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A MAXIMUM PRINCIPLE FOR BURGERS' EQUATION WITH UNIMODAL MOVING AVERAGE DATA*

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Abstract. The paper is devoted to a study of the extremal rearrangement property of statistical solutions of Burgers' equation with initial input generated by the Brownian motion or by a Poisson process.

1. Introduction. The non-linear diffusion equation

$$u_t + uu_x = \tfrac{1}{2}u_{xx}, \tag{1.1}$$

$t > 0$, $x \in \boldsymbol{R}$, $u = u(t, x)$, $u(0, x) = u_0(x)$, also called the *Burgers equation*, with random initial data has been studied for a long time (see, e.g., [3]). It describes propagation of non-linear hyperbolic waves, and has been considered as a model equation for various physical phenomena from the hydrodynamic turbulence (see, e.g., [12] and [5]) to evolution of the density of matter in the Universe (see [15] and [8]). Due to the non-linearity, its solutions enter several different stages (depending on the viscosity parameter which in the present paper is chosen to be 1/2), including that of shock waves formation.

Several mathematical papers written over the last few years have studied the question of large-time scaling limits for solutions of the Burgers equation with random initial data (see, e.g., [2], [18], [20], and [7]), the structure of shocks [16], and connections between the so-called intermediate asymptotics in Burgers' turbulence and the theory of extremal processes (see [14] and [17]).

In a recent paper [11], the authors discovered an extremal rearrangement property of statistical solutions of the Burgers equation with initial velocity potential data of the form

$$U_0(x) \equiv -\int_{-\infty}^{x} u_0(y)\,dy = \sum_{i=1}^{N} c_i \xi_{\lfloor x \rfloor - i}. \tag{1.2}$$

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Here $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. Roughly speaking, the property can be formulated as follows. Let ξ_i's be independent, identically distributed, with either Gaussian or Poisson distributions. Then, among all the permutations of the components of the coefficient vector $(c_1, \ldots, c_N)$, the permutation satisfying the condition

$$c_1 \leqslant c_N \leqslant c_2 \leqslant c_{N-1} \leqslant \ldots$$

guarantees the maximum variance (energy density) of the limiting solution random field.

In this note we study a continuous-time version of the above extremal problem with initial input of the form

$$U_0(x) = \int_{\boldsymbol{R}} h(x-y)\, dM(y), \tag{1.3}$$

where $h(y) \geqslant 0$ is a continuous function with compact support, and $M(y)$ is either the Brownian motion or a Poisson process defined on the whole real line. Although the continuous-time phenomenon is similar to the one encountered for discrete moving averages, it turns out that the passage from discrete to continuous initial data requires non-trivial modifications in the proofs. We do not know if our results extend to processes other than the Brownian motion and the Poisson process.

The following result parallels Theorem 1 of Hu and Woyczyński [11]. Notice, however, that the scaling in the continuous case is different from the scaling in the discrete case.

THEOREM 1.1. *Let $u = u(t, x)$ be a solution of* (1.1) *with initial data* (1.3). *Then, for each $x \in \boldsymbol{R}$,*

$$t^{3/4} u(t, x\sqrt{t}) \to N\left(0, \frac{1}{\sqrt{2\pi}} \sigma(h)\right) \tag{1.4}$$

in probability as $t \to \infty$, where,

(i) *if $M(y)$ is the Brownian motion,*

$$\sigma(h) = \int_{\boldsymbol{R}} \left(\exp\left(\int_{\boldsymbol{R}} h(x+y) h(y)\, dy\right) - 1\right) dx; \tag{1.5}$$

(ii) *if $M(y)$ is the Poisson process,*

$$\sigma(h) = \int_{\boldsymbol{R}} \left(\exp\left(\int_{\boldsymbol{R}} (e^{h(x+y)} - 1)(e^{h(y)} - 1)\, dy\right) - 1\right) dx. \tag{1.6}$$

The next result shows that the variance $\sigma(h)$ in (1.4) is an increasing functional of kernel h.

THEOREM 1.2. *If $h_1(x) \leqslant h_2(x)$ for all $x \in \operatorname{supp}(h)$, then $\sigma(h_1) \leqslant \sigma(h_2)$.*

Finally, our main result shows that the maximum rearrangement principle, proved in [11] (Theorem 3) via the Schur convexity arguments (see, e.g., [9]) also has a parallel for continuous-time moving averages. The result is again in the spirit of domination principles developed in [13].

THEOREM 1.3. *Suppose $h(\cdot) \geqslant 0$ is a symmetric-unimodal continuous function on $\boldsymbol{R}$ with compact support. Then*

$$\sigma(h) = \max_{g \in \mathcal{M}_h} \sigma(g), \tag{1.7}$$

where $\mathcal{M}_h$ is the family of all continuous functions equimeasurable with h.

The detailed definitions of notions used above, and some auxiliary results, are given in Section 2. Section 3 describes the relevant general domination principle, and the limit behavior needed in the proof of Theorem 1.1 is established in Section 4. Section 5 contains proofs of Theorems 1.1 and 1.2, and of Theorem 1.3 – the main result of this paper. Finally, in Section 6 we provide some additional comments on the domination principle from Section 3.

2. Preliminaries. Let us begin with establishing the notation that will be used throughout the paper. Let $f(\cdot)$ be a non-negative continuous function on $\boldsymbol{R}$, with compact support, and let $m(A)$ be the Lebesgue measure of the set $A \in \boldsymbol{R}$.

DEFINITION 2.1. Functions f_1, f_2 on $\boldsymbol{R}$ are said to be *equimeasurable* if, for all $c > 0$,

$$m(\{y: f_1(y) \geqslant c\}) = m(\{y: f_2(y) \geqslant c\}).$$

DEFINITION 2.2. A function f is said to be *unimodal* if, for any $c > 0$, $\{y: f(y) \geqslant c\}$ is an interval.

Notice that this definition has been structured so that it can be easily adjusted for functions of several variables by, say, replacing "an interval" by "a convex set." Elsewhere, we plan to extend results of this paper in that direction.

DEFINITION 2.3. A function f is said to be *symmetric-unimodal* if $f(\cdot)$ is a unimodal function and if there exists t_0 such that, for all $x \in \boldsymbol{R}$, $f(t_0 - x) = f(t_0 + x)$.

