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ON THE ALMOST SURE APPROXIMATION AND CONVERGENCE OF LINEAR OPERATORS IN L_2-SPACES

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Abstract. We prove some results concerning the almost sure approximation of contractions in L_2 by projections, unitary operators or partial isometries. In particular, a unitary operator for which no subsequence of ergodic means converges almost surely is constructed.

1. Introduction. There are several classical theorems concerning the almost sure approximation in L_2. Let us mention here the results on (monotone) sequences of projections, like theorems on the convergence of orthogonal series or martingales. On the other hand, it is well known that every contraction (positive contraction) in an infinite-dimensional separable Hilbert space is a limit in the weak operator topology of some sequence of unitary operators (projections) (see [2]). In this paper we discuss the possibilities of approximation in the following sense. Let A, A_n be bounded linear operators acting in some $L_2(X, \mathscr{F}, \mu)$. We say that A_n *converges to* A *almost surely* ($A_n \to A$ a.s.) if the following condition is satisfied:

$$A_n f \to Af \quad \mu\text{-almost surely for each } f \in L_2.$$

In the sequel, A will be a contraction in L_2 and for A_n we shall take some "regular" operators like orthogonal projections, partial isometries or unitary operators. For the sake of simplicity, we shall formulate all the results for the case $H = L_2(0, 1)$ (with the ordinary Lebesgue measure on the unit interval). In fact, the theorems are true in much more general situations, at least for L_2 over a standard measure space.

In the sequel, $H = L_2(0, 1)$. $\mathrm{Proj}(H)$ is the set of all orthogonal projections in H. As usual, $B(H)$ will denote the algebra of all bounded linear operators acting in H.

2. Almost sure approximation. To present some typical methods, we start with simple results concerning the approximation of positive contractions. A stronger result is proved in Theorem 2.6.

2.1. PROPOSITION. *Let A be a positive contraction in H. Then there exist sequences $\{P_n\}$ of orthogonal projections and $\{U_n\}$ of unitary operators in $L_2(0, 1)$, such that*

(i) $P_n \to A$ *a.s. as* $n \to \infty$,

(ii) $U_n \to A$ *a.s. as* $n \to \infty$.

Proof. (i) We define a sequence Q_n of orthogonal projections by putting

$$Q_n^{\perp} = 1 - Q_n\colon f \to \chi_{[1/(n+1),\, 1/n)} f,$$

where χ_Z denotes the indicator of Z. Let $V_n \in B(H)$ be a partial isometry such that

$$V_n^* V_n = Q_n \quad \text{and} \quad V_n V_n^* = Q_n^{\perp}.$$

We put

$$P_n = Q_n A Q_n + \sqrt{Q_n A Q_n (I-A) Q_n}\, V_n^* + V_n \sqrt{Q_n A Q_n (I-A) Q_n} + V_n Q_n (I-A) Q_n V_n^*.$$

It is not difficult to observe that P_n is an orthogonal projection in H. Moreover, for any $f \in H$, we have

$$Q_n A Q_n f = Q_n A f - Q_n A Q_n^{\perp} f \xrightarrow{\text{a.s.}} Af$$

since $\sum_n \|Q_n A Q_n^{\perp} f\|^2 \leqslant \|A\|^2 \|f\|^2 \leqslant \|f\|^2 < +\infty$.

Putting

$$B_n = \sqrt{Q_n A Q_n (I-A) Q_n}\, V_n^*,$$

we have

$$\sum_n \|B_n f\|^2 \leqslant \sum_n \|V_n^* f\|^2 = \sum_n \|V_n^* Q_n^{\perp} f\|^2 = \sum_n \|Q_n^{\perp} f\|^2 \leqslant \|f\|^2 < +\infty.$$

Consequently, $B_n f \to 0$ a.s. Moreover,

$$\operatorname{supp}\{V_n \sqrt{Q_n A Q_n (I-A) Q_n}\, f\} \subset [1/(n+1),\, 1/n),$$

$$\operatorname{supp}\{V_n Q_n (I-A) Q_n V_n^* f\} \subset [1/(n+1),\, 1/n).$$

Finally, $P_n f \xrightarrow{\text{a.s.}} Af$.

