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A GENERALIZED BINOMIAL MODEL AND OPTION PRICING FORMULAE FOR SUBORDINATED STOCK-PRICE PROCESSES

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Abstract. The first half of the paper is intended as a short survey on discrete- and continuous-time option pricing. In the second part, we develop new concepts and derive new results for option valuations within a generalized binomial model with random upturns and downturns, characterizing the equivalent portfolio, the trading strategy, and the call option valuation. Motivated by the Mandelbrot–Taylor Paretian stable model for stock returns we apply the generalized binomial model to obtain – in the limit – call valuation formulae for subordinated stock-price processes.

1. Introduction and a survey on option pricing. One of the striking applications of stochastic calculus is the recent progress in the security markets with option pricing. It takes its roots in the seminal works of Arrow and Debreu on asset pricing (see [17], Chapter 7), and of Black and Scholes [5] and Merton [52].

The most prominent type of option contract is the *call option*; it gives the buyer the right to buy specific number of shares of a company from the option writer at a specific purchase price (known as *exercise price* K or striking price) at any time up to and including a specific date (known as the *expiration*). The option is *European* if it can only be exercised on its expiration date, and it is called *American* if it can be exercised any time throughout its expiration date.

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1. *Discrete-time option pricing: binomial option pricing formulae.* We start with an illustration of the nature of the Black and Scholes [5] arguments for valuation European call option on nondivident-paying stocks by considering the situation where the stock-price movements are described by a multiplicative binomial process over discrete time; the "binomial" approach to option pricing seems to be independently presented by Sharpe [73], by Cox, Ross and Rubinstein [14] – whose arguments we shall follow – and by Rendleman and Bartter [67].

Suppose the current stock-price is $S = S_0$ (the known stock price at $t_0 = 0$) and let τ be the length of calendar time representing the expiration of the call. In the binomial model, the elapsed time between successive stock-price changes is discrete and equals $h = \tau/N$, where N is the number of periods prior to expiration. At the end of the $(k+1)$-st period, the stock is going "upward" $S_{k+1} = US_k$ with probability q, and "downward" $S_{k+1} = DS_k$ with probability $1-q$; letting R to be the $1+$"*the interest rate on a default-free loan over one period*". To obtain meaningful values for U, D and R, we assume that $U > R > 1 > D$. Therefore, the successive movements of the stock-price are given recursively by

$$S_{k+1} = S_k(\xi_{k+1} U + (1-\xi_{k+1})D),$$

where ξ_k are i.i.d. Bernoulli with success probability q. To see the Cox–Ross–Rubinstein approach to a call valuation, suppose that $N = 1$ and let C be the current value of the call; then at the end of the period (the expiration time) the call value with striking price K will be $C_U = (US-K)^+$ ($(a)^+ := \max(0, a)$) with probability q and $C_D = (DS-K)^+$ with probability $1-q$. One can easily check that if there are to be no riskless arbitrage opportunities, the current value of the call must be equal to the current value of an equivalent portfolio $S\Delta + B$ containing Δ shares of the stock and an amount B in riskless bonds; the Δ and B are chosen to equate the end of the period values of the call for each possible outcome, that is, $C_U = US\Delta + RB$ with probability q and $C_D = DS\Delta + RB$ with probability $1-q$. With these values for Δ and B the call value $C = S\Delta + B$ equals

$$C = (pC_U + (1-p)C_D)/R,$$

where p is the *riskless interest rate*, satisfying $pUS + (1-p)DS = RS$. Using the same recursive arguments for any $N \in \mathbf{N}$ we write the general valuation *binomial option pricing formula*

$$C = C(N) = R^{-N} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} (U^j D^{N-j} S - K)^+ \tag{1.1}$$

$$= S\mathrm{Bi}(a_N;\, N,\, p') - KR^{-N}\mathrm{Bi}(a_N;\, N,\, p)$$

with

$$\mathrm{Bi}(a_N;\, N,\, p) = \sum_{j=a_N}^{N} \binom{N}{j} p^j (1-p)^{N-j},$$

where $p = (R-D)/(U-D)$ and $p' = (U/R)p$, and a_N is the smallest nonnegative integer greater than $\log(K/SD^N)/\log(U/D)$. If $a^N > N$, then $C = 0$.

The simple binomial formula produced so far assumed that we are dealing with options on a nondividend-paying stock. For various kinds of generalizations involving the impact of dividends, dependence of the ups and the downs from the level of the stock price, multinomial models for price movements, etc., we refer to [15], [38], [68], [36], and the references therein.

We have implicitly taken it that the options being valued are European. Merton [52] (see also Smith [75] for a lucid review of his seminal work) has shown that if the stock pays no dividends, European and the American call options are equally valued, since in this case the American option will not be exercised before the expiration date. Consequently, the Black–Scholes formula can be used to value the American call option on nondividend-paying stocks. In the presence of dividends, the American call option can be worth more than the European since there is a positive probability for an early exercise. For further details on American options, see [71], [69], [61], and [27].

2. *Continuous-time option pricing.* As trading takes place almost continuously, the elapsed time $h = \tau/N$ goes to zero and one needs to adjust the N-dependent values of U, D, R and q in order to obtain meaningful limiting values of the call. Cox et al. [14] have chosen

$$U = \exp(\sigma\sqrt{\tau/N}), \qquad D = \exp(-\sigma\sqrt{\tau/N}),$$

$$q = \tfrac{1}{2}(1+(\mu/\sigma)\sqrt{\tau/N}), \qquad R^N = R_0,$$

in the binomial option pricing formula (1.1) for $C = C(N)$; now as N goes to infinity, the limiting value of the call $C = \lim_{N\to\infty} C(N)$ equals the *Black–Scholes formula*

$$C = S\Phi(x) - KR_0^{-\tau}\Phi(x-\sigma\sqrt{\tau}), \qquad \text{where } x = \frac{\log(S/KR^{-\tau})}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau},$$

and Φ is the standard normal distribution function.