LEMMA 2.1. *Suppose $f(t)$ is a continuous function with compact support such that* $\operatorname{supp}(f) \subset [a, b]$. *Then there exists a symmetric-unimodal and continuous function $\hat{f} \in \mathcal{M}_f$ such that*

$$\hat{f}((b-a)/2 + a - x) = \hat{f}((b-a)/2 + a + x) \quad \textit{for all } x \geqslant 0.$$

Proof. Define $l(c) = m(\{y: f(y) > c\})$ for $c \geqslant 0$. It is easy to see that $l(c)$ is continuous and strictly decreasing in the interval $[0, \max f(y)]$. We can extend $l(\cdot)$ to the interval $[0, \infty)$ by assuming $l(x) = 0$ when $x \in [\max f(y), \infty)$.

Introduce a function $\hat{f}(\cdot)$ satisfying the symmetry condition

$$\hat{f}((b-a)/2+a-x) = \hat{f}((b-a)/2+a+x) \quad \text{for all } x \geqslant 0,$$

and defined as follows: If $x = (b-a)/2+a+l(z)/2$ with $z \in [0, \max f(y)]$, then we set $\hat{f}(x) = z$. If $x \geqslant (b-a)/2+a+l(0)/2$, then we set $\hat{f}(x) = 0$. The function $l(x)$ is continuous and strictly decreasing for $x \in [0, \max f(y)]$. Therefore $\hat{f}(\cdot)$ is well defined, symmetric-unimodal and continuous on $\boldsymbol{R}$. It is easy to check that $\hat{f} \in \mathcal{M}_f$. ∎

In the rest of this paper, we shall always denote by $\hat{f}$ this special symmetric-unimodal function in $\mathcal{M}_f$.

The property of being symmetric-unimodal is preserved under convolutions. The following result is due to Wintner [19].

Proposition 2.1. *Suppose f_1 and f_2 are two symmetric-unimodal functions such that their convolution is well defined on $\boldsymbol{R}$. Then*

$$g(y) = \int_{-\infty}^{\infty} f_1(y-x) f_2(x)\, dx$$

is symmetric-unimodal.

Now, let

$$R_f(x) = \int f(x-y) f(-y)\, dy.$$

It is easy to check that $R_f(-x) = R_f(x)$. Since $R_f(x)$ is an even function, we can restrict our attention to $x \in [0, \infty)$.

Another function $\tilde{R}_f(x)$, related to $R_f(x)$, will be used in the remainder of this note. It is defined as follows. Let

$$\tilde{R}_f(x) = \begin{cases} c & \text{if } x = m([0, \infty) \cap \{y\colon R_f(y) > c\}); \\ 0 & \text{if } x \geqslant m([0, \infty) \cap \{y\colon R_f(y) > c\}). \end{cases}$$

Using an argument similar to that of the proof of Lemma 2.1, it is easy to check that the $\tilde{R}_f(\cdot)$ is well defined on $[0, \infty)$, continuous and decreasing.

3. A domination property. In this section, we will prove the main domination property of this note. The notation is that of Section 2.

Proposition 3.1 (domination property). *Suppose f is a continuous and symmetric-unimodal function with compact support. Then, for any $g \in \mathcal{M}_f$ and any $x > 0$,*

$$\int_0^x \tilde{R}_g(y)\, dy \leqslant \int_0^x R_f(y)\, dy \tag{3.1}$$

and

$$\int_0^\infty \tilde{R}_g(y)\, dy = \int_0^\infty R_f(y)\, dy. \tag{3.2}$$

To prove the above proposition, we need the following four lemmas.

LEMMA 3.1. *If f is a continuous and symmetric-unimodal function, then, for* $x \in [0, \infty)$,

$$\tilde{R}_f(x) = R_f(x). \tag{3.3}$$

Proof. Since $f(x)$ is symmetric-unimodal, we know that $f(-x)$ is symmetric-unimodal as well. By Proposition 2.1, $R_f(x)$ is also symmetric-unimodal. Due to the fact that $R_f(x)$ is even, $R_f(x)$ is decreasing in $x \in [0, \infty)$. In view of the construction of $\tilde{R}_f(x)$ given in Section 2, the result follows. ∎

LEMMA 3.2. *For any $g \in \mathcal{M}_f$, the equality* (3.2) *holds true.*

Proof. We first notice that

$$\int_0^\infty \tilde{R}_g(y)\,dy = \int_0^\infty R_g(y)\,dy.$$

Indeed, this is so because

$$\begin{aligned}\int_0^\infty \tilde{R}_g(y)\,dy &= \int_0^{\max(\tilde{R}_g(y))} m(\{y\colon \tilde{R}_g(y) > c\})\,dc \\ &= \int_0^{\max(R_g(y))} m(\{y\colon R_g(y) > c\})\,dc = \int_0^\infty R_g(y)\,dy.\end{aligned}$$

Next, it is easy to check that

$$\begin{aligned}\int_{-\infty}^{\infty} R_g(y)\,dy &= \left(\int g(y)\,dy\right)^2 = \left(\int_0^{\max(g(y))} m(\{y\colon g(y) > c\})\,dc\right)^2 \\ &= \left(\int_0^{\max(f(y))} m(\{y\colon f(y) > c\})\,dc\right)^2 = \left(\int f(y)\,dy\right)^2 = \int_{-\infty}^{\infty} R_f(y)\,dy.\end{aligned}$$

Since both $R_g(y)$ and $R_f(y)$ are even, the lemma follows. ∎

Before we state Lemma 3.3, some additional notation is needed. Let Π denote the permutation group on $\{1, 2, \ldots, n\}$. For a vector $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\pi \in \Pi$, write

$$\pi\vec{a} = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}).$$

Denote by $(a_{[1]}, a_{[2]}, \ldots, a_{[n]})$ a special non-increasing rearrangement of $(a_1, a_2, \ldots, a_n)$ satisfying $a_{[1]} \geqslant a_{[2]} \geqslant \ldots \geqslant a_{[n]}$. The following result can be found in [11].

LEMMA 3.3. *Let $n \in Z^+$, $\vec{c} = (c_1, c_2, \ldots, c_n) \in R^n$, with $c_i \geqslant 0$ and let, for* $k = 1, \ldots, n-1$,

$$R_k(\vec{c}) = \sum_{i=1}^{n-k} c_i c_{i+k}. \tag{3.4}$$

If $\pi_0 \in \Pi$ *is a permutation such that*

$$c_{\pi_0(1)} \leqslant c_{\pi_0(n)} \leqslant c_{\pi_0(2)} \leqslant c_{\pi_0(n-1)} \leqslant c_{\pi_0(3)} \leqslant c_{\pi_0(n-2)} \leqslant \ldots, \tag{3.5}$$

then, for $k = 1, \ldots, n-2$,

$$\max_{\{l_1,\ldots,l_k\}} \sum_{j=1}^{k} R_{l_j}(\bar{c}) = \sum_{i=1}^{k} R_{[i]}(\bar{c}) \leqslant \sum_{i=1}^{k} R_i(\pi_0 \bar{c}), \tag{3.6}$$

where $\{l_1, \ldots, l_k\} \subset \{1, 2, \ldots, n-1\}$, *and*

$$\sum_{i=1}^{n-1} R_{[i]}(\bar{c}) = \sum_{i=1}^{n-1} R_i(\pi_0 \bar{c}). \tag{3.7}$$

We will use Lemma 3.3 in the proof of Proposition 3.1 by finding first a way to discretize our continuous moving average problem.