(ii) We define a sequence U_n of unitary operators by putting

$$U_n = \sqrt{Q_n A^2 Q_n} - \sqrt{Q_n (I-A^2) Q_n}\, V_n^* + V_n \sqrt{Q_n (I-A^2) Q_n} + V_n \sqrt{Q_n A^2 Q_n}\, V_n^*.$$

The proof that $U_n \to A$ a.s. is similar to that of (i) and can be omitted. ∎

2.2. COROLLARY. *Let A be a contraction. Then there exists a sequence of partial isometries $\{V_n\}$ in H such that $V_n \to A$ a.s.*

Proof. It is enough to apply Proposition 2.1 (ii) to the operator $|A^*| = \sqrt{AA^*}$ by putting $V_n = U_n V$, where $A = |A^*| V$ is the polar decomposition of A. ∎

Let us remark that in [1] we proved more than Proposition 2.1 (i). In [1] it was shown that the orthogonal projections P_n can be taken as finite dimensional. However, in this case, the proof is much more complicated and is based on the following theorem:

THEOREM A (Ciach et al. [1]). *Let $\{A_n\}$ be a sequence of finite-dimensional self-adjoint operators. Suppose that $A_n \xrightarrow{s.} A$ as $n \to \infty$. Then there exists an increasing sequence $\{n_i\}$ such that $A_{n_i} \to A$ a.s. as $i \to \infty$.*

The above theorem embraces as a very special case the following well-known result:

THEOREM B (Marcinkiewicz [4]). *If $\{\varphi_v\}$ is an orthonormal sequence in $L_2(0, 1)$, then there exists a sequence $\{n_i\}$ of positive integers such that the subsequence of partial sums*

$$S_{n_i}(x) = \sum_{v=1}^{n_i} a_v \varphi_v(x) \qquad (i = 1, 2, \ldots)$$

converges almost surely on $(0, 1)$ for each sequence $\{a_v\}$ such that $\sum_v |a_v|^2 < \infty$.

Theorem A is a necessary tool in our further considerations.

Let us remark that Theorem A implies the following result, the strongest one in this direction (cf. Section 3).

THEOREM C. *Let $\{A_n\}$ be a sequence of finite-dimensional operators. If $A_n \xrightarrow{s.} A$, then there exists an increasing sequence $\{n_i\}$ such that $A_{n_i} \xrightarrow{a.s.} A$.*

2.3. LEMMA. *Let $\{A_n\}$ be a sequence of finite-dimensional operators. If $A_n \xrightarrow{s.} A$, then there exist an increasing sequence $\{n(k)\}$ and a sequence of finite-dimensional operators $\{B_k\}$ such that*

$$A_{n(k)} - B_k \xrightarrow{s.} A, \qquad (A_{n(k)} - B_k)^* \xrightarrow{s.} A^*, \qquad B_k \xrightarrow{a.s.} 0.$$

Proof. By assumption, one can define the indices $1 = n(1) < n(2) < \ldots$ and finite-dimensional projections $P_1 \leqslant P_2 \leqslant \ldots$, satisfying

$$(1) \qquad A_{n(k)} P_k = A_{n(k)}, \qquad \|(A_{n(k+1)} - A) P_k\| < 2^{-k}, \qquad P_k \xrightarrow{s.} 1.$$

Let us put $B_k = A_{n(k)} P_{k-1}^{\perp}$, $k \geqslant 2$. Then, for any $f \in H$,

$$\sum_{k=2}^{\infty} \|B_k f\|^2 = \sum_{k=2}^{\infty} \|A_{n(k)} P_k P_{k-1}^{\perp} f\|^2 \leqslant \sum_{k=2}^{\infty} \|A_{n(k)}\| \, \|(P_k - P_{k-1}) f\|^2 < \infty;$$

thus $B_k f \to 0$ a.s. It is evident that, by (1), $A_{n(k+1)} - B_{k+1} = A_{n(k+1)} P_k \to A$ $*$-strongly. ∎

Proof of Theorem C. Take $C_{n(k)} = A_{n(k)} - B_k$ as in Lemma 2.3. By Theorem A, for some $\{n''(k)\} \subset \{n'(k)\} \subset \{n(k)\}$, we have

$$C_{n'(k)} + C^*_{n'(k)} \xrightarrow{\text{a.s.}} A + A^*, \qquad C_{n''(k)} - C^*_{n''(k)} \xrightarrow{\text{a.s.}} A - A^*,$$

and, finally,

$$C_{n''(k)} \xrightarrow{\text{a.s.}} A, \qquad A_{n''(k)} \xrightarrow{\text{a.s.}} A. \ \blacksquare$$