The alternative derivation of the continuous-trading call valuation obtained in the work of Black and Scholes was based on the assumption that the stock value follows a log-normal diffusion process $dS(t) = S(t)\big(\mu dt + \sigma dW(t)\big)$, $S(0) = s$, where $S(t)$ is the value of the stock, μ is the drift term, $\sigma > 0$ is the volatility coefficient, and $W(t)$ is a Brownian motion in $\boldsymbol{R}$. (W is defined on the complete probability space $(\Omega, \boldsymbol{F}, \boldsymbol{P})$, where $\{\boldsymbol{F}_t\}$ stands for the $\boldsymbol{P}$-augmentation of the natural filtration generated by W.) The non-risky asset, the bond, with price $B(t)$ is given by $dB(t) = B(t)r(t)dt$, $B(0) = 1$. More generally, one can assume that r, μ and σ are time-dependent; in this case, some regularity conditions are required: $r(t)$, $\mu(t)$ and $\sigma(t)$ are progressively measurable with respect to $\{F_t\}$, bounded uniformly in $\boldsymbol{R} \times \Omega$ and $\sigma(t)$ is strictly positive (greater than some $\varepsilon > 0$) for all t; see [40] and [16].

The European option contract is equivalent to a payment of $(S(\tau)-K))^+$ at the expiration τ. Black and Scholes [5] asserted that – in the case of constant μ, σ and r – there is a unique rational value for the option independent of the investor's risk attitude; with

$$f(x, t) = x\Phi(g(x, t)) - Ke^{-rt}\Phi(h(x, t)),$$

$$\text{where } g(x, t) = \{\ln(x/K) + (r + \tfrac{1}{2}\sigma)t\}/\sigma\sqrt{t}, \; h(x, t) = g(x/t) - \sigma\sqrt{t},$$

this unique rational value is $f(S(0), \tau)$. The arbitrage arguments used by Black and Scholes [5] and Merton [52] have become the starting point for option pricing valuation in deterministic bond price. The further studies include:

(i) Cox and Ross [13] give the risk-neutral approach to option pricing – as an analogue of Modigliani–Miller [58] theory.

(ii) Harrison and Kreps [32], Harrison and Pliska [33], [34], Kreps [43] show that a price process is arbitrage free if it is, after renormalization, a martingale with respect to some equivalent probability measure.

(iii) Duffie [19], [20] explored an indirect solution of the Black–Scholes partial differential equation via the Feynman–Kac formula to extend the Merton continuous-time asset pricing model [52].

(iv) The arbitrage arguments for option pricing with stochastic interest rate environment were extended in Duffie [19], Dybvig [21], Heath et al. [35], Kopp and Elliott [41], Turnball and Miln [76], and Cheng [11].

The European option may be viewed as one example of *European contingent claim* (ECC), which is a financial instrument consisting of payment $\boldsymbol{B}$ at terminal time τ; we assume that $\boldsymbol{B}$ is a nonnegative F_τ-measurable random variable with finite moment of order greater than 1 (see [40]). To define a hedging strategy against the ECC, let $X(t)$ be the wealth of an investor at time t, $\pi(t)$ the amount he invests in the stock, and $c(t)$ is the consumption rate process. (Here the regularity conditions are: (i) $\pi(t)$ is progressively measurable with respect to $\{F_t\}$ and square-integrable on $[0, \tau]$ almost everywhere; (ii) $\int_0^\tau c(t)dt < \infty$ almost everywhere.) The wealth process $X(t)$ is then determined by the equation

$$dX(t) = \pi(t)\{\mu(t)dt + \sigma(t)dW(t)\} - c(t)dt + \{X(t) - \pi(t)\}r(t)dt.$$

The nondegeneracy condition ($\sigma > \varepsilon > 0$) implies the existence of the measure $P^*(A) = \mathbb{E}\{Z(\tau)1_A\}$ equivalent to $\boldsymbol{P}$, where Z is the exponential martingale:

$$Z(t) = \exp\Big[-\int_0^t \theta(s)dW(s) - \tfrac{1}{2}\int_0^t \theta(s)^2 ds\Big], \qquad \theta(s) = \sigma(t)^{-1}[\mu(t) - r(t)].$$

By the Girsanov theorem [29], under the new probability space $(\Omega, \boldsymbol{F}_\tau, \boldsymbol{P}^*)$, the process $W^*(t) = W(t) + \int_0^t \theta(s)ds$ is a Brownian motion, and the discounted stock price process $B(t)^{-1}S(t)$ is a nonnegative supermartingale under $\boldsymbol{P}^*$. The pair (π, c) is now called *admissible* for the initial capital $s \geqslant 0$ if for the wealth process X, almost surely, $X(\tau) > 0$ for $\tau > t > 0$. A *hedging strategy* against the ECC is an admissible pair (π, c) (with initial wealth $s > 0$) which at the terminal time is valued as ECC; $X(t) = \boldsymbol{B}$ almost surely. The *fair price* $\boldsymbol{v}$ at time $t = 0$ for the ECC is then the smallest value of s for which a hedging strategy exists; its explicit value is, in fact, given by

$$\boldsymbol{v} = \mathbf{E}^*\Big[\boldsymbol{B}\exp\Big(-\int_0^\tau r(u)du\Big)\Big].$$

Moreover, there exists a hedging strategy with consumption rate $c = 0$ and wealth process

$$X(t) = \mathbf{E}^*\Big[\boldsymbol{B}\exp\Big(-\int_t^\tau r(u)du\Big)\Big|\boldsymbol{F}_t\Big], \qquad X(0) = \boldsymbol{v};$$

for further extensions of the method we refer to [59] and [16].

