI introduced in Section 3. Therefore, as $n \to \infty$,

$$n^{-p/2} \sum_{\substack{j_1,\ldots,j_p=1\\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_p}(X_{j_p}) \xrightarrow{\mathscr{L}} \sum_{k=0}^{[p/2]} C_{p,k}(-1)^k\,\delta_p^s(\overrightarrow{\mathrm{Tr}}^k(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}))$$
$$= I_p(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}),$$

the last step following from Theorem 6.1 of [7]. Hence (4.16) is proved for $u = p$.

Finally, consider (4.17). Then, just as in (4.19), we have

$$n^{-u/2} \sum_{\substack{j_1,\ldots,j_b,j_{2b+1},\ldots,j_p=1 \\ j_l \neq j_m}}^{n} \phi_{i_1} \otimes \ldots$$

$$\ldots \otimes \phi_{i_p}(X_{j_1}, X_{j_1}, X_{j_2}, X_{j_2}, \ldots, X_{j_b}, X_{j_b}, X_{j_{2b+1}}, \ldots, X_{j_p})$$

$$= n^{-p/2} \sum_{k=0}^{[(p-2b)/2]} \Big\{ C_{p-2b,k}(-1)^k \sum_{\pi} \frac{1}{p!} \Big(\sum_{j_1,\ldots,j_{k+b},j_{2k+2b+1},\ldots,j_p=1}^{n} \phi_{i_{\pi(1)}} \otimes \ldots$$

$$\ldots \otimes \phi_{i_{\pi(p)}}(X_{j_1}, X_{j_1}, \ldots, X_{j_{k+b}}, X_{j_{k+b}}, X_{j_{2k+2b+1}}, \ldots, X_{j_p}) \Big) \Big\} + o_p(1).$$

And now the proof for (4.17) for $u = p$ follows exactly along the lines of the proof for (4.16). Hence, assuming that (a) holds for $u = p$ and (4.16), (4.17) hold for $u = p-1$, we have shown that (a) holds for $u = p+1$ and (4.16), (4.17) hold for $u = p$. The proof therefore is complete by induction on u. ∎

LEMMA 4.11. *Let f_p be as in Theorem 4.3. Then*

$$(4.21) \qquad n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} f_{p,c}^{(r)}(X_{j_1}^{(s_1)}, \ldots, X_{j_b}^{(s_b)}) \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

whenever $\max(s_1, \ldots, s_b) > 2$, *where* $s_1, \ldots, s_b$ *are positive integers with* $\sum_{i=1}^{b} s_i = p-r$.

Proof. Let $\{\phi_i\}$ be a CONS for $L^2[0, 1]$ with $\phi_1 \equiv 1$. Define

$$(4.22) \qquad a_{i_1,\ldots,i_b}^{(r)} = \int_{[0,1]^b} f_{p,c}^{(r)}(x_1^{(s_1)}, \ldots, x_b^{(s_b)}) \phi_{i_1}(x_1) \ldots \phi_{i_b}(x_b) \, dx_1 \ldots dx_b,$$

where $x_k^{(s_k)} = (x_k, x_k, \ldots)$ is an s_k-dimensional vector for $k = 1, 2, \ldots, b$. Then

$$(4.23) \qquad f_{p,c}^{(r)}(x_1^{(s_1)}, \ldots, x_b^{(s_b)}) = \sum_{i_1,\ldots,i_b=1}^{\infty} a_{i_1,\ldots,i_b}^{(r)} \phi_{i_1}(x_1) \ldots \phi_{i_b}(x_b),$$

where the summation converges in $L^2[0, 1]^b$. Since X_i's are i.i.d. uniform variates, we infer from (4.23) that

$$(4.24) \qquad n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} f_{p,c}^{(r)}(X_{j_1}^{(s_1)}, \ldots, X_{j_b}^{(s_b)})$$

$$= \sum_{i_1,\ldots,i_b=1}^{\infty} a_{i_1,\ldots,i_b}^{(r)} n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_b}(X_{j_b}),$$

where the summation converges in $L^2(\Omega)$. Note that from (4.16) we obtain

$$n^{-b/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_b}(X_{j_b}) \xrightarrow{\mathscr{L}} I_b(\phi_{i_1} \otimes \ldots \otimes \phi_{i_b}) \quad \text{as } n \to \infty.$$