2.4. COROLLARY. *Let $\{A_n\}$ be a sequence of finite-dimensional operators. If $A_n \xrightarrow{\text{s.}} A$ and $A_n^* \xrightarrow{\text{s.}} A^*$, then there exists an increasing sequence $\{n_i\}$ such that $A_{n_i} \xrightarrow{\text{a.s.}} A$ and $A_{n_i}^* \xrightarrow{\text{a.s.}} A^*$.*

2.5. EXAMPLE. Let $\{A_n\}$ be a sequence of finite-dimensional normal contractions such that $A_n \xrightarrow{\text{s.}} A$ and $A_n^* \xrightarrow{\text{s.}} A^*$. We construct a sequence of finite-dimensional partial isometries $\{V_n\}$ in H such that $V_n \to A$ a.s. and $V_n^* \to A$ a.s. By Corollary 2.4, without loss of generality we may assume that $A_n \to A$ a.s. and $A_n^* \to A^*$ a.s. Let

$$A_n f = \sum_{j=1}^{k_n} \lambda_j^{(n)} (f, \varphi_j^{(n)}) \varphi_j^{(n)},$$

where $\{\varphi_1^{(n)}, \ldots, \varphi_{k_n}^{(n)}\}$ is an orthonormal sequence, $|\lambda_j^{(n)}| \leqslant 1$. Let us denote by $\{f_1^{(n)}, \ldots, f_{k_n}^{(n)}\}$ an orthonormal sequence in the space $\operatorname{lin}\{\varphi_1^{(n)}, \ldots, \varphi_{k_n}^{(n)}\}^{\perp}$. Let us put, for $f \in H$,

$$V_n f = \sum_{j=1}^{k_n} [\lambda_j^{(n)} (f, \varphi_j^{(n)}) \varphi_j^{(n)} - \sqrt{1 - |\lambda_j^{(n)}|^2}\, (f, \varphi_j^{(n)}) f_j^{(n)}].$$

It is clear that V_n is a finite-dimensional partial isometry. It is not difficult to observe that $V_n \xrightarrow{\text{a.s.}} A$, $V_n^* \xrightarrow{\text{a.s.}} A^*$ if

$$\operatorname{supp} f_j^{(n)} \subset (1/(n+1), 1/n], \qquad n = 1, 2, \ldots;\ j = 1, 2, \ldots, k_n.$$

It is not hard to show the existence of such a double sequence $f_j^{(n)}$. In fact, for any projections P and Q, we have the relation

$$P - P \wedge Q \sim Q^{\perp} - Q^{\perp} \wedge P^{\perp} = Q^{\perp} - (Q \vee P)^{\perp}$$

(see [5]), where $\sim$ denotes the unitary equivalence. If Q is a finite-dimensional projection on the subspace $\operatorname{lin}\{\varphi_1^{(n)}, \ldots, \varphi_{k_n}^{(n)}\}$ and P is an infinite-dimensional projection on the subspace

$$K = \{f \in H;\ \operatorname{supp} f \subset (1/(n+1), 1/n]\},$$

then $P \wedge Q^{\perp}$ is infinite dimensional, as $P - P \wedge Q^{\perp} \sim Q - Q \wedge P^{\perp}$.

Now we shall prove the following general result:

2.6. THEOREM. *Let A be a contraction. Then there exists a sequence $\{U_n\}$ of unitary operators in H such that $U_n \to A$ a.s. and $U_n^* \to A^*$ a.s.*

Theorem 2.6 results immediately from the following two lemmas:

2.7. LEMMA. *Let A be a contraction. Then there exists a sequence $\{W_n\}$ of finite-dimensional partial isometries in H such that $W_n \to A$ a.s. and $W_n^* \to A^*$ a.s.*