Without loss of generality, we assume $\operatorname{supp}(f) \subset [0, 1]$. Some auxiliary functions and notation will be used. Let, for $x \geqslant 0$,

$$f_n(x) := \sum_{i=0}^{n-1} f(i/n) I_{[i/n,(i+1)/n)}(x)$$

and

$$R_n(x) := \frac{1}{n} \sum_{i=1}^{n} f((\lfloor xn \rfloor)/n + i/n) f(i/n).$$

If $\pi_0 \{f(i/n)\}_{i=0}^{n-1} = \{f(\pi_0(i)/n)\}_{i=0}^{n-1}$ is a rearrangement of $\{f(i/n)\}_{i=0}^{n-1}$ such that $\pi_0\{f(i/n)\}_{i=1}^{n}$ satisfies condition (3.5), then we put

$$\hat{f}_n(x) := \sum_{i=0}^{n-1} f(\pi_0(i)/n) I_{[i/n,(i+1)/n)}(x)$$

and

$$\hat{R}_n(x) := \frac{1}{n} \sum_{i=1}^{n} f(\pi_0(\lfloor xn \rfloor + i)/n) f(\pi_0(i)/n).$$

For the sake of simplicity we are adopting here a convention to the effect that $\pi(k) = k$ whenever $k > n$. We shall keep this convention throughout the rest of this paper.

From the definition of $R_n(x)$ we know that $R_n(x) = R_n(k/n)$ when $k/n \leqslant x < (k+1)/n$. Introduce a special permutation $\bar{\pi} \in \Pi$ such that

$$\bar{\pi}\{R_n(k/n)\}_{k=0}^{n-1} = \{R_n(\bar{\pi}(k)/n)\}_{k=0}^{n-1}$$

is a rearrangement of $\{R_n(k/n)\}_{k=0}^{n-1}$, satisfying the monotonicity condition

$$R_n(\bar{\pi}(1)/n) \geqslant R_n(\bar{\pi}(2)/n) \geqslant R_n(\bar{\pi}(3)/n) \geqslant \ldots \tag{3.8}$$

Finally, define

$$\bar{R}_n(x) := R_n(\bar{\pi}\lfloor xn \rfloor). \tag{3.9}$$

LEMMA 3.4. *In the above notation, as $n \to \infty$, we have*

(i) $f_n(x) \to f(x)$ *uniformly for* $x \in [0, 1]$;

(ii) $R_n(x) \to \int f(x+y) f(y)\,dy$ *uniformly for* $x \in [0, 1]$;

(iii) $\bar{R}_n(x) \to \tilde{R}_f(x)$, *where* $\tilde{R}_f(x)$ *is defined in Section* 2;

(iv) $\hat{f}_n(x) \to \hat{f}(x)$ *uniformly for all* $x \in [0, 1]$, *where* $\hat{f}(x)$ *is defined in Lemma* 2.1 (*with* $a = 0$, $b = 1$);

(v) $\hat{R}_n(x) \to \int \hat{f}(x+y)\hat{f}(y)\,dy$, *where* $\hat{f}$ *is the same as in* (iv).

Proof. (i) follows from the uniform continuity of any continuous function with compact support. To verify (ii) use the definition of integral and the fact that $f(x+y)f(y)$ is continuous and has a compact support on $\boldsymbol{R} \times \boldsymbol{R}$.

To verify (iii) we need the following steps. Notice that $R_f(x) = \int f(x+y) f(y)\,dy$ is continuous, even and has compact support. Hence, by the definition of $\tilde{R}_f(x)$, we see that $\tilde{R}_f(x)$ is a continuous decreasing function on $[0, \infty)$, which is strictly decreasing on $[0, m(\{y\colon R_f(y) > 0\})]$.

From property (ii) we infer that, for a fixed $c > 0$ and for any $\varepsilon > 0$, there exists an N such that, for all $n \geqslant N$,

$$\{x\colon R_f(x) \geqslant c-\varepsilon\} \supset \{x\colon R_n(x) \geqslant c\} \supset \{x\colon R_f(x) \geqslant c+\varepsilon\}. \tag{3.10}$$

Therefore, using the continuity of $R_f(x)$, we get

$$m(\{x\colon R_f(x) \geqslant c\}) \geqslant \lim_n m(\{x\colon R_n(x) \geqslant c\}) \geqslant m(\{x\colon R_f(x) \geqslant c\}). \tag{3.11}$$

Notice that

$$m(\{y\colon R_n(y) \geqslant c\}) = \frac{\#\{k\colon R_n(k/n) \geqslant c\}}{n}. \tag{3.12}$$

Since $R_f(x)$ is continuous, for each $c \in (0, \max R_f(x))$ we have

$$m(\{x\colon R_f(x) \geqslant c-\varepsilon\}) > m(\{x\colon R_f(x) \geqslant c\}) > m(\{x\colon R_f(x) \geqslant c+\varepsilon\}).$$

Therefore, for any $\varepsilon > 0$ there exist $N(\varepsilon)$ such that for all $n \geqslant N$ we have

$$\frac{\#\{k\colon R_n(k/n) \geqslant c-\varepsilon\}}{n} \geqslant m(\{x\colon R_f(x) \geqslant c\}) \geqslant \frac{\#\{k\colon R_n(k/n) \geqslant c+\varepsilon\}+1}{n}.$$

From the definition of $\bar{R}_n(\cdot)$ we know that $\bar{R}_n(\cdot)$ is decreasing, and

$$\bar{R}_n\left(\frac{\#\{k\colon R_n(k/n) \geqslant d\}}{n}\right) \geqslant d, \qquad \bar{R}_n\left(\frac{\#\{k\colon R_n(k/n) \geqslant d\}+1}{n}\right) < d.$$

Therefore, we get

$$c+\varepsilon > \bar{R}_n\left(\frac{\#\{k\colon R_n(k/n) \geqslant c+\varepsilon\}+1}{n}\right) \geqslant \bar{R}_n(m(\{x\colon R_f(x) \geqslant c\}))$$

and

$$\bar{R}_n\big(m(\{x:\ R_f(x) \geqslant c\})\big) \geqslant \bar{R}_n\left(\frac{\#\{k:\ R_n(k/n) \geqslant c-\varepsilon\}}{n}\right) > c-\varepsilon.$$

In other words,

$$\underline{\lim}_{n\to\infty} \bar{R}_n(x) \geqslant c \geqslant \overline{\lim}_{n\to\infty} \bar{R}_n(x), \tag{3.13}$$

where $x = m(\{x:\ R_f(x) \geqslant c\})$. This proves statement (iii).