Therefore, since $p-r$ is strictly greater than b, we have

$$n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_b}(X_{j_b}) \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Therefore, for all $M \geqslant 1$,

$$\sum_{i_1,\ldots,i_b=1}^{M} a^{(r)}_{i_1,\ldots,i_b} n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_b}(X_{j_b}) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{4.25}$$

Furthermore, arguments as in Lemma 4.8 show that as $M \to \infty$

$$\sum_{i_1,\ldots,i_b=1}^{M} a^{(r)}_{i_1,\ldots,i_b} n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_b}(X_{j_b}) \tag{4.26}$$
$$\to \sum_{i_1,\ldots,i_b=1}^{\infty} a^{(r)}_{i_1,\ldots,i_b} n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_b=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1})\ldots\phi_{i_b}(X_{j_b}) =: A_n$$

in L^2, uniformly in n. Therefore combining (4.25) and (4.26), we see that $A_n \xrightarrow{P} 0$ as $n \to \infty$. ∎

LEMMA 4.12. *Let f_p be as in Theorem 4.3. Then*

$$n^{-(p-r)/2} \sum_{j_1,\ldots,j_{p-r}=1}^{n} f^{(r)}_{p,c}(X_{j_1}, \ldots, X_{j_{p-r}}) \xrightarrow{\mathscr{L}} \delta_{p-r}(f^{(r)}_{p,c}) \tag{4.27}$$

for all $r = 0, 1, \ldots, p-1$.

Proof. For the sake of simplicity of notation we will assume that $p-r$ is odd. The case when $p-r$ is even can be treated similarly. Furthermore, we can assume that $p-r$ is greater than one or else the result follows from the central limit theorem. Observe first that from Lemma 4.11 and a combinatorial argument we obtain

$$n^{-(p-r)/2} \sum_{j_1,\ldots,j_{p-r}=1}^{n} f^{(r)}_{p,c}(X_{j_1}, \ldots, X_{j_{p-r}}) \tag{4.28}$$
$$= n^{-(p-r)/2} \Big(\sum_{\substack{j_1,\ldots,j_{p-r}=1 \\ j_l \neq j_q}}^{n} f^{(r)}_{p,c}(X_{j_1}, \ldots, X_{j_{p-r}}) + \sum_{k=1}^{[(p-r)/2]} \Big(C_{p-r,k}$$
$$\times \sum_{\substack{j_1,\ldots,j_k,j_{2k+1},\ldots,j_{p-r}=1 \\ j_l \neq j_q}}^{n} f^{(r)}_{p,c}(X_{j_1}, X_{j_1}, \ldots, X_{j_k}, X_{j_k}, X_{j_{2k+1}}, \ldots, X_{j_{p-r}})\Big)\Big) + o_p(1)$$
$$=: B_n + o_p(1).$$

Expanding the function $f_{p,c}^{(r)}(x_1, x_1, \ldots, x_k, x_k, x_{2k+1}, \ldots, x_{p-r})$ in terms of the CONS $\{\phi_i\}$ we have

(4.29)

$$n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_k,j_{2k+1},\ldots,j_{p-r}=1 \\ j_l \neq j_q}}^{n} f_{p,c}^{(r)}(X_{j_1}, X_{j_1}, \ldots, X_{j_k}, X_{j_k}, X_{j_{2k+1}}, \ldots, X_{j_{p-r}})$$

$$= n^{-(p-r)/2} \sum_{i_1,\ldots,i_k=1}^{\infty} \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{\infty} \Big(b_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}^{(k)}$$

$$\times \sum_{\substack{j_1,\ldots,j_k,j_{2k+1},\ldots,j_{p-r}=1 \\ j_l \neq j_q}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})\Big),$$

the above series converging in $L^2(\Omega)$ for each fixed n, where $b_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}^{(k)}$ are given by (4.12) and (4.14). Note that the inner summation in (4.29) runs from 2 to ∞, since $\phi_1 \equiv 1$ and $\int_0^1 f_{p,c}^{(r)}(x, y)\, dy = 0$ for all $x \in [0, 1]^{p-r-1}$.