In the models above the market models were assumed to be frictionless; *hedging strategies in the presence of transaction costs* were studied in:

(i) Gilster and Lee [28] and Leland [45] derived call formulae for options, that are revised at finite number of times;

(ii) Dybvig and Ross [22], Prisman [64], and Ross [70] studied multi-period market models in the presence of taxes;

(iii) Bensaid et al. [4] develop hedging strategy in the binomial model with proportional transaction costs;

(iv) Jouini and Kallal [39] compute arbitrage bounds for contingent claim valuations for markets with transaction costs using the martingale approach;

(v) Figlewski [24] provides numerical analysis to investigate the influence of transaction costs in hedging;

(vi) Grannan and Swindle [31] develop hedging strategies to minimize transaction costs using diffusion limits.

Another feature of the above contingent claim valuation model is that the financial market we are dealing with, is *complete*, that is, any claim is *redundant* – it can be replicated by a self-financing strategy based only of the stock-price process $\{S(t)\}_{0\leqslant t\leqslant\tau}$ on the underlying probability space $(\Omega, \boldsymbol{F}, \boldsymbol{P})$. In other words, any contingent claim can be represented as a stochastic integral with respect to the semimartingale S; recall that in the absence of arbitrage opportunities the general hedging method (cf. [32]–[34]) implies the existence of equivalent probability measure $\boldsymbol{P}^*$, and consequently S is a semimartingale with respect to $\boldsymbol{P}$. In contrast, if the market is *incomplete*, a contingent claim is not necessarily a stochastic integral of S, that is, there exist nonredundant claims, which will carry on an intrinsic risk. Fölmer and Sondermann [26] introduced the notion of a risk-minimizing strategy, and, by making use of the projection technique due to Kunita and Watanabe [44], they showed that, in the martingale case $P = P^*$, there exists such a strategy and it is unique. Possible extensions to be the general incomplete model, where S is only a semimartingale with respect to P, were studied by Schweizer [72] (where a risk minimizing strategy was defined in a local sense), and by Fölmer and Schweizer [25] in the case where S is a semimartingale with continuous paths. For further generalizations of the notion of incomplete market, see [16].

3. *Stable models for asset returns and option pricing.* The form of the distribution of stock-price changes continues to be one of the most controversial issues in modelling functionals of security market prices. Recall that a basic assumption in viewing the Black–Scholes formula as the limiting case of the binomial option pricing formula was that the price changes were in domain of attraction of the normal law. While earlier theories, starting with Bachelier's theory [1] on speculative prices, had been based on the normality of the law of price changes, more researchers who have studied series of price changes reject the normality assumption (see, e.g., Mandelbrot [47], [48], and Fama [23]; for a list of more than 50 papers in the area we refer to Cox and Rubinstein [15], pp. 482–484; for a recent survey see Mittnik and Rachev [56], [57]). The typical sample density has excess kurtosis — more observations around the mean and in the tails than the normal. We list some of the alternative non-normal models proposed for fitting stock-price-changes data:

(i) *Mixture of normals*: Brada and Van Tassell [10], Press [63], Grander and Morgenstern [30], Clark [12], and Boness et al. [9];

(ii) *Student distribution*: Praetz [62] and Blattberg and Gonedes [6];

(iii) *Stable Paretian model*: in their seminal works, Mandelbrot [47], [48] and Fama [23] argue that the departures from normality should be explained by the assumption that the price changes are stable with a parameter of stability $\alpha < 2$, in particular, they have infinite variance;

(iv) *Mixture of stable laws*: Barnea and Downes [3];

(v) *Geometric stable laws*: Mittnik and Rachev [53]–[57], Kozubowski [42];

(vi) *Generalized convolutions*: Panorska [60];

(vii) *ARCH and GARCH-type models*: Bollerslev [7], Vries [77], Bollerslev et al. [8], Baillie and Bollerslev [2].

The rest of the section is intended as a short discussion of a call option formula under the assumption that the stock-price process follows the Mandelbrot-Taylor stable Paretian model. In [65], option pricing formulae are given for price processes with marginals having mixtures of normals, Student t-distributions, geometric stable and other type distributions for asset returns.

Mandelbrot and Taylor [50] argued strongly in favour of the stable Paretian distribution over the normal law for modelling the distribution of asset returns, and a substantial body of subsequent empirical studies supported the stable Paretian model (see the discussion in [56]). This model gives rise to alternative option pricing formulae.