Proof. Let $\{A_n\}$ be a sequence of finite-dimensional contractions in H such that $A_n \xrightarrow{s.} A$ and $A_n^* \xrightarrow{s.} A^*$. By Corollary 2.4, without loss of generality we may assume that $A_n \xrightarrow{a.s.} A$ and $A_n^* \xrightarrow{a.s.} A^*$. Let $A_n = |A_n^*| V_n$ be the polar decomposition of A_n and let

$$K_n = \{f \in H;\ \operatorname{supp} f \subset (1/(n+1),\, 1/n]\}.$$

Suppose that $H_n \subset K_n \wedge \operatorname{ran}(|A_n^*|)^{\perp}$, $\dim \operatorname{ran}(|A_n^*|) = \dim H_n$, and let $T_n^* T_n = P_n$, $T_n T_n^* = Q_n$, where P_n and Q_n are the orthogonal projections onto the spaces $\operatorname{ran}(|A_n^*|)$ and H_n, respectively, and T_n is a partial isometry. It is not difficult to observe that

$$S_n = |A_n^*| + T_n \sqrt{I - |A_n^*|^2}$$

is a finite-dimensional partial isometry since

$$\begin{aligned}\|S_n P_n f\|^2 &= \||A_n^*| f\|^2 + \|T_n P_n \sqrt{I-|A_n^*|^2}\, P_n f\|^2 \\ &= (|A_n^*|^2 P_n f,\, P_n f) + ((I-|A_n^*|^2) P_n f,\, P_n f) = \|P_n f\|^2\end{aligned}$$

and $S_n P_n^{\perp} f = 0$.

Finally, $W_n = S_n V_n$ is a partial isometry since $V_n V_n^* = P_n$ and $S_n^* S_n = P_n$ and, for any $f \in H$, we have

$$\begin{aligned} W_n f = S_n V_n f &= (|A_n^*| + T_n \sqrt{I-|A_n^*|^2})\, V_n f \\ &= A_n f + T_n \sqrt{I-|A_n^*|^2}\, V_n f \xrightarrow{a.s.} Af \end{aligned}$$

since $\operatorname{supp}(T_n \sqrt{I-|A_n^*|^2}\, V_n f) \subset (1/(n+1),\, 1/n]$. Moreover,

$$\begin{aligned} W_n^* f = V_n^* S_n^* f &= V_n^* (|A_n^*| + \sqrt{I-|A_n^*|^2}\, T_n^*) f \\ &= A_n^* f + V_n^* \sqrt{I-|A_n^*|^2}\, T_n^* f \xrightarrow{a.s.} A_n^* f \end{aligned}$$

because

$$\sum_n \|V_n^* \sqrt{I-|A_n^*|^2}\, T_n^* f\|^2 \leqslant \sum_n \|T_n^* f\|^2 = \sum_n \|Q_n f\|^2 \leqslant \|f\|^2. \ \blacksquare$$

2.8. LEMMA. *Suppose that $P_n, Q_n \in \operatorname{Proj}(H)$ and $P_n^{\perp}, Q_n^{\perp}$ are finite dimensional for $n = 1, 2, \ldots$ Then there exists a sequence $\{S_n\}$ of partial isometries in H such that*

$$S_n^* S_n = Q_n, \quad S_n S_n^* = P_n \quad \textit{and} \quad S_n \xrightarrow{a.s.} 0, \quad S_n^* \xrightarrow{a.s.} 0.$$

Proof. Let R_n be an orthogonal projection such that

$$R_n(H) \subset P_n(H) \cap Q_n(H) \cap K_n,$$

where $K_n = \{f \in H;\ \operatorname{supp} f \subset (1/(n+1), 1/n]\}$ and $\dim R_n(H) = \dim(P_n - R_n)(H) = \dim(Q_n - R_n)(H) = \infty$. Suppose that S'_n and S''_n are partial isometries such that

$$S'^*_n S'_n = R_n, \quad S'_n S'^*_n = P_n - R_n, \quad S''^*_n S''_n = Q_n - R_n, \quad S''_n S''^*_n = R_n.$$

It is clear that $S_n = S'_n + S''_n$ is a partial isometry and $S^*_n S_n = Q_n$, $S_n S^*_n = P_n$. Moreover, for any $f \in H$, we have $S''_n f \xrightarrow{\text{a.s.}} 0$ since $\operatorname{supp}(S''_n f) \subset (1/(n+1), 1/n]$ and $S'_n f \xrightarrow{\text{a.s.}} 0$ since $\sum_n \|S'_n f\|^2 = \sum_n \|R_n f\|^2 \leqslant \|f\|^2 < +\infty$. In consequence, $S_n \xrightarrow{\text{a.s.}} 0$. Analogously, $S^*_n \xrightarrow{\text{a.s.}} 0$. ■