Using (4.16), we observe that if exactly s elements (say, $1, \ldots, s$) amongst $\{\phi_{i_1}, \ldots, \phi_{i_k}\}$ equal ϕ_1 ($\equiv 1$), where $s < k$, then as $n \to \infty$ the expression

$$n^{-(p-r)/2} \sum_{\substack{j_1,\ldots,j_k,j_{2k+1},\ldots,j_{p-r}=1 \\ j_l \neq j_q}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})$$

converges to zero in probability, and if all k elements $\{\phi_{i_1}, \ldots, \phi_{i_k}\}$ are identically equal to 1, then this expression converges in distribution to $I_{p-r-2k}(\phi_{i_{2k+1}} \otimes \ldots \otimes \phi_{i_{p-r}})$. Therefore, if M is a fixed positive integer, we infer from Lemma 4.5 that for $k = 1, 2, \ldots, [(p-r)/2]$, as $n \to \infty$,

(4.30)

$$n^{-(p-r)/2} \sum_{i_1,\ldots,i_k=1}^{M} \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{M} \Big(b_{i_1,\ldots,i_k,i_{2k+1},\ldots,i_{p-r}}^{(k)}$$

$$\times \sum_{\substack{j_1,\ldots,j_k,j_{2k+1},\ldots,j_{p-r}=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_k}(X_{j_k}) \phi_{i_{2k+1}}(X_{j_{2k+1}}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})\Big)$$

$$\xrightarrow{\mathscr{L}} \sum_{i_{2k+1},\ldots,i_{p-r}=2}^{M} b_{1,1,\ldots,1,i_{2k+1},\ldots,i_{p-r}}^{(k)} I_{p-r-2k}(\phi_{i_{2k+1}} \otimes \ldots \otimes \phi_{i_{p-r}}).$$

In a similar fashion we can show that as $n \to \infty$

(4.31)

$$n^{-(p-r)/2} \sum_{i_1,\ldots,i_{p-r}=2}^{M} b_{i_1,\ldots,i_{p-r}}^{(0)} \sum_{\substack{j_1,\ldots,j_{p-r}=1 \\ j_l \neq j_m}}^{n} \phi_{i_1}(X_{j_1}) \ldots \phi_{i_{p-r}}(X_{j_{p-r}})$$

$$\xrightarrow{\mathscr{L}} \sum_{i_1,\ldots,i_{p-r}=2}^{M} b_{i_1,\ldots,i_{p-r}}^{(0)} I_{p-r}(\phi_{i_1} \otimes \ldots \otimes \phi_{i_{p-r}}).$$

Therefore once more from Lemma 4.5 we have

$$(4.32)\quad n^{-(p-r)/2}\Big(\sum_{i_1,\dots,i_{p-r}=2}^{M} b^{(0)}_{i_1,\dots,i_{p-r}} \sum_{\substack{j_1,\dots,j_{p-r}=1\\ j_l\neq j_m}}^{n} \phi_{i_1}(X_{j_1})\dots\phi_{i_{p-r}}(X_{j_{p-r}})$$

$$+\sum_{k=1}^{[(p-r)/2]} C_{p-r,k} \sum_{i_1,\dots,i_k=1}^{M} \sum_{i_{2k+1},\dots,i_{p-r}=2}^{M} b^{(k)}_{i_1,\dots,i_k,i_{2k+1},\dots,i_{p-r}}$$

$$\times \sum_{\substack{j_1,\dots,j_k,j_{2k+1},\dots,j_{p-r}=1\\ j_l\neq j_m}}^{n} \phi_{i_1}(X_{j_1})\dots\phi_{i_k}(X_{j_k})\,\phi_{i_{2k+1}}(X_{j_{2k+1}})\dots\phi_{i_{p-r}}(X_{j_{p-r}})\Big)$$