A. *The main ingredients of the Mandelbrot–Taylor model.* The return process $(W(T))_{T \geqslant 0}$ on a time scale measured in volume of transactions is assumed to be the Brownian motion with zero drift and variance v^2. The cumulative volume $(T(t))_{T \geqslant 0}$ – the number of transactions up to physical time t – is assumed to follow a positive $(\alpha/2)$-stable stochastic process with characteristic function (ch.f.)

$$\mathbf{E}e^{i\theta T(t)} = \exp\{-\nu t|\theta|^{\alpha/2}(1-i(\theta/|\theta|)\tan(\pi\alpha/4))\} \quad (1<\alpha<2,\ \nu>0). \tag{1.2}$$

The subordinated process $Z(t) = W(T(t))$ representing the return process on the physical time scale is now the α-stable Lévy motion with ch.f.

$$\mathbf{E}e^{i\theta Z(t)} = \exp\{-t|\sigma\theta|^{\alpha}\}, \tag{1.3}$$

where σ is a function of ν, v and α, see [50]. Since returns are defined as the consecutive differences of the logarithms of the prices, the process $S(t) = \exp\{Z(t)\}$ is the price process in the Mandelbrot–Taylor model.

B. *The discrete version of the Mandelbrot–Taylor model* (see [66]). Similar to the price tree in the binomial option pricing formula, the consecutive movements of the price are determined by

$$S_k \stackrel{\mathrm{d}}{=} S \sum_{i=1}^{k} U_i^{\delta_i} D_i^{(1-\delta_i)},$$

where δ_i's are i.i.d. Bernoulli($\frac{1}{2}$) independent of U_i and D_i. In contrast to the standard binomial option pricing model (where $U_i = U = \text{const}$, $D_i = D = \text{const}$) we assume that U_i and D_i are random, $U_i = \exp\{\sigma|X_i^{(n)}|\}$, $D_i = U_i^{-1}$, where n represents the number of movements until the terminal time T of a call, and $\{X_i^{(n)},\ i = 1, \ldots, n\}$ are i.i.d. symmetric Pareto r.v.'s with $P(|X_i^{(n)}| > x) = n^{-1}x^{-\alpha}$, $x \geqslant n^{-1/\alpha}$, $1 < \alpha < 2$. For the discrete time price process $(S_k)_{k \geqslant 0}$, $S_0 = S$, we then write

$$\log(S_k/S) \stackrel{\mathrm{d}}{=} \sigma \sum_{i=1}^{k} X_i^{(n)},$$

and thus the process

$$Z_n(t) = \log(S_k/S), \quad \tau\frac{k-1}{n} < t \leqslant \tau\frac{k}{n},\ k = 1, \ldots, n\ (\xi_n(0) = 0)$$

converges weakly to a symmetric α-stable Lévy motion $Z(t)$ on $D[0, \tau]$ with ch.f. given in (1.3). The random "riskless interest rate" in the i-th period is given by

$$R_i = \tfrac{1}{2}(U_i + D_i). \tag{1.4}$$

C. *The option pricing formula for stock returns governed by the Lévy motion.* Formula (1.4) ensures the martingale property of the sequence $S_k^* = S_k/R_1 \ldots R_k$ with respect to the filtration generated by the random ups and downs:

$$\mathbf{E}(S_k^* \mid \sigma(U_1, \ldots, U_{k-1}, D_1, \ldots, D_{k-1})) = S_{k-1}^* \tfrac{1}{2}\mathbf{E}(U_k + D_k)/R_k = S_{k-1}^*.$$

Therefore, (1.4) provides a riskless measure, and the option pricing formula is given by

$$C_n = \mathbf{E}c^{(n)} = \mathbf{E}(S_n - K)^+/(R_1 \ldots R_n),$$

where

$$c^{(n)} = \frac{2^{-n}}{R_1 \ldots R_n}\{(U_1 \ldots U_n S - K)^+ + \ldots + [(U_1 \ldots U_{n-1} D_n S - K)^+ + \ldots$$
$$+ (D_1 U_2 \ldots U_n S - K)^+] + \ldots + (D_1 \ldots D_n S - K)^+\}.$$

In [66] it is shown that $R_1 \ldots R_n$ does not converge to a constant in contrast to the classical Black-Scholes formula, where $R_1 \ldots R_n$ is set to be $R^n = R_0^{-\tau}$.

The next theorem provides an expression for the limit $C = \lim_{n\to\infty} C_n$. Suppose Z_i's are i.i.d. uniforms on (0, 1) and ε_i's are Rademacher random signs independent of Z_i's. Then $X_i^{(n)} \stackrel{d}{=} \varepsilon_i n^{-1/\alpha} Z_i^{-1/\alpha}$, and rearranging $(X_1^{(n)}, \ldots, X_n^{(n)})$ in an increasing absolute order, say $(X_{1,n}^{(n)}, \ldots, X_{n,n}^{(n)})$, we observe that the latter order statistics have the same joint distribution as

$$\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha}(\varepsilon_1 \Gamma_1^{-1/\alpha}, \ldots, \varepsilon_n \Gamma_n^{-1/\alpha}),$$

where $\Gamma_1, \Gamma_2, \ldots$ are Poisson arrivals with intensity 1, independent of ε_i's. We can rewrite C_n as

$$C_n = \mathbf{E}\frac{(S\exp\{\sigma(X_1^{(n)} + \ldots + X_n^{(n)})\} - K)_+}{2^{-n}\sum_{i=1}^{n}(\exp\{\sigma|X_i^{(n)}|\} + \exp\{-\sigma|X_i^{(n)}|\})}.$$

THEOREM 1 (Rachev and Samorodnitsky [66]). *Letting $n \to \infty$ in the "discretized" Mandelbrot–Taylor model implies $C_n \to C$, where*

$$C = \mathbf{E}\frac{(S\exp\{\sum_{i=1}^{\infty}\sigma\varepsilon_i\Gamma_i^{-1/\alpha}\} - K)_+}{\mathrm{E}[\exp\{\sum_{i=1}^{\infty}\sigma\varepsilon_i\Gamma_i^{-1/\alpha}\} \mid (e_i)_{i=1}^{\infty}]}.$$

Janicki and Weron [37], pp. 198–202, have analyzed alternative (numerical) approaches to determine the European call option value under the Mandelbrot–Taylor stock-price model. Starting with a numerical analysis of the price process driven by the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dZ(t))$$

with Z being an α-stable motion, they were able to construct approximations for the density of $C_T = (S(\tau) - K)^+$. This continuous-time option pricing approach is certainly of interest since it relies on a more realistic stock-price process than that in the Black–Scholes model. What still needs to be done in the Rachev–Samorodnitsky model is the justification of the "average-hedging" arguments.