To verify (iv), we will proceed as follows. In this case, $\hat{f}(x)$ is defined as in Lemma 2.1, with $a = 0$ and $b = 1$, symmetric about the line $x = 1/2$. Notice that, for each $c \in (0, \max f(y))$,

$$\{x:\ \hat{f}(x) \geqslant c\} = \big[1/2 - m(\{y:\ f(y) > c\})/2,\ 1/2 + m(\{y:\ f(y) > c\})/2\big],$$

and $\{x:\ \hat{f}_n(x) \geqslant c\}$ is one of the intervals:

$$\big[k/n,\ (n-k)/n\big) \quad \text{or} \quad \big[k/n,\ (n-k+1)/n\big) \quad \text{or} \quad \big[k/n,\ (n-k-1)/n\big),$$

where $k/n < 1/2$ and $k = \min\{i:\ f(i/n) \geqslant c\}$. Since $\{\hat{f}_n(k/n)\}$ is a special rearrangement of $\{f_n(k/n)\}$ which satisfies (3.5) and since $\hat{f} \in \mathcal{M}_f$, we infer that, for all $c > 0$,

$$\frac{k}{n} \to \frac{1}{2} - \frac{m(\{y:\ \hat{f}(y) > c\})}{2}, \qquad \frac{n-k}{n} \to \frac{1}{2} + \frac{m(\{y:\ \hat{f}(y) > c\})}{2},$$

and

$$\frac{n-2k}{n} \to m(\{y:\ \hat{f}(y) > c\})$$

as $n \to \infty$. As in the proof of (iii), we have

$$\underline{\lim}_{n\to\infty} \hat{f}_n(x) \geqslant c \geqslant \overline{\lim}_{n\to\infty} \hat{f}_n(x),$$

so that $\hat{f}_n(x) \to \hat{f}(x)$.

Finally, observe that $\hat{f} \in \mathcal{M}_f$ is a continuous function with compact support, that $\{\hat{f}_n(k/n)\}$ is a rearrangement of $\{f_n(k/n)\}$, and that $f_n(x)$ uniformly converges to a continuous function (property (i)). This gives the uniform convergence in (iv).

To verify (v), we just need to show that

$$\frac{1}{n}\sum_{i=0}^{n}\left(f\left(\frac{\pi_0(\lfloor xn\rfloor + i)}{n}\right) f(\pi_0(i)/n) - \hat{f}\left(\frac{\lfloor xn\rfloor + i}{n}\right)\hat{f}(i/n)\right) \to 0 \tag{3.14}$$

as $n \to \infty$. This follows in view of the uniform convergence in (iv). ∎

Proof of Proposition 3.1. Using all of the above facts we infer that, for all x, $k = \lfloor xn \rfloor$,

$$\begin{aligned}
\int_0^x \tilde{R}_g(y)\,dy &= \lim_{n\to\infty} \int_0^x \bar{R}_n(y)\,dy \\
&\leqslant \lim_{n\to\infty} \left(\left(\frac{1}{n^2} \max_{\{l_1,\ldots,l_k\}} \sum_{j=1}^{k} \sum_{i=1}^{n} g\big((l_j+i)/n\big)\, g(i/n) \right) + o(1) \right) \\
&\leqslant \lim_{n\to\infty} \left(\frac{1}{n^2} \Big(\sum_{j=1}^{k} \sum_{i=1}^{n} g\big(\pi_0(j+i)/n\big)\, g\big(\pi_0(i)/n\big)\Big) + o(1) \right) \\
&= \lim_{n\to\infty} \int_0^x \hat{R}_n(y)\,dy = \int_0^x \Big(\int \hat{g}(y+z)\,\hat{g}(z)\,dz \Big)\,dy \\
&= \int_0^x R_{\hat{g}}(y)\,dy,
\end{aligned}$$

where the first inequality follows from (ii), the third line by Lemma 3.3, the fourth line by (v), and the last line by Lemma 3.1.

By construction, $\hat{g}$ is also continuous, symmetric-unimodal and belongs to $\mathcal{M}_f$. Since $R_{\hat{g}}(y)$ [resp. $R_f(y)$] is a (modified) convolution of $\hat{g}$ [resp. f] with itself (which is even), we get $R_{\hat{g}}(y) = R_f(y)$. ∎

4. The limiting behavior. Explicit solutions of the Burgers equation (1.1) can be obtained via the Hopf–Cole transformation (cf., e.g., [10], [3], and [20]):

$$u(t, x) = t^{-1} \frac{Z(t, x)}{I(t, x)}, \tag{4.1}$$

where

$$Z(t, x) = \int_{\mathbf{R}} (x-y) \exp\left(U_0(y) - \frac{(x-y)^2}{2t} \right) dy, \tag{4.2}$$

$$I(t, x) = \int_{\mathbf{R}} \exp\left(U_0(y) - \frac{(x-y)^2}{2t} \right) dy, \tag{4.3}$$

and $U_0(y) \equiv \int_{-\infty}^{y} u_0(x)\,dx$ is the initial velocity potential. As stated in the Introduction, the initial velocity potential is assumed here to be of the form (1.3).

LEMMA 4.1. (i) *If $M(y)$ is the Brownian motion, then, in probability,*

$$\lim_{t\to\infty} t^{-1/2} I(t, x\sqrt{t}) = \sqrt{2\pi} \exp\big(\tfrac{1}{2} \textstyle\int h(s)^2\,ds\big). \tag{4.4}$$

(ii) *If $M(y)$ is the Poisson process, then, in probability,*

$$\lim_{t\to\infty} t^{-1/2} I(t, x\sqrt{t}) = \sqrt{2\pi} \exp\big(\textstyle\int (e^{h(s)} - 1)\,ds\big). \tag{4.5}$$