$$\xrightarrow{\mathscr{L}} \sum_{i_1,\dots,i_{p-r}=2}^{M} b^{(0)}_{i_1,\dots,i_{p-r}} I_{p-r}(\phi_{i_1}\otimes\dots\otimes\phi_{i_{p-r}}) + \sum_{k=1}^{[(p-r)/2]} C_{p-r,k}$$

$$\times \sum_{i_{2k+1},\dots,i_{p-r}=2}^{M} b^{(k)}_{1,1,\dots,1,i_{2k+1},\dots,i_{p-r}} I_{p-r-2k}(\phi_{i_{2k+1}}\otimes\dots\otimes\phi_{i_{p-r}}).$$

Note that since the conditions of Lemma 4.7 are easily satisfied in this case, making the obvious identifications, M can be replaced by ∞ in (4.32). Substituting the expression for the Fourier coefficients $b^{(k)}_{i_1,\dots,i_k,i_{2k+1},\dots,i_{p-r}}$ in (4.12) and (4.14), we see that the last expression in (4.32) (with $M=\infty$) is equal to

$$\sum_{i_1,\dots,i_{p-r}=2}^{\infty} \Big[\!\Big[\int_{[0,1]^{p-r}} f^{(r)}_{p,c}(x_1,\dots,x_{p-r})\,\phi_{i_1}(x_1)\dots\phi_{i_{p-r}}(x_{p-r})\,dx_1\dots dx_{p-r}\Big]\!\Big]$$

$$\times I_{p-r}(\phi_{i_1}\otimes\dots\otimes\phi_{i_{p-r}}) + \sum_{k=1}^{[(p-r)/2]} C_{p-r,k}$$

$$\times \sum_{\substack{i_{2k+1},\dots\\ \dots,i_{p-r}=2}}^{\infty} \Big[\!\Big[\int_{[0,1]^{p-r-2k}} \Big(\int_{[0,1]^k} f^{(r)}_{p,c}(x_1,x_1,\dots,x_k,x_k,x_{2k+1},\dots,x_{p-r})\,dx_1\dots dx_k\Big)$$

$$\times \phi_{i_{2k+1}}(x_{2k+1})\dots\phi_{i_{p-r}}(x_{p-r})\,dx_{2k+1}\dots dx_{p-r}\Big]\!\Big] I_{p-r-2k}(\phi_{i_{2k+1}}\otimes\dots\otimes\phi_{i_{p-r}}),$$

which is the same as

$$I_{p-r}(f^{(r)}_{p,c}) + \sum_{k=1}^{[(p-r)/2]} C_{p-r,k} I_{p-r-2k}\big(\tau_k(f^{(r)}_{p,c})\big).$$

Note also that from (4.29) it is seen that the left-hand side of (4.32) (with $M=\infty$) is equal to the summand B_n on the right-hand side of (4.28). Therefore, as $n\to\infty$,

$$(4.33)\quad B_n \xrightarrow{\mathscr{L}} I_{p-r}(f^{(r)}_{p,c}) + \sum_{k=1}^{[(p-r)/2]} C_{p-r,k} I_{p-r-2k}\big(\tau_k(f^{(r)}_{p,c})\big).$$

And finally from (4.28) we infer that (4.33) implies

$$n^{-(p-r)/2} \sum_{j_1,\dots,j_{p-r}=1}^{n} f_{p,c}^{(r)}(X_{j_1},\dots,X_{j_{p-r}})$$
$$\xrightarrow{\mathscr{L}} \sum_{k=0}^{[(p-r)/2]} C_{p-r,k} I_{p-r-2k}(\tau_k(f_{p,c}^{(r)})) = \delta_{p-r}(f_{p,c}^{(r)}). \blacksquare$$

We are now ready to prove our main theorem.