Proof of Theorem 2.6. By Lemmas 2.7 and 2.8 where $W_n W^*_n = P_n^\perp$ and $W^*_n W_n = Q_n^\perp$, we define $U_n = W_n Q_n^\perp + S_n Q_n$. ■

Keeping the notation of Lemma 2.7, we have the following

2.9. PROPOSITION. *Let us assume that $\|A\| \leqslant \frac{1}{2}$. The finite-dimensional partial isometries $W_n \mapsto A$, $W^*_n \mapsto A^*$ a.s. can be chosen as the canonical linear combinations of four finite-dimensional mutually orthogonal projections $P_n^{(k)}$, $k = 1, 2, 3, 4$:*

$$W_n = P_n^{(1)} - P_n^{(2)} + iP_n^{(3)} - iP_n^{(4)}.$$

Proof. Let $A = A^{(1)} - A^{(2)} + iA^{(3)} - iA^{(4)}$, $A^{(k)} \geqslant 0$, $\sum_{k=1}^4 A^{(k)} \leqslant I$, and let $0 \leqslant A_n^{(k)}$ ($A_n^{(k)}$ finite dimensional) with $\operatorname{ran}(A_n^{(k)}) \subset H_n$ (H_n finite dimensional) be such that $A_n^{(k)} \to A^{(k)}$ a.s. as $n \to \infty$ and $\sum_{k=1}^4 A_n^{(k)} \leqslant I$. By the Naĭmark Dilation Theorem, for every n there exist finite-dimensional Hilbert space $\tilde{H}_n$ and mutually orthogonal projections $P_n^{(k)}$, $k = 1, 2, 3, 4$, acting in $H_n \oplus \tilde{H}_n$ and such that $Q_n P_n^{(k)} Q^*_n = A_n^{(k)}|_{H_n}$ $(k = 1, 2, 3, 4)$, where Q_n: $H_n \oplus \tilde{H}_n \to H_n$ and $\tilde{Q}_n$: $H_n \oplus \tilde{H}_n \to \tilde{H}_n$ are canonical projections. Passing, if necessary, to an isomorphic image, we can assume that

$$\tilde{H}_n \subset K_n = \{f \in H;\ \operatorname{supp} f \subset (1/(n+1), 1/n]\}$$

(this is possible because $\dim \tilde{H}_n < \infty$ and $\dim H_n < \infty$).

Let us put

$$P_n^{(k)} f = Q_n f = \tilde{Q}_n f = 0 \quad \text{for } f \in (H_n \oplus \tilde{H}_n)^\perp.$$

Then

$$\begin{aligned} P_n^{(k)} &= (Q_n + \tilde{Q}_n) P_n^{(k)} (Q_n + \tilde{Q}_n) \\ &= A_n^{(k)} + \tilde{Q}_n P_n^{(k)} Q_n + Q_n P_n^{(k)} \tilde{Q}_n + \tilde{Q}_n P_n^{(k)} \tilde{Q}_n \xrightarrow{\text{a.s.}} A^{(k)} \quad (k = 1, 2, 3, 4). \ ■ \end{aligned}$$

We conclude this section with the following

2.10. EXAMPLE. We construct a sequence $\{U_n\}$ of unitary operators which is convergent almost surely to the normal contraction A in $L_2(0, 1)$, where

$(Af)(\lambda) = g(\lambda)f(\lambda)$, $f \in H$, for some $g \in L_\infty(0, 1)$. Let

$$\gamma_n\colon \left[\frac{1}{n}, 1\right) \to \left[\frac{1}{n+1}, \frac{1}{n}\right), \quad \gamma_n(\lambda) = \frac{1}{n} + \frac{\lambda - 1}{n^2 - 1},$$

and, for $Z \subset [1/n, 1)$, let us set $F_n(Z) = E(\gamma_n(Z))$, where $E(A)f = \chi_A f, f \in H$. It is not difficult to check that, putting

$$U_n = \int_{1/n}^{1} g(\lambda)dE(\lambda) + \int_{1/n}^{1} \overline{g(\lambda)}\, dF_n(\lambda)$$
$$+ \int_{1/n}^{1} \sqrt{1 - |g(\lambda)|^2}\, d\big(V_n(\lambda) - V_n^*(\lambda)\big) + E\left[\left(0, \frac{1}{n+1}\right]\right],$$

where $V_n(\cdot)$ is an operator measure satisfying $V_n(Z)V_n^*(Z) = F_n(Z)$, $V_n^*(Z)V_n(Z) = E(Z)$, we obtain unitary operators U_n such that $U_n \to A$ a.s.