Proof. (i) If $M(t)$ is the Brownian motion, then

$$\mathbf{E}\frac{1}{\sqrt{t}}I(t, x\sqrt{t}) = \frac{1}{\sqrt{t}}\int \exp\left(-\frac{(x\sqrt{t}-y)^2}{2t}\right)\mathbf{E}\exp\left(\int h(y-z)\,dM(z)\right)dy$$

$$= \int \exp\left(-\frac{(x-y)^2}{2}\right)\exp\left(\frac{1}{2}\int h(s)^2\,ds\right)dy = \sqrt{2\pi}\exp\left(\frac{1}{2}\int h(s)^2\,ds\right)$$

and

$$\mathbf{E}\left(\frac{1}{\sqrt{t}}I(t, x\sqrt{t})\right)^2 = \mathbf{E}\left(\frac{1}{\sqrt{t}}\int \exp\left(-\frac{(x\sqrt{t}-y)^2}{2t}\right)\exp\left(\int h(y-z)\,dM(z)\right)dy\right)^2$$

$$= \iint \exp\left(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\right)$$

$$\times \mathbf{E}\exp\left(\int\left(h(\sqrt{t}u_1-z)+h(\sqrt{t}u^2-z)\right)dM(z)\right)du_1\,du_2$$

$$= \iint \exp\left(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\right)$$

$$\times \exp\left(\int\left(h(\sqrt{t}u_1-z)+h(\sqrt{t}u_2-z)\right)^2 dz\right)du_1\,du_2$$

$$= \exp\left(\int h^2(s)\,ds\right)\iint \exp\left(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\right)$$

$$\times \exp\left(\int h(\sqrt{t}u_1-z)\,h(\sqrt{t}u_2-z)\,dz\right)du_1\,du_2.$$

Since $h(\cdot)$ has compact support as long as $u_1 \neq u_2$, we have

$$\lim_{t\to\infty} h(\sqrt{t}u_1-z)\,h(\sqrt{t}u_2-z) = 0.$$

Notice that the Lebesgue measure of the set $\{(u_1, u_2), u_1 = u_2\}$ is zero. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to\infty}\mathbf{E}\left(\frac{1}{\sqrt{t}}I(t, x\sqrt{t})\right)^2 = 2\pi\exp\left(\int h^2(s)\,ds\right). \tag{4.6}$$

Therefore $\lim_{t\to\infty}\operatorname{Var}\left(t^{-1/2}I(t, x\sqrt{t})\right) = 0$, and an application of the Chebyshev inequality gives Lemma 4.1 in the case of the Brownian motion.

(ii) In the case where $M(y)$ is the Poisson process we have

$$\mathbf{E}\frac{1}{\sqrt{t}}I(t, x\sqrt{t}) = \frac{1}{\sqrt{t}}\int \exp\left(-\frac{(x\sqrt{t}-y)^2}{2t}\right)\mathbf{E}\exp\left(\int h(y-z)\,dM(z)\right)dy$$

$$= \int \exp\left(-\frac{(x-y)^2}{2}\right)\exp\left(\int (e^{h(s)}-1)\,ds\right)dy = \sqrt{2\pi}\exp\left(\int (e^{h(s)}-1)\,ds\right)$$

and

$$\mathbf{E}\Big(\frac{1}{\sqrt{t}} I(t, x\sqrt{t})\Big)^2 = \mathbf{E}\Big(\frac{1}{\sqrt{t}} \int \exp\Big(-\frac{(x\sqrt{t}-y)^2}{2t}\Big) \exp\big(\int h(y-z)\,dM(z)\big)\,dy\Big)^2$$

$$= \iint \exp\Big(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\Big)$$

$$\times \mathbf{E}\exp\big(\int (h(\sqrt{t}u_1-z)+h(\sqrt{t}u_2-z))\,dM(z)\big)\,du_1\,du_2$$

$$= \iint \exp\Big(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\Big)$$

$$\times \exp\big(\int (\exp(h(\sqrt{t}u_1-z)+h(\sqrt{t}u_2-z))-1)\,dz\big)\,du_1\,du_2$$

$$= \exp\big(2\int (e^{h(s)}-1)\,ds\big) \iint \exp\Big(-\frac{(x-u_1)^2+(x-u_2)^2}{2}\Big)$$

$$\times \exp\big(\int (\exp(h(\sqrt{t}u_1-z))-1)$$

$$\times (\exp(h(\sqrt{t}u_2-z))-1)\,dz\big)\,du_1\,du_2.$$

Since $e^{h(\cdot)}-1$ has compact support as long as $u_1 \neq u_2$, we have

$$\lim_{t\to\infty} \big(\exp(h(\sqrt{t}u_1-z))-1\big)\big(\exp(h(\sqrt{t}u_2-z))-1\big) = 0.$$

Again, the Lebesgue measure of the set $\{(u_1, u_2), u_1 = u_2\}$ is zero, and an application of the Lebesgue Dominated Convergence Theorem gives

$$\lim_{t\to\infty} \mathbf{E}\Big(\frac{1}{\sqrt{t}} I(t, x\sqrt{t})\Big)^2 = 2\pi \exp\big(2\int (e^{h(s)}-1)\,ds\big). \tag{4.7}$$

Therefore, again $\lim_{t\to\infty} \mathrm{Var}\big(t^{-1/2} I(t, x\sqrt{t})\big) = 0$, and another application of the Chebyshev inequality yields Lemma 4.1 for the case of the Poisson process. ∎

LEMMA 4.2. *Let Z be a solution of the Burgers equation described in* (4.1). *Then*

$$\mathbf{E}\big(t^{-3/4} Z(t, x\sqrt{t})\big) = 0 \tag{4.8}$$

and

$$\mathrm{Var}\big(t^{-3/4} Z(t, x\sqrt{t})\big) = \sqrt{2\pi}\,\tilde{\sigma}(h), \tag{4.9}$$

where, for $M(y)$ being the Brownian motion

$$\tilde{\sigma}(h) = \exp\big(\int h^2(s)\,ds\big) \int \big(\exp\big(\int h(v-s)\,h(-s)\,ds\big)-1\big)\,dv, \tag{4.10}$$

and for $M(y)$ being the Poisson process

$$\tilde{\sigma}(h) = \exp\left(2\int (e^{h(s)}-1)\,ds\right)\int \left(\exp\left(\int (e^{h(v-s)}-1)(e^{h(-s)}-1)\,ds\right)-1\right)dv. \tag{4.11}$$