Proof of Theorem 4.3. Note initially that from Lemma 4.4 we have

$$(4.34)\quad V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx = \frac{1}{n^p} \sum_{j_1,\dots,j_p=1}^{n} f_p(X_{j_1},\dots,X_{j_p}) - \int_{[0,1]^p} f_p(x)\,dx$$
$$= \frac{1}{n^p} \sum_{j_1,\dots,j_p=1}^{n} \sum_{\pi} \sum_{r=0}^{p-1} \frac{1}{r!\,(p-r)!} f_{p,c}^{(r)}(X_{j_{\pi(1)}},\dots,X_{j_{\pi(p-r)}})$$
$$= \sum_{r=0}^{p-1} \frac{p!}{r!\,(p-r)!} \frac{1}{n^{(p-r)/2}} \left(\frac{1}{n^{(p-r)/2}} \sum_{j_1,\dots,j_{p-r}=1}^{n} f_{p,c}^{(r)}(X_{j_1},\dots,X_{j_{p-r}}) \right).$$

The second equality in the above set of equations follows from equation (4.3). Using (4.27) we obtain

$$n^{1/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!]$$
$$= \sum_{r=0}^{p-1} \frac{p!}{r!\,(p-r)!} \frac{n^{1/2}}{n^{(p-r)/2}} \left(\frac{1}{n^{(p-r)/2}} \sum_{j_1,\dots,j_{p-r}=1}^{n} f_{p,c}^{(r)}(X_{j_1},\dots,X_{j_{p-r}}) \right)$$
$$\xrightarrow{\mathscr{L}} 0 + p\delta_1(f_{p,c}^{(p-1)}) = \delta_1(f_{p,c}^{(p-1)}).$$

Moreover, if $f_{p,c}^{(p-1)}, f_{p,c}^{(p-2)}, \dots, f_{p,c}^{(p-k)}$ are identically 0, then

$$(4.35)\quad n^{(k+1)/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!]$$
$$= \sum_{r=0}^{p-k-2} \binom{p}{r} \frac{1}{n^{(p-k-1-r)/2}} \left(\frac{1}{n^{(p-r)/2}} \sum_{j_1,\dots,j_{p-r}=1}^{n} f_{p,c}^{(r)}(X_{j_1},\dots,X_{j_{p-r}}) \right)$$
$$+ \binom{p}{k+1} \left(\frac{1}{n^{(k+1)/2}} \sum_{j_1,\dots,j_{k+1}=1}^{n} f_{p,c}^{(p-k-1)}(X_{j_1},\dots,X_{j_{k+1}}) \right).$$

From (4.27) it follows that the first term on the right-hand side of (4.35) converges to 0 in probability and the second term converges in distribution to

$$\binom{p}{k+1} \delta_{k+1}(f_{p,c}^{(p-k-1)})$$

as $n \to \infty$. Therefore we have shown that

$$n^{(k+1)/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \xrightarrow{\mathscr{L}} \binom{p}{k+1} \delta_{k+1}(f_{p,c}^{(p-k-1)}) \quad \text{as } n \to \infty. \blacksquare$$

The next result is due to Hoeffding (see [3], Theorem 7.4 and remarks on p. 306). We record it here (in our notation) since it is a natural corollary to Theorem 4.3.

COROLLARY 4.13. *Let f_p be as in Theorem 4.3. Suppose that $\|f_{p,c}^{(p-1)}\| \neq 0$. Then both*

$$n^{1/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \quad \textit{and} \quad n^{1/2} [\![U_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!]$$

are asymptotically normal with mean 0 *and variance* $p^2 \|f_{p,c}^{(p-1)}\|^2$.

Proof. Applying Theorem 4.3 we have

$$n^{1/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \xrightarrow{\mathscr{L}} \binom{p}{1} \delta_1(f_{p,c}^{(p-1)}) = p I_1(f_{p,c}^{(p-1)}). \tag{4.36}$$

By Lemma 5.7.3 of [10] we obtain

$$n^{1/2} [\![V_n(f_p) - U_n(f_p)]\!] \to 0 \quad \text{as } n \to \infty. \tag{4.37}$$

Therefore from (4.36) and (4.37) we have

$$n^{1/2} [\![U_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \xrightarrow{\mathscr{L}} p I_1(f_{p,c}^{(p-1)}).$$

The proof is completed by observing that $p I_1(f_{p,c}^{(p-1)})$ is normally distributed with mean 0 and variance $p^2 \|f_{p,c}^{(p-1)}\|^2$. $\blacksquare$

The following proposition relates the asymptotic distribution of V-statistics with that of a linear combination of U-statistics.