In the next two sections we shall derive an alternative call option pricing formula on the Mandelbrot–Taylor price process using a generalized binomial formula and hedging arguments throughout the derivation (Section 2); then we use an approximation technique to arrive at a continuous-time call pricing formula (Section 3).

2. Option pricing for the generalized binomial model. We assume that the movement of stock prices of a particular stock follows $S_{n+1} = S_n U_{n+1}$ if $\xi_{n+1} = 1$, and $S_{n+1} = S_n D_{n+1}$ if $\xi_{n+1} = 0$, that is, for $n \geqslant 0$,

$$S_{n+1} = S_n(\xi_{n+1} U_{n+1} + (1-\xi_{n+1}) D_{n+1}). \tag{2.1}$$

Here, S_0 is the current market price of the underlying stock, $U_N = (U_1, \ldots, U_N)$, $D_N = (D_1, \ldots, D_N)$ are random sequences with finite mean (but they might have infinite variance) describing the values of the upward and downward moves, the vector $\xi_N = (\xi_1, \ldots, \xi_N)$,

$$P(\xi_i = 0 \text{ or } 1) = 1, \tag{2.2}$$

describes the probability for ups and downs. We assume that

$$P(0 < D_i < (1+r_i) < U_i < \infty) = 1, \tag{2.3}$$

where r_i is the rate of interest on a default-free loan over the i-th period (an amount x in secure bonds on the $(i-1)$-st day fetches an interest $r_i x$ on the i-th day). Finally, N stands for the number of periods until the expiration time τ.

Our problem is to obtain a notion of a fair price of a (European option) contract which gives the buyer the "option" to buy a prescribed number of shares on the N-th day at a striking price K. Thus, the "expected" gain on this option is $\mathrm{E}(S_N - K)^+$.

Recall that in the binomial model (see (1.1)), where U_i and D_i are degenerate random variables, ξ_i's are i.i.d., that the price is not equal to the present worth of the gain, namely $(R_1 \ldots R_N)^{-1}\, \mathrm{E}(S_N - K)^+$ with $R_i := 1 + r_i$. In fact, the price does not depend on $P(\xi_i = 1)$! As we have already noticed, this is so because the investor can also invest directly on the stock. So if the stock is expected to go up ($P(\xi_i = 1)$ is high), and the broker set a high price on the option, the investor may instead invest his money on the stock directly.

In the model (2.1), the distribution of the triple (U_N, D_N, ξ_N) is quite arbitrary. The only assumption in addition to (2.2) and (2.3) is

$$P(\xi_n = 1 \mid (U_N, D_N)) = P(\xi_n = 1 \mid (U_n, D_n)), \tag{2.4}$$

that is, having observed the vectors $U_n = (U_1, \ldots, U_n)$ and $D_n = (D_1, \ldots, D_n)$, ξ_n does not contain any additional information on $U_{n+1}, D_{n+1}, U_{n+2}, D_{n+2}, \ldots, U_N, D_N$.

We assume that the random variables U_{n+1}, D_{n+1} are observed on the n-th day, and thus can be used to decide the amount π_{n+1} an investor may want to invest on the stock on the n-th day. Thus, an investment strategy is given by $\{f_n\}$, where f_n is a function of

$$u_n = (u_1, \ldots, u_n) \in R_+^n, \quad d_n = (d_1, \ldots, d_n) \in R_+^n, \quad \varepsilon_{n-1} = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \{0, 1\}^{n-1},$$

and having observed U_n, D_n and ξ_{n-1}, the investor invests $\pi_n = f_n(U_n, D_n, \varepsilon_{n-1})$ on the $(n-1)$-st day.

Consider an investor with initial wealth $X_0 = x$ who chooses an investment strategy $\{f_n\}$. If there is no inflow or outflow of funds, any excess (shortfall) funds after investing in the stock are put in the bank (borrowed) at the appropriate rate of interest (r_i on the i-th day), the worth X_{n+1} of the "portfolio" at time $n+1$ satisfies

$$\begin{aligned} X_{n+1} &= \pi_{n+1}\frac{S_{n+1}}{S_n} + (X_n - \pi_{n+1})R_{n+1} \\ &= R_1 \ldots R_{n+1}\left\{x + \sum_{j=0}^{n} \frac{\pi_{j+1}}{R_0 \ldots R_j}\left(\frac{S_{j+1}}{S_j R_{j+1}} - 1\right)\right\}, \qquad R_0 = 1. \end{aligned} \tag{2.5}$$