2.11. Remark. For each sequence $\{V_n\}$ of partial isometries in H, $V_n \to A$ a.s. implies that $V_n \to A$ weakly. Indeed, for any $f, g \in H$, the functions $(V_n f)g$ are uniformly integrable and $(V_n f)g \to (Af)g$ a.s. This immediately implies that $V_n \to A$ weakly.

3. Sequences having no a.s. convergent subsequences. We start with the following observation. Let r_n be Rademacher functions on $(0, 1)$; then for $A_n f = (f, r_1)r_n$ we have $A_n \to 0$ weakly and $A_{n(k)} \to 0$ a.s. for *no* increasing sequence $\{n(k)\}$. Thus the assumption in Theorems A and C, concerning the strong convergence $A_n \to A$, is necessary in a rather obvious way.

In Theorems A and C, the assumption that the operators A_n are finite dimensional cannot be omitted (in Theorem B of Marcinkiewicz this assumption, for $A_n = \sum_{k=1}^{n} (\cdot, \varphi_k)\varphi_k$, is automatically satisfied). To show this, we construct a suitable counterexample based on the following idea of Menshov (cf. [3], Lemma 1, p. 295):

3.1. Lemma. *There exists a constant $c_0 > 0$ such that, for any $N = 1, 2, \ldots$, one can find an orthonormal sequence $\{\psi_s^N\}_{s=1,\ldots,N}$ of functions on $(0, 1)$ such that*

$$\lambda\{\omega \in (0, 1)\colon \max_{1 \leqslant j \leqslant N} \Big|\sum_{s=1}^{j} \psi_s^N(\omega)\Big| > c_0 N^{1/2} \log_2 N\} \geqslant \frac{1}{4}$$

and, moreover,

$$\psi_s^N(\omega) \in D_{4N}, \qquad \int_{(0,1)} \psi_s^N(\omega)d\omega = 0, \qquad s = 1, \ldots, N,$$

where

(2) $$D_N = \Big\{\sum_{s=1}^{N} \alpha_s \chi_{((s-1)/N,\, s/N)};\ \alpha_s \in \boldsymbol{C}\Big\}.$$

3.2. Counterexample. There exists a sequence of projections $P_n \nearrow I$ such that, for any increasing sequence $\{n(k)\}$, one can choose a vector f such that $P_{n(k)} f$ does not converge a.s.; moreover, $P_{n(k)} f$ does not converge on a set of Lebesgue measure 1.

3.3. Lemma. *Fix $\varepsilon > 0$. There exists $N(\varepsilon)$ such that, for any $N \geqslant N(\varepsilon)$ and $k \geqslant 1$, one can find an orthogonal sequence*

$$\{f_1, \ldots, f_N\} \subset D_{4Nk} \cap D_k^{\perp},$$

with D_k defined in (2), *such that*

$$\lambda\{\omega \in (0, 1): \max_{1 \leqslant j \leqslant N} \Big|\sum_{s=1}^{j} f_s(\omega)\Big| > 1\} \geqslant \frac{1}{4}, \qquad \sum_{j=1}^{N} \|f_j\|^2 \leqslant \varepsilon,$$

$\int f_j = 0$ *and f_j is stochastically independent of f for $j = 1, \ldots, N$, $f \in D_k$.*

Proof. Choose $N(\varepsilon)$ satisfying $\big(c_0 \log_2 N(\varepsilon)\big)^2 > 1/\varepsilon$. In Lemma 3.1, let $N > N(\varepsilon)$ and let $\tilde{\psi}_j$ be an extension of ψ_j^N with a period 1 on the whole $\boldsymbol{R}$. Let us write

$$f_j(\omega) = \tilde{\psi}_j(k\omega)/c_0 N^{1/2} \log_2 N, \qquad \omega \in (0, 1).$$