Proof. A verification of (4.8) is trivial, and we omit it. Let us check (4.9). In the case where $M(t)$ is a Brownian motion we have

$$\begin{aligned}
&\mathbf{E}\left(t^{-3/4}\int(\sqrt{t}x-y)\exp\left(-\frac{(\sqrt{t}x-y)^2}{2t}\right)\exp\left\{\int h(y-u)\,dM(u)\right\}dy\right)^2 \\
&= \mathbf{E}\left(t^{1/4}\int(x-y)\exp\left(-\frac{(x-y)^2}{2}\right)\exp\left\{\int h(\sqrt{t}y-u)\,dM(u)\right\}dy\right)^2 \\
&= t^{1/2}\iint(x-y_1)(x-y_2)\exp\left(-\frac{(x-y_1)^2+(x-y_2)^2}{2}\right) \\
&\qquad\times\exp\left\{\int\left(h(\sqrt{t}y_1-u)+h(\sqrt{t}y_2-u)\right)^2du\right\}dy_1\,dy_2 \\
&= \exp\left(\int h^2(u)\,du\right)t^{1/2}\iint\left(x^2-x(y_1+y_2)+\frac{(y_1+y_2)^2-(y_1-y_2)^2}{4}\right) \\
&\qquad\times\exp\left(-\left\{x^2-(y_1+y_2)x+\frac{(y_1+y_2)^2+(y_1-y_2)^2}{4}\right\}\right) \\
&\qquad\times\exp\left(\int h(\sqrt{t}(y_1-y_2)-u)\,h(-u)\,du\right)dy_1\,dy_2.
\end{aligned} \tag{4.12}$$

Moreover,

$$\begin{aligned}
&\exp\left(\int h^2(u)\,du\right)t^{1/2}\iint\left(x^2-x(y_1+y_2)+\frac{(y_1+y_2)^2-(y_1-y_2)^2}{4}\right) \\
&\qquad\times\exp\left(-\left\{x^2-(y_1+y_2)x+\frac{(y_1+y_2)^2+(y_1-y_2)^2}{4}\right\}\right)=0.
\end{aligned} \tag{4.13}$$

Subtracting (4.13) from (4.12), we get

$$\begin{aligned}
&\mathbf{E}\left(t^{-3/4}Z(t,x\sqrt{t})\right)^2 = \exp\left(\int h^2(u)\,du\right)t^{1/2}\iint\left(x^2-x(y_1+y_2)\right. \\
&\left.+\frac{(y_1+y_2)^2-(y_1-y_2)^2}{4}\right)\exp\left(-\left\{x^2-(y_1+y_2)x+\frac{(y_1+y_2)^2+(y_1-y_2)^2}{4}\right\}\right) \\
&\qquad\times\left(\exp\left(\int h(\sqrt{t}(y_1-y_2)-u)\,h(-u)\,du\right)-1\right)dy_1\,dy_2.
\end{aligned}$$

Changing variables $v_1 = y_1+y_2$, $v_2 = \sqrt{t}(y_1-y_2)$, we obtain

$$\begin{aligned}
&\mathbf{E}\left(t^{-3/4}Z(t,x\sqrt{t})\right)^2 = 2\exp\left(\int h^2(s)\,ds\right)\iint\left(x^2-xv_1+v_1^2/4-v_2^2/4t\right) \\
&\qquad\times\exp\left(-(x^2-v_1x+v_1^2/4)-v_2^2/4t\right)\left(\exp\left(\int h(v_2-u)\,h(-u)\,du\right)-1\right)dv_1\,dv_2.
\end{aligned}$$

Since $\exp\left(\int h(v_2-u)h(-u)\,du\right)-1$ has compact support, by the Lebesgue Dominated Convergence Theorem we get

$$\lim_{t\to\infty}\mathrm{E}\left(t^{-3/4}Z(t,x\sqrt{t})\right)^2 = 2\exp\left(\int h^2(s)\,ds\right)\iint(x^2-xv_1+v_1^2/4)$$
$$\times\exp(-(x^2-v_1x+v_1^2/4))\left(\exp\left(\int h(v_2-u)h(-u)\,du\right)-1\right)dv_1\,dv_2.$$

Finally, observe that

$$\int(x^2-xv_1+v_1^2/4)\exp\left(-(x^2-v_1x+v_1^2/4)\right)dv_1 = \sqrt{\pi}/\sqrt{2},$$

which completes the proof of Lemma 4.2 in the Brownian motion case.

In the case where $M(t)$ is the Poisson process we proceed in a similar fashion:

$$\mathrm{E}\left(t^{-3/4}Z(t,x\sqrt{t})\right)^2 = \exp\left(2\int(e^{h(u)}-1)\,du\right)t^{1/2}\iint\Bigg(x^2-x(y_1+y_2)$$
$$+\frac{(y_1+y_2)^2-(y_1-y_2)^2}{4}\Bigg)\exp\left(-\left(x^2-(y_1+y_2)x+\frac{(y_1+y_2)^2+(y_1-y_2)^2}{4}\right)\right)$$
$$\times\left(\exp\left(\int\left(\exp\left(h(\sqrt{t}(y_1-y_2)-u)\right)-1\right)\left(\exp(h(-u))-1\right)du\right)-1\right)dy_1\,dy_2.$$

Changing variables as above, after a computation similar to that in the Brownian case, we complete the proof in the case of the Poisson processes as well. ∎

The following rate-of-convergence result in the central limit theorem for dependent random variables is due to Bulinski [1].

PROPOSITION 4.1. *Let $\{X_j(t), j\in U(t)\}$ be an $m(t)$-dependent field on a finite set $U(t)\subset \mathbf{Z}^d$, and let, for some $s\in(2, 3]$ and all $t>0$,*

$$\sup_{j\in U(t)}\left(\mathrm{E}|X_j(t)|^s\right)^{1/s} = C_s(t) < \infty.$$

Then

$$\sup_{x\in\mathbf{R}}\left|P\left(\delta^{-1}(t)\sum_{j\in U(t)}(X_j(t)-\mathrm{E}X_j(t))\leqslant x\right)-\Phi(x)\right|$$
$$\leqslant k_0|U(t)|\,M_s^s(t)\,m^{d(s-1)}(t)+M_s(t)\,m^d(t)\log^{(d-1)/2}|U(t)|$$
$$+|U(t)|^{1/2}M_s^2(t)\,m^{2d/s}(t),$$

where $\delta(t) = \left(\operatorname{Var}\sum_{j\in U(t)}X_j(t)\right)^{1/2}>0$, $k_0=k_0(d)$, $|U(t)|$ is a number of points in $U(t)$, $M_s(t)=\delta^{-1}(t)\,C_s(t)$, and $\Phi(x)$ is the distribution function of $N(0, 1)$.

LEMMA 4.3. *The distribution of $t^{3/4}Z(t, x\sqrt{t})$ weakly converges to $N(0, \sqrt{2\pi}\,\tilde{\sigma}(h))$ as $t\to\infty$.*

Proof. Without loss of generality we assume that $\operatorname{supp}(f)\subset[0, 1]$.