To define what will be a "fair price" x^* of the option we introduce the following functions: for $(u_n, d_n, \varepsilon_n) \in R_+^n \times R_+^n \times \{0, 1\}^n$,

$$h_n(u_n, d_n, \varepsilon_n) = S_0 \sum_{j=1}^{n} (\varepsilon_j u_j + (1-\varepsilon_j) d_j), \qquad v_n(u_n, d_n, \varepsilon_n) = \left(h_n(u_n, d_n, \varepsilon_n) - K\right)^+,$$

and given an investment strategy $\{f_n\}$, let

$$\begin{aligned} g_n(x, u_n, d_n, \varepsilon_n) = R_1 \ldots R_n \Bigg\{ x + \sum_{j=0}^{n-1} (R_0 \ldots R_j)^{-1} f_{j+1}(u_{j+1}, d_{j+1}, \varepsilon_j) \\ \times \left[\varepsilon_{j+1}\frac{u_{j+1}}{R_{j+1}} + (1-\varepsilon_{j+1})\frac{d_{j+1}}{R_{j+1}} - 1\right]\Bigg\}. \end{aligned}$$

Then we have $S_n = h_n(U_n, D_n, \varepsilon_n)$ for the stock price at time n, and with initial wealth x and investment strategy $\{f_j\}$, the worth X_n of the portfolio at time n is given by $X_n = g_n(x, U_n, D_n, \varepsilon_n)$.

By analogy with the binomial case (the Cox–Ross–Rubinstein model: U_n, D_n are constants, and ξ_n are i.i.d.; U_n, D_n and ξ_n depend on n only), we would like to define the price of the option to be x^* if there exists an investment strategy $\{f_j\}$ such that

$$\mathrm{E}g_N(x^*, U_N, D_N, \eta_N) = \mathrm{E}\big(v_N(U_N, D_N, \eta_N)\big) \tag{2.6}$$

for all $\{0, 1\}$-valued random variables $\{\eta_i\}$ with

$$P\big(\eta_n = 1 \mid (U_N, D_N)\big) = P\big(\eta_n = 1 \mid (U_n, D_n)\big). \tag{2.7}$$

In particular, taking η_i to be degenerate, $\eta_i = \varepsilon_i \in \{0, 1\}$ for all i,

$$\mathrm{E}g_N(x^*, U_N, D_N, \varepsilon_N) = \mathrm{E}v_N(U_N, D_N, \varepsilon_N) \tag{2.8}$$

for all $\varepsilon_i \in \{0, 1\}$. When $\{U_i, D_i\}$ are degenerate, (2.8) is the same as saying that $\{f_j\}$ is a hedging strategy.

We will prove below that (2.6) determines x^* uniquely and show that for this x^* there exists a strategy $\{f_j\}$ which ensures (2.6). We will also obtain a formula for x^*. Once this is done, one can argue as in the binomial case that x^* must be the *fair price of the option.*

LEMMA 1. (i) *Suppose that there exists a strategy $\{f_j\}$ such that* (2.6) *holds. Then*

$$x^* = \mathrm{E}v_N(U_N, D_N, \xi_N^*)/(R_1 \ldots R_N), \tag{2.9}$$

where ξ_i^* are $\{0, 1\}$-valued random variables such that

$$P\big(\xi_1^* = \varepsilon_i,\ i \leqslant N \mid (U_N, D_N)\big) = \prod_{i=1}^{N} p_i^{\varepsilon_i}(1-p_i)^{1-\varepsilon_i}$$

with $p_i = (R_i - D_i)/(U_i - D_i)$.

(ii) *There exists an explicit strategy $\{f_j\}$ which yields* (2.6).

Proof. (i) Note that $\{\xi_i^*\}$ satisfies (2.7), and hence (2.6) holds for $\eta_i = \xi_i^*$. Also, by the choice of p_i,

$$\begin{aligned}\mathrm{E}[\xi_{j+1}^* U_{j+1} + (1-\xi_{j+1}^*)D_{j+1} \mid (U_{j+1}, D_{j+1})] \\ = p_{j+1}U_{j+1} + (1-p_{j+1})D_{j+1} = R_{j+1}.\end{aligned}$$

Using this "riskless" property of p_j's, we obtain $\mathrm{E}g_N(x^*, U_N, D_N, \xi_N^*) = R_1 \ldots R_N x^*$. Now (2.9) follows from (2.6).

(ii) Let us define

$$S_n^* = S_0 \prod_{j=1}^{n} (\xi_j^* U_j + (1-\xi_j^*)D_j).$$

Then the result of part (i) can be phrased

$$x^* = (R_1 \dots R_N)^{-1} \mathbb{E}(S_N^* - K)^+. \tag{2.10}$$

Now we construct an explicit strategy $\{f_j\}$ which yields (2.6). The strategy is obtained by taking a suitable conditional form of the hedging strategy in the binomial case.

The strategy $\{f_j\}$ is going to be defined by backward induction. We need to introduce more notation. Let

$$F_{n-1}(x, y; \mathbf{u}_{n-1}, \mathbf{d}_{n-1}) = P(U_n \leqslant x, D_n \leqslant y \mid \mathbf{U}_{n-1} = \mathbf{u}_{n-1}, \mathbf{D}_{n-1} = \mathbf{d}_{n-1})$$

be the conditional distribution of U_n, D_n. Let $\{w_n^1, w_n^0, v_n^*, f_n, v_{n-1}\}$ for $n = N, N-1, \dots, 1$ be defined by backward induction as follows (note v_N has been defined earlier):

(a) $$w_n^1(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) = v_n(\mathbf{u}_n, \mathbf{d}_n, (\boldsymbol{\varepsilon}_{n-1}, 1)),$$

i.e. w_n^1 stands for the call value if at the end of the n-th interval the stock is "up";