All the required properties of f_j, $j = 1, \ldots, N$, are rather obvious. ■

Construction of Counterexample 3.2. Fix $\varepsilon_r > 0$, $\sum_{r=1}^{\infty} \varepsilon_r < \infty$. Let $N_r > N(\varepsilon_r)$, $N_r \nearrow \infty$, according to Lemma 3.3. Denote by $\mathscr{L}_r$ the set of all increasing sequences of indices of length N_r. Let $\{c_i\}$ be the sequence exhausting all elements of $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \ldots$, i.e., $c_i \in \mathscr{L}_{r(i)}$ and

$$c_i = (n_1^i, \ldots, n_{N_{r(i)}}^i) \tag{3}$$

for any $i = 1, 2, \ldots$ Using Lemma 3.3, one can define by induction indices $k(1) = 1$, $k(i+1) = 4N_{r(i)} k(i)$, and orthogonal sequences $\{f_1^{(i)}, \ldots, f_{N_{r(i)}}^{(i)}\}$ satisfying the following conditions: $f_j^{(i)}$ and $f_j^{(i')}$ are orthogonal and independent for $i \neq i'$, and

$$\lambda\{\omega \in (0, 1): \max_{1 \leqslant j \leqslant N_{r(i)}} \Big|\sum_{s=1}^{j} f_s^{(i)}(\omega)\Big| > 1\} \geqslant \frac{1}{4}, \qquad \sum_{j=1}^{N_{r(i)}} \|f_j^{(i)}\|^2 \leqslant \varepsilon_{r(i)}.$$

Denote by $\hat{f}$ the projection $(\cdot, f/\|f\|^2) f$ for any $0 \neq f \in H$, and let

$$P_n = \sum_{\substack{i \geqslant 1 \\ 1 \leqslant j \leqslant N_{r(i)} \\ n_j^i < n}} \hat{f}_j^{(i)} + \Big(1 - \sum_{\substack{i \geqslant 1 \\ 1 \leqslant j \leqslant N_{r(i)}}} \hat{f}_j^{(i)}\Big)$$

according to (3). Let now an increasing sequence $\{n(k)\}$ be given. Obviously, $\{n(k)\}$ can be divided into finite sequences of lengths $N_1, N_2, \ldots$:

$$\{n(k)\} = \{n_1^{i(1)}, \ldots, n_{N_1}^{i(1)}; n_1^{i(2)}, \ldots, n_{N_2}^{i(2)}; \ldots\} \tag{4}$$

with $r(i(1)) = 1$, $r(i(2)) = 2, \ldots$ For the vector

$$f = \sum_{r=1}^{\infty} (f_1^{(i(r))} + \ldots + f_{N_r}^{(i(r))}),$$

we have

$$\|f\|^2 \leqslant \sum_{r=1}^{\infty} \varepsilon_r < \infty,$$

$$\lambda\{\omega;\ \max_{1 \leqslant j \leqslant N_{r+1}} |(P_{n_j^{i(r+1)}} - P_{n_{N_r}^{i(r)}}) f(\omega)| > 1\} = \lambda\{\omega;\ \max_{1 \leqslant j \leqslant N_{r+1}} |\sum_{s=1}^{j} (f_s^{(i(r+1))}(\omega)| > 1\}.$$

Let

$$Z_r = \{\omega;\ \max_{1 \leqslant j \leqslant N_{r+1}} |(P_{n_j^{i(r+1)}} - P_{n_{N_r}^{i(r)}}) f(\omega)| > 1\}.$$

Then $\lambda(Z_r) \geqslant \frac{1}{4}$ and $Z_1, Z_2, \ldots$ are independent. Thus, by the Borel–Cantelli lemma and by (4), $\{P_{n(k)} f\}$ does not converge on a set of Lebesgue measure 1. ∎

It is well known that, in general, the individual ergodic theorem does not hold for an arbitrary unitary operator U in $L_2(0, 1)$. The natural question arises whether, for such an operator U, there exists an increasing sequence $\{n(k)\}$ of indices such that

$$\sigma_{n(k)}(U) f = \frac{1}{n(k)} \sum_{s=1}^{n(k)-1} U^s f \xrightarrow{\text{a.s.}} \bar{f} \tag{5}$$

for any $f \in L_2(0, 1)$. The answer is negative. As an important application of our Counterexample 3.2, we construct the following

3.4. COUNTEREXAMPLE. There exists a unitary operator U in $L_2(0, 1)$ such that, for every increasing sequence $\{n(k)\}$, there exists a vector $f \in L_2(0, 1)$ such that $\sigma_{n(k)}(U) f$ does not converge a.s.