Throughout this proof $g_1 \asymp g_2$ means that

$$0 < \liminf_{t\to\infty} \frac{g_1(t)}{g_2(t)} \leqslant \limsup_{t\to\infty} \frac{g_1(t)}{g_2(t)} < \infty .$$

Let

$$H(t) = t^{1/4} \int_{-\infty}^{-t^{1/8}} (x-y) \exp\left(-\frac{(x-y)^2}{2}\right) \exp\left(\int h(\sqrt{t}y-u)\, dM(u)\right) dy.$$

Then

$$\begin{aligned} \mathbf{E}\,|H(t)| \asymp t^{1/4} \int_{-\infty}^{-t^{1/8}} |x-y| \exp\left(-\frac{(x-y)^2}{2}\right) dy \\ \asymp t^{1/4} \int_{-\infty}^{-t^{1/8}} |x-y| \exp\left(-\frac{(x-y)^2}{2}\right) dy \asymp t^{1/4} \exp(-t^{1/4}), \end{aligned}$$

and

$$\mathbf{E}\,|H(t)|^2 \asymp t^{1/2} \left(\int_{-\infty}^{-t^{1/8}} |x-y| \exp\left(-\frac{(x-y)^2}{2}\right) dy \right)^2 \asymp t^{1/2} \exp(-2t^{1/4}).$$

Using Chebyshev's inequality, we get $H(t) \to 0$ in probability as $t \to \infty$. Similarly, we can get as $t \to \infty$ the convergence in probability:

$$t^{1/4} \int_{t^{1/8}}^{\infty} (x-y) \exp\left(-\frac{(x-y)^2}{2}\right) \exp\left(\int h(\sqrt{t}y-u)\, dM(u)\right) dy \to 0.$$

For the remainder we have

$$t^{1/4} \int_{-t^{1/8}}^{t^{1/8}} (x-y) \exp\left(-\frac{(x-y)^2}{2}\right) \exp\left(\int h(\sqrt{t}y-u)\, dM(u)\right) dy \asymp \sum_{-t^{5/8} \leqslant k \leqslant t^{5/8}} \eta_k(t),$$

where

$$\eta_k(t) = t^{1/4} \int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}} (x-y) \exp\left(-\frac{(x-y)^2}{2}\right) \exp\left(\int h(\sqrt{t}y-u)\, dM(u)\right) dy.$$

Since $\operatorname{supp}(h) \subset [0, 1]$, $\eta_k(t)$ is a 2-dependent sequence. Using Proposition 4.1, we get

$$|U(t)| = 2t^{5/8}, \quad C_3(t) = \max_k \left(\mathbf{E}\,|\eta_k(t)|^3\right)^{1/3} \asymp t^{-1/4}, \quad \delta(t) \asymp 1, \quad M_3(t) \asymp t^{-1/4}.$$

Since $d = 1$, we have

$$k_0\, |U(t)|\, M_3^3(t)\, 2^2 + |U(t)|^{1/2} M_3^2(t)\, 2^{2/3} \asymp t^{5/8} t^{-3/4} + t^{5/16} t^{-1/2} \asymp t^{-1/8},$$

so that

$$t^{-3/4} Z(t, x\sqrt{t}) \to N(0, \sqrt{2\pi}\, \tilde{\sigma}(h)). \quad \blacksquare$$

5. Proofs of the main results. In this section, we prove our main theorems. Suppose $R_1(x)$ and $R_2(x)$ are integrable functions with compact support on $[0, C]$, where $C > 0$. Let $\tilde{R}_1(x)$ and $\tilde{R}_2(x)$ be defined as in Section 2. The following result is due to Burkill [4].

PROPOSITION 5.1. *Suppose that $\tilde{R}_1(x)$ and $\tilde{R}_2(x)$ satisfy the following relations:*

$$\int_0^x \tilde{R}_1(y)\,dy \leqslant \int_0^x \tilde{R}_2(y)\,dy, \quad 0 \leqslant x \leqslant C,$$

and

$$\int_0^C \tilde{R}_1(y)\,dy = \int_0^C \tilde{R}_2(y)\,dy.$$

Then, for all convex continuous functions ϕ,

$$\int_0^C \phi(\tilde{R}_1(y))\,dy \leqslant \int_0^C \phi(\tilde{R}_2(y))\,dy.$$

Now, proofs of the three theorems formulated in Section 1 can be given in quick succession.

Proof of Theorem 1.1. Since

$$t^{3/4} u(t, x\sqrt{t}) = \frac{t^{-3/4} Z(t, x\sqrt{t})}{t^{-1/2} I(t, x\sqrt{t})},$$

by Lemmas 4.1 and 4.4 we have

$$\frac{t^{-3/4} Z(t, x\sqrt{t})}{t^{-1/2} I(t, x\sqrt{t})} \to N\left(0, \frac{1}{\sqrt{2\pi}}\sigma(h)\right). \ ■$$

Proof of Theorem 1.2. Notice that for all $x, y \in \boldsymbol{R}$ we have

$$h_1(y+x)\,h_1(x) \leqslant h_2(y+x)\,h_2(x)$$

in the case of the Brownian motion, and

$$\big(\exp(h_1(y+x))-1\big)\big(\exp(h_1(x))-1\big) \leqslant \big(\exp(h_2(y+x))-1\big)\big(\exp(h_2(x))-1\big)$$

in the case of the Poisson processes. Since $\phi(x) = e^x - 1$ is an increasing function, the proof is complete. ■

Proof of Theorem 1.3. In the case where $M(y)$ is the Brownian motion, since $\phi(x) = e^x - 1$ is convex and $R_h(x) = \int h(x+y)\,h(y)\,dy$ is continuous and has compact support, Propositions 3.1 and 5.1 give us the desired result.