PROPOSITION 4.14. *Let f_p be as in Theorem 4.3. Suppose that $f_{p,c}^{(p-j)} = 0$ a.e. in $[0, 1]^j$ for all $j = 1, 2, \ldots, r-1$; $r \leqslant p$. Then*

$$n^{r/2} [\![V_n(f_p) - \int_{[0,1]^p} f_p(x)\,dx]\!] \tag{4.38}$$

has the same asymptotic distribution as

$$\binom{p}{r} \sum_{k=0}^{[r/2]} C_{r,k}\, n^{(r-2k)/2}\, U_n\big(\tau^k(f_{p,c}^{(p-r)})\big). \tag{4.39}$$

The proof makes use of the Hu–Meyer formula (equation (2.24)) and the fact that the asymptotic distribution of U_n is a multiple Wiener integral.

5. Asymptotic distribution of von Mises differentiable statistical functions. In this section we recall Filippova's result [2] on the asymptotic distribution of a differentiable statistical function and give an alternative proof using results from Section 4. Filippova obtained the limit as an MSI with respect to a Brownian bridge. In this section we obtain the limit as an MSI with respect to a Wiener process and show that the two integrals are the same almost surely.

We will assume in this section the setup of Section 4. Let $\mathscr{G}$ be the class of all real-valued measurable functions defined on R. Let F be the distribution function of the i.i.d. $\mathscr{U}[0, 1]$ variates $X_1, \ldots, X_n$. Denote the empirical distribution function based on $X_1, \ldots, X_n$ by F_n. A real-valued function T on $\mathscr{G}$ will be referred to as a *statistical function*. The following definitions are taken from [2].

DEFINITION 5.1. A statistical functional T is called *p times differentiable at the point* $G \in \mathscr{G}$ with respect to the set $\mathscr{G}_1 \subset \mathscr{G}$, which is assumed to be star shaped at the point G (i.e., if $G_1 \in \mathscr{G}_1$, then $G + t(G_1 - G) \in \mathscr{G}_1$ for all $t \in [0, 1]$), if the following conditions are satisfied:

(1) For any $t \in [0, 1]$, $m = 1, 2, \ldots, p$, and any $G_1 \in \mathscr{G}_1$,

$$\frac{d^m}{dt^m} T[G + t(G_1 - G)]$$

exists.

(2) There exist functions $T^{(m)}(G)\colon R^m \to R$, $m = 1, 2, \ldots, p$, such that for any $G_1 \in \mathscr{G}_1$, the relation

$$\frac{d^m}{dt^m} T[G + t(G_1 - G)]|_{t=0} = \int_{R^m} T^m(G)(y)\,dy \tag{5.1}$$

holds.

DEFINITION 5.2. A statistical functional T is called a *von Mises functional of order p at the point F* if:

(1) there exists a star shaped set $\mathscr{G}_1 \subset \mathscr{G}$ at the point F such that

$$\lim_{n \to \infty} P(F_n \in \mathscr{G}_1) = 1;$$

(2) the functional T is p times differentiable (in the sense of Definition 5.1) at F with respect to $\mathscr{G}_1$;

(3) for any $\varepsilon > 0$, $\delta > 0$, and $m = 1, \ldots, p$,

$$\lim_{n \to \infty} P\left(n^{(m/2)-\delta} \sup_{t \in [0,1]} \left|\frac{d^m}{dt^m} T[F + t(F_n - F)]\right| > \varepsilon\right) = 0. \tag{5.2}$$

Filippova's theorem ([2], Theorem 4) gives the asymptotic distribution of $n^{p/2}[T(F_n) - T(F)]$ (where T is a von Mises functional of order $p+1$ with $T^{(m)}(F)$ identically equal to zero for $m = 1, 2, \ldots, p-1$) as an MSI with respect to a Brownian bridge. We now give an alternative proof of Filippova's theorem and in addition we obtain the limit as an MSI with respect to the Wiener process.