We start with two simple observations

$$\left|\sum_{l=0}^{m-1} e^{i\delta l}\right| \leqslant \frac{2}{|1 - e^{i\delta}|} = \frac{1}{|\sin(\delta/2)|}, \qquad 0 < \delta < \pi,$$

$$\left|\frac{1}{m} \sum_{l=0}^{m-1} e^{i\delta l} - 1\right| \leqslant 2 \sin\frac{\delta_1}{2}, \qquad 0 < m\delta \leqslant \delta_1 < \pi.$$

By induction, we can define $\pi/2 = \delta(1) > \delta(2) > \ldots > 0$ and $N(1) < N(2) < \ldots$, satisfying

$$\left|\frac{1}{m} \sum_{l=0}^{m-1} e^{i\delta l}\right| < \frac{1}{2^h} \quad \text{for } m > N(h),\ \delta(h) \leqslant \delta \leqslant \frac{\pi}{2}, \tag{6}$$

$$\left|\frac{1}{m}\sum_{l=0}^{m-1} e^{i\delta l}-1\right| < \frac{1}{2^h} \quad \text{for } 1 \leqslant m \leqslant N(h),\ 0<\delta<\delta(h+1), \tag{7}$$

for any $h = 1, 2, \ldots$ We define

$$U = \sum_{s=1}^{\infty} e^{i\delta(s)}(P_s - P_{s-1}),$$

where $P_0 = 0$, and $P_1, P_2, \ldots$ is the sequence of orthogonal projections from Counterexample 3.2.

Fix an increasing sequence $\{n(k)\}$. Then there exist a subsequence $\{m(k)\} \subset \{n(k)\}$ and an increasing sequence $\{h(k)\}$, such that

$$N(h(k)) < m(k) \leqslant N(h(k+1)).$$

By Counterexample 3.2, the sequence

$$\{P_{h(k)}f\} \text{ does not converge a.s.} \tag{8}$$

for some $f \in H$. It is enough to prove that, according to (5), $\sigma_{m(k)}(U)f$ does not converge a.s.

It is clear that

$$\sigma_{m(k)}(U)f = \sum_{s=1}^{\infty}\left[\frac{1}{m(k)}\sum_{l=0}^{m(k)-1} e^{i\delta(s)l}\right](P_s - P_{s-1})f.$$

Thus, as $P_s \to 1$ strongly, we have

$$\|\sigma_{m(k)}(U)f - P_{h(k+1)}^{\perp}f\|^2 = \left\|\sum_{s=1}^{h(k+1)}\left[\frac{1}{m(k)}\sum_{l=0}^{m(k)-1} e^{i\delta(s)l}\right](P_s-P_{s-1})f\right\|^2$$

$$+\left\|\sum_{s=h(k+1)+1}^{\infty}\left[\frac{1}{m(k)}\sum_{l=0}^{m(k)-1} e^{i\delta(s)l}-1\right](P_s-P_{s-1})f\right\|^2.$$

Inequalities (6) and (7) imply

$$\left|\frac{1}{m(k)}\sum_{l=0}^{m(k)-1} e^{i\delta(s)l}\right|^2 < 2^{-2h(k)} \leqslant 2^{-2k} \quad \text{for } 1 \leqslant s \leqslant h(k),$$

$$\left|\left(\frac{1}{m(k)}\sum_{l=0}^{m(k)-1} e^{i\delta(s)l}\right)-1\right|^2 < 2^{-2h(k)} \leqslant 2^{-2k} \quad \text{for } s > h(k+1),$$

and we have

$$\sum_{k=1}^{\infty} \|\sigma_{m(k)}(U)f - P_{h(k+1)}^{\perp} f\|^2$$

$$\leqslant \sum_{k=1}^{\infty} \Big(\sum_{s=1}^{h(k)} 2^{-2k} \|(P_s - P_{s-1})f\|^2 + \sum_{s=h(k)+1}^{h(k+1)} \|(P_s - P_{s-1})f\|^2$$

$$+ \sum_{s=h(k+1)+1}^{\infty} 2^{-2k} \|(P_s - P_{s-1})f\|^2\Big) \leqslant \infty.$$

Thus $\sigma_{m(k)}(U)f - P_{h(k+1)}^{\perp} f \to 0$ a.s. as $k \to \infty$ and $\sigma_{m(k)}(U)f$ does not converge a.s. by (8). ■

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