In the case where $M(t)$ is the Poisson process, consider $f(x) = e^{h(x)} - 1$. It is easy to check that $f(x)$ is continuous, symmetric-unimodal and has compact support if and only if $h(x)$ is continuous, symmetric-unimodal with the same compact support. Also, $e^x - 1$ is a strictly increasing function. Since $h \geqslant 0$, we get also $f \geqslant 0$. If $g \in \mathcal{M}_h$, then by a simple computation we have $e^g - 1 \in \mathcal{M}_f$. Therefore, as in the Brownian motion case, applying Propositions 3.1 and 5.1 to the function $f(\cdot)$, we obtain the result in the Poisson process case. ∎

6. A remark on the domination property. In Section 3, the domination principle of Proposition 3.1 showed that, for all $x \geqslant 0$,

$$\int_0^x \tilde{R}_g(y)\,dy \leqslant \int_0^x R_f(y)\,dy \quad \text{and} \quad \int_0^\infty \tilde{R}_g(y)\,dy = \int_0^\infty R_f(y)\,dy,$$

where f is a non-negative continuous symmetric-unimodal function with compact support, and $g \in \mathcal{M}_f$. In this section we will show that, as a matter of fact, the left and the right-hand sides in inequality (6.1) are not too far from each other if their averages are not too far from each other. More exactly, their difference can be uniformly majorized by the difference of their averages. Define

$$\varrho(f) := \int_{-\infty}^{\infty} \int_x^\infty \tilde{R}_f(y)\,dy\,dx.$$

PROPOSITION 6.1. *Let $\tilde{R}_f$ and R_f be as in Section 2. Then, for any $g \in \mathcal{M}_f$,*

$$\sup_{0 \leqslant x \leqslant \infty} \left| \int_0^x \tilde{R}_g(y)\,dy - \int_0^x R_f(y)\,dy \right| \leqslant \sqrt{2R_f(0)\big(\varrho(g) - \varrho(f)\big)}, \tag{6.1}$$

where the constant $R_f(0)$ does not depend on g.

Proof. Since f is symmetric-unimodal, $\tilde{R}_f = R_f$. Let

$$\Delta(x) = \int_0^x R_f(y)\,dy - \int_0^x \tilde{R}_g(y)\,dy.$$

Obviously, $\Delta(x) \geqslant 0$ for $x \geqslant 0$. For $x_2 \geqslant x_1$,

$$\Delta(x_2) - \Delta(x_1) = \int_{x_1}^{x_2} R_f(y)\,dy - \int_{x_1}^{x_2} \tilde{R}_g(y)\,dy \leqslant \int_{x_1}^{x_2} R_f(y)\,dy \leqslant R_f(0)(x_2 - x_1)$$

or

$$\Delta(x_1) \geqslant \Delta(x_2) - R_f(0)(x_2 - x_1).$$

Let $x_1 = 0$ in the inequality above. Since $\Delta(0) = 0$, we have

$$x - \Delta(x)/R_f(0) \geqslant 0.$$

For any $x_2 > 0$,

$$\varrho(g) - \varrho(f) = \int_0^\infty \Delta(x)\,dx \geqslant \int_{x_2 - \Delta(x_2)/R_f(0)}^{x_2} \Delta(x)\,dx$$

$$\geqslant \int_{x_2 - \Delta(x_2)/R_f(0)}^{x_2} \big(\Delta(x_2) - R_f(0)(x_2 - x)\big)\,dx = \Delta^2(x_2)/\big(2R_f(0)\big).$$

Therefore, for any $x \geqslant 0$, we get

$$\Delta(x) \leqslant \sqrt{2R_f(0)\big(\varrho(g)-\varrho(f)\big)}. \quad ■$$

PROPOSITION 6.2. *Let $\varrho(g)$ and $\varrho(f)$ be the same as in Proposition 6.1. Assume that R_f satisfies the Lipschitz condition with constant M_f, i.e., for any $y_1, y_2 \geqslant 0$, we have*

$$|R_f(y_1)-R_f(y_2)| \leqslant M_f|y_1-y_2|.$$

Then, for any $g \in \mathcal{M}_f$,

$$\sup_{0\leqslant y\leqslant\infty} |\tilde{R}_g(y)-R_f(y)| \leqslant \sqrt{4M_f\sqrt{2R_f(0)\big(\varrho(g)-\varrho(f)\big)}}, \tag{6.2}$$

where the constants do not depend on g.

Proof. For $y_1 > y_2$, we have

$$\begin{aligned}&\big(\tilde{R}_g(y_1)-R_f(y_1)\big)-\big(\tilde{R}_g(y_2)-R_f(y_2)\big)\\ &= \tilde{R}_g(y_1)-\tilde{R}_g(y_2)-\big(R_f(y_1)-R_f(y_2)\big) \leqslant R_f(y_2)-R_f(y_1) \leqslant M_f(y_1-y_2).\end{aligned}$$

Therefore, for $y_1 > y_2$, we have

$$\tilde{R}_g(y_2)-R_f(y_2) \geqslant \tilde{R}_g(y_1)-R_f(y_1)-M_f(y_1-y_2) \tag{6.3}$$

or

$$R_f(y_1)-\tilde{R}_g(y_1) \geqslant R_f(y_2)-\tilde{R}_g(y_2)-M_f(y_1-y_2). \tag{6.4}$$

Observe that we have $\tilde{R}_g(0)-R_f(0) = 0$. Indeed, we can divide both sides of (3.1) by x, and let $x \to 0$. Hence we get

$$\tilde{R}_g(0) \leqslant R_f(0).$$

Since $R_f(0) \leqslant \tilde{R}_g(0)$ (because $\tilde{R}_g$ is a rearrangement of R_g), and since for $g \in \mathcal{M}_f$ we have

$$R_g(0) = \int g(-y)g(-y)\,dy = \int f(-y)f(-y)\,dy = R_f(0),$$

our observation follows.

Taking $y_2 = 0$ in (6.3), we get

$$y \geqslant \big(\tilde{R}_g(y)-R_f(y)\big)/M_f.$$

If $\tilde{R}_g(y)-R_f(y) \geqslant 0$, using (6.3), we obtain

$$\begin{aligned}2\sup_{0\leqslant y\leqslant\infty} \Delta(y) &\geqslant \int_{y-(\tilde{R}_g(y)-R_f(y))/M_f}^{y} \big(\tilde{R}_g(y_2)-R_f(y_2)\big)\,dy_2 \\ &\geqslant \int_{y-(\tilde{R}_g(y)-R_f(y))/M_f}^{y} \big(\tilde{R}_g(y)-R_f(y)-M_f(y-y_2)\big)\,dy_2 = \big(\tilde{R}_g(y)-R_f(y)\big)^2/(2M_f).\end{aligned} \tag{6.5}$$

If $\tilde{R}_g(y)-R_f(y)\leqslant 0$, using (6.4), we have

$$(6.6)\qquad 2\sup_{0\leqslant y\leqslant\infty}\Delta(y)\geqslant\int_y^{y+(R_f(y)-\tilde{R}_g(y))/M_f}\big(R_f(y_1)-\tilde{R}_g(y_1)\big)\,dy_1$$

$$\geqslant\int_y^{y+(R_f(y)-\tilde{R}_g(y))/M_f}\big(R_f(y)-\tilde{R}_g(y)-M_f(y_1-y)\big)\,dy_2$$

$$=\big(\tilde{R}_g(y)-R_f(y)\big)^2/(2M_f).$$

Combining (6.5), (6.6) and (6.1), we get (6.2). ∎

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