(b) $$w_n^0(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) = v_n(\mathbf{u}_n, \mathbf{d}_n, (\boldsymbol{\varepsilon}_{n-1}, 0)),$$

i.e. w_n^0 stands for the call value if at the end of the n-th interval the stock is "down";

(c) $$f_n(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) = \frac{w_n^1(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) - w_n^0(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1})}{u_n - d_n},$$

i.e. f_n stands for the neutral hedge ratio;

(d) $$v_n^*(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) = w_n^1(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) \frac{R_n - d_n}{u_n - d_n} + w_n^0(\mathbf{u}_n, \mathbf{d}_n, \boldsymbol{\varepsilon}_{n-1}) \frac{u_n - R_n}{u_n - d_n},$$

i.e. v_n^* stands for the conditional mean value of the call given the ups and the downs at the end of the n-th interval;

(e) $$v_{n-1}(\mathbf{u}_{n-1}, \mathbf{d}_{n-1}, \boldsymbol{\varepsilon}_{n-1}) = \int R_n^{-1} v_n^*((\mathbf{u}_{n-1}, x), (\mathbf{d}_{n-1}, y), \boldsymbol{\varepsilon}_{n-1}) F_{n-1}(dx, dy; \mathbf{u}_{n-1}, \mathbf{d}_{n-1}),$$

i.e. v_{n-1} is the value of the call at the end of the $(n-1)$-st interval.

Let $\{\eta_n\}$ be a sequence of random variables satisfying (2.6). Writing $\boldsymbol{\eta}_n = (\eta_1, \eta_2, \dots, \eta_n)$, we obtain $\hat{\pi}_n(\eta_n U_n + (1-\eta_n) D_n) + v_n^*(\mathbf{U}_n, \mathbf{D}_n, \boldsymbol{\eta}_{n-1}) - R_n \hat{\pi}_n = v_n(\mathbf{U}_n, \mathbf{D}_n, \boldsymbol{\eta}_n)$, where $\hat{\pi}_n = f_n(\mathbf{U}_n, \mathbf{D}_n, \boldsymbol{\eta}_{n-1})$. Taking expectation, and noting that

$$\mathbb{E} v_n^*(\mathbf{U}_n, \mathbf{D}_n, \boldsymbol{\eta}_{n-1}) = R_n \mathbb{E} v_{n-1}(\mathbf{U}_{n-1}, \mathbf{D}_{n-1}, \boldsymbol{\eta}_{n-1})$$

in view of (2.7), we get

$$\mathrm{E}\big[\hat{\pi}_n\big(\eta_n U_n+(1-\eta_n)D_n\big)+R_n v_{n-1}(U_{n-1}, D_{n-1}, \eta_{n-1})-\hat{\pi}_n R_n\big] = \mathrm{E}[v_n(U_n, D_n, \eta_n)]$$

or equivalently, writing $\hat{v}_n = v_n(U_n, D_n, \eta_n)$, we get

$$\mathrm{E}\hat{\pi}_n\big[(\eta_n U_n+(1-\eta_n)D_n)/R_n-1\big] = \mathrm{E}(\hat{v}_n/R_n-\hat{v}_{n-1}).$$

Thus

$$\mathrm{E}\sum_{n=1}^{N}\frac{\hat{\pi}_n}{R_1\ldots R_{n-1}}\big[(\eta_n U_n+(1-\eta_n)D_n)/R_n-1\big] = \mathrm{E}\left(\frac{\hat{v}_N}{R_1\ldots R_N}-v_0\right)$$

or

$$\mathrm{E}g_N(v_0, U_N, D_n, \eta_n) = \mathrm{E}v_N(U_N, D_N, \eta_N).$$

Thus the strategy given by the functions $\{f_N\}$ defined by (a)–(e) satisfies (2.6). ∎

By Lemma 1, v_0 must be equal to x^*. It is easy to see this here directly. It is a direct consequence of the definitions that

$$v_{n-1}(U_{n-1}, D_{n-1}, \xi^*_{n-1}) = R_n^{-1}\mathrm{E}[v_n(U_n, D_n, \xi^*_n) \mid (U_{n-1}, D_{n-1}, \xi_{n-1})],$$

and hence

$$v_0 = (R_1\ldots R_N)^{-1}\mathrm{E}[v_N(U_N, D_N, \xi^*_N)] = (R_1\ldots R_N)^{-1}\mathrm{E}(S^*_N-K)^+.$$

In view of the preceding discussion, we can define x^* given by (2.9) as the price of the option. We will call S^*_n to be the *"risk free" stock-price process associated with* S_n.

3. Option pricing for the generalized Mandelbrot—Taylor model. We now consider a generalized version of the Mandelbrot–Taylor model for movement of stock prices and obtain a formula for the price of an option.

Let $(W(t))_{t\geqslant 0}$ be a standard Brownian motion defined on $(\Omega_1, \mathscr{F}_1, \mathscr{P}_1)$ and let $(T(t))_{t\geqslant 0}$ be a positive $(\alpha/2)$-stable motion with $0<\alpha<2$, defined on $(\Omega_0, \mathscr{F}_0, \mathscr{P}_0)$, with ch.f. given by (1.2). We consider the product space

$$(\Omega, \mathscr{F}, \mathscr{P}) = (\Omega_0, \mathscr{F}_0, \mathscr{P}_0)\otimes(\Omega_1, \mathscr{F}_1, \mathscr{P}_1),$$

and now $\big((W(t))_{t\geqslant 0},\ (T(t))_{t\geqslant 0}\big)$ regarded as processes on $(\Omega, \mathscr{F}, \mathscr{P})$ are independent. Let $r(s)$ be the interest rate at time s on riskless bonds. This means that one unit invested at time u in bonds yields $\exp\{\int_u^t r(s)ds\}$ units at time t.