THEOREM 5.3. *Let T be a von Mises function of order $p+1$. Suppose that $T^{(m)}(F) \equiv 0$, $m = 1, 2, \ldots, p-1$. Denote $T^{(p)}(F)$ by $p!f_p$. Suppose that f_p satisfies the conditions in* (4.6). *Then*

$$n^{p/2}(T(F_n)-T(F)) \xrightarrow{\mathscr{L}} \delta_p(f_{p,c}) = \delta_p^0(f_p), \tag{5.3}$$

where $f_{p,c}$ is the centering of f_p.

Proof. Observe initially that from Theorem 2 of [2] we have

$$n^{p/2}(T(F_n)-T(F)) = n^{p/2} \int_{[0,1]^p} f_p(x_1, \ldots, x_p) \prod_{j=1}^{p} d[F_n(x_j)-x_j]+o_p(1). \tag{5.4}$$

Note that when $p = 1$, the theorem follows by applying the central limit theorem. Therefore, without loss of generality assume $p > 1$. Then

$$\begin{aligned} \int_{[0,1]^p} f_p(x_1, \ldots, x_p) \prod_{j=1}^{p} d[F_n(x_j)-x_j] &= \frac{1}{n^p} \sum_{j_1,\ldots,j_p=1}^{n} f_p(X_{j_1}, \ldots, X_{j_p}) \\ &+ \sum_{r=1}^{p-1} (-1)^r \binom{p}{r} \frac{1}{n^{p-r}} \sum_{j_1,\ldots,j_{p-r}=1}^{n} \int_{[0,1]^r} f_p(X_{j_1}, \ldots, X_{j_{p-r}}, y)\, dy \\ &+ (-1)^p \int_{[0,1]^p} f_p(y)\, dy. \end{aligned} \tag{5.5}$$

A further simplification yields the following equality:

$$\begin{aligned} &\int_{[0,1]^p} f_p(x_1, \ldots, x_p) \prod_{j=1}^{p} d[F_n(x_j)-x_j] \\ &= \frac{1}{n^p}\Bigg(\sum_{j_1,\ldots,j_p=1}^{n} f_p(X_{j_1}, \ldots, X_{j_p}) \\ &+ \sum_{r=1}^{p-1} \frac{(-1)^r}{r!(p-r)!} \sum_{\pi} \int_{[0,1]^r} f_p(X_{j_{\pi(1)}}, \ldots, X_{j_{\pi(p-r)}}, y)\, dy + (-1)^p \int_{[0,1]^p} f_p(y)\, dy \Bigg). \end{aligned}$$

From equation (4.3) it follows that the expression on the right-hand side equals

$$n^{-p} \sum_{j_1,\ldots,j_p=1}^{n} f_{p,c}(X_{j_1}, \ldots, X_{j_p}).$$

Therefore, in view of Lemma 4.12, we have

$$\begin{aligned} n^{p/2} &\int_{[0,1]^p} f_p(x_1, \ldots, x_p) \prod_{j=1}^{p} d[F_n(x_j)-x_j] \\ &= \frac{1}{n^{p/2}} \sum_{j_1,\ldots,j_p=1}^{n} f_{p,c}(X_{j_1}, \ldots, X_{j_p}) \xrightarrow{\mathscr{L}} \delta_p(f_{p,c}). \end{aligned} \tag{5.6}$$

Finally, we show now that $\delta_p(f_{p,c})$ is the same as $\delta_p^0(f_p)$. Using (4.3) we have

$$\delta_p(f_{p,c}) = \delta_p(f_p) + \sum_{\pi}\sum_{r=1}^{p-1} \frac{(-1)^r}{r!(p-r)!}\delta_p\Big(\int_{[0,1]^r} f_p(x_{\pi(1)}, \ldots, x_{\pi(p-r)}, y_r)\,dy_r\Big) \tag{5.7}$$
$$+\delta_p\Big((-1)^p \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p\Big)$$
$$= \delta_p(f_p) + \sum_{r=1}^{p-1} \frac{(-1)^r p!}{r!(p-r)!} W_1^r\, \delta_{p-r}\Big(\int_{[0,1]^r} f_p(x_1, \ldots, x_{p-r}, y_r)\,dy_r\Big)$$
$$+ (-1)^p W_1^p \int_{[0,1]^p} f_p(y_1, \ldots, y_p)\,dy_1 \ldots dy_p$$
$$= \delta_p^0(f_p).$$

The last equality follows from Proposition 3.3. Hence, taking the limit in (5.4) as $n \to \infty$ and using (5.7) and (5.6), we have the result. ∎

COROLLARY 5.4. *Let T be a von Mises functional of order 3. Suppose that $T^{(1)}(F) \equiv 0$. Denote $T^{(2)}(F)$ by $2f_2$. Suppose that f_2 satisfies the conditions in (4.6) with p replaced by 2. Let $f_{2,c}$ be the centering of f_2. Then*

$$n\big(T(F_n) - T(F)\big) \xrightarrow{\mathscr{L}} \sum_{k=1}^{\infty} \lambda_k [\chi_{1,k}^2 - 1] + \int_0^1 f_{2,c}(x, x)\,dx,$$

where $\{\chi_{1,k}^2\}$ is a sequence of independent χ_1^2 variates and λ_k are the eigenvalues of the integral operator corresponding to $f_{2,c}$.

Proof. Note that, by Theorem 5.3,

$$n\big(T(F_n) - T(F)\big) \xrightarrow{\mathscr{L}} \delta_2(f_{2,c}). \tag{5.8}$$

Again, from Proposition 2.9 we obtain

$$\delta_2(f_{2,c}) = I_2(f_{2,c}) + \int_0^1 f_{2,c}(x, x)\,dx. \tag{5.9}$$

Let $\{e_i\}$ be a CONS of $L^2[0, 1]$ such that e_i is an eigenvector corresponding to the eigenvalue λ_i. Then

$$I_2(f_{2,c}) = I_2\Big(\sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i\Big) = \sum_{i=1}^{\infty} \lambda_i I_2(e_i \otimes e_i) = \sum_{i=1}^{\infty} \lambda_i [(I_1(e_i))^2 - 1]$$
$$\overset{\mathscr{L}}{\equiv} \sum_{i=1}^{\infty} \lambda_i [\chi_{1,i}^2 - 1].$$

Substituting this in (5.9), we have the result from (5.8). ∎

Remark 5.5. This result is different from Serfling's Theorem 6.4.1 in [10], which says that the asymptotic limit is an infinite linear combination of independent χ_1^2 variates. The proof there seems to contain an error since it assumes that $\int_0^1 f_{2,c}(x, x)\,dx$ equals the trace of the integral operator induced by $f_{2,c}$. This is in general false as discussed in Section 2. Clearly, if $\int_0^1 f_{2,c}(x, x)\,dx$ does equal the trace of the integral operator associated with $f_{2,c}$, then

$$\int_0^1 f_{2,c}(x, x)\,dx = \sum_{i=1}^{\infty} \lambda_i,$$

and therefore

$$n\big(T(F_n) - T(F)\big) \xrightarrow{\mathscr{L}} \sum_{k=1}^{\infty} \lambda_k [\chi_{1,k}^2 - 1] + \int_0^1 f_{2,c}(x, x)\,dx = \sum_{k=1}^{\infty} \lambda_k \chi_{1,k}^2. \tag{5.10}$$

The statement (5.10) is true if, e.g., there exists a CONS $\{\phi_i\}$ of $L^2[0, 1]$ such that the Fourier coefficients of $f_{2,c}$ in the expansion with respect to the basis $\{\phi_i \otimes \phi_j\}$ are summable or, more generally, if the sufficient conditions given in Proposition 2.13 with f_p replaced by $f_{2,c}$ hold.

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