Let the stock price of a certain stock be modelled as

$$(3.1)\qquad S_t = S_0 \exp\{\int_0^t r(s)ds + Z(t)\},$$

where $Z(t) = W(T(t)) + \int_0^t \mu(s)dT(s)$, and $\mu(s)$ is a continuous function. When $\mu = 0$, $\log S_t$ becomes a stable motion, as desired, in the Mandelbrot–Taylor model (1.2), (1.3). Thus $\mu(u)$ can be interpreted as "drift." We wish to find the price of an option on this stock with expiration time τ and striking price K. Let us introduce an auxiliary process

$$(3.2)\qquad S_t^* = S_0 \exp\{\int_0^t r(s)ds + W_0(T(t))\},$$

where $W_0(s) = W(s) - s/2$. Our main result is:

THEOREM 2. *The price of the option on* $\{S(t)\}_{0 \leqslant t \leqslant \tau}$ *with terminal time* τ *and striking price* K *is*

$$(3.3)\qquad x^* = \mathbf{E}\big[\exp\{-\int_0^\tau r(s)ds\}(S_\tau^* - K)^+\big].$$

Remark. Let us first make it clear that we cannot interpret the price as in the Black–Scholes formula, that is, we cannot demand or postulate the existence of a hedging strategy. Instead, here the interpretation is via a discrete approximation of the S-process. To be precise, we will construct approximations $(S_t(n, m))_{t \geqslant 0}$ to the stock-price process $(S_t)_{t \geqslant 0}$ such that the price x_{nm}^* of the option on $(S_t(n, m))_{t \geqslant 0}$ with terminal time τ and striking price K converges to x^* given by (3.3). This is the reason to call x^* the *price of the option for the stock-price process* $S(t)$.

Proof. Construction of $(S_t(n, m))_{t \geqslant 0}$. We divide the time interval $[0, \tau]$ into nm intervals, and for t in the interval

$$\frac{i-1}{n} + \frac{j-1}{nm} \leqslant \frac{t}{\tau} < \frac{i-1}{n} + \frac{j}{nm} \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$$

we define

$$(3.4)\quad S_t(n, m) = S_0 \prod_{l=1}^{i-1} \prod_{k=1}^{m} \{\xi_{lk} U_{lk} + (1 - \xi_{lk}) D_{lk}\} \prod_{k=1}^{j-1} \{\xi_{ik} U_{ik} + (1 - \xi_{ik}) D_{ik}\},$$

$$S_\tau(n, m) = S_{\tau-}(n, m),$$

where

$$U_{lk} = U_{lk}(n, m) = \left[1 + \left(\theta_l^n \frac{1}{\sqrt{m}}\right) \wedge \frac{1}{2}\right] \exp\{r_{lk}(n, m)\},$$

$$D_{lk} = D_{lk}(n, m) = \left[1 - \left(\theta_l^n \frac{1}{\sqrt{m}}\right) \wedge \frac{1}{2}\right] \exp\{r_{lk}(n, m)\},$$

$$\theta_l^n = \left\{T\left(\frac{l-1}{n}\right) - T\left(\frac{l-2}{n}\right) \vee 0\right\}^{1/2},$$

$$r_{lk} = r_{lk}(n, m) = \int_0^{\tau/(nm)} r\left[\tau\left(\frac{l-1}{n} + \frac{k-1}{nm} + u\right)\right] du,$$

and $\{\xi_{ik} = \xi_{ik}(n, m)\}$ are $\{0, 1\}$-valued random variables satisfying

$$P\big(\xi_{lk} = 1 \mid T((\tau i)/n)\colon 0 \leqslant i \leqslant n\big) = q_{lk};$$

$\{\xi_{lk}\colon 1 \leqslant l \leqslant n, 1 \leqslant k \leqslant m\}$ are conditionally independent given $\{T((\tau i)/n)\colon 0 \leqslant i \leqslant n\}$, and

$$q_{lk}(n, m) = \frac{1}{2} + \frac{1}{2}\left[\theta_l^n \frac{1}{\sqrt{m}} \mu\left[\tau\left(\frac{i-1}{n} + \frac{j-1}{nm}\right)\right]\right] \wedge 1.$$

We will first show that the process $(S_t(n, m))_{t \geqslant 0}$ converges weakly to the process $(S_t)_{t \geqslant 0}$ in $D[0, \infty]$. To do this, for

$$\tau\left(\frac{i-1}{n} + \frac{j-1}{nm}\right) \leqslant t < \left(\frac{i-1}{n} + \frac{j}{nm}\right)\tau$$

we write

$$\hat{t} := \left(\frac{i-1}{n} + \frac{j-1}{nm}\right)\tau, \qquad \log S_t(n, m) = \int_0^{\hat{t}} r(u) du + X_t,$$

where

$$X_t = X_t(n, m) = \sum_{l=1}^{i-1} \sum_{k=1}^{m} [\xi_{lk} \log(1 + v_{lk}) + (1 - \xi_{lk}) \log(1 - v_{lk})]$$
$$+ \sum_{k=1}^{j-1} [\xi_{ik} \log(1 + v_{ik}) + (1 - \xi_{ik}) \log(1 - v_{ik})]$$

and

$$v_{lk}(n, m) = \left(\theta_l^n \frac{1}{\sqrt{m}}\right) \wedge \frac{1}{2}.$$

Conditioned on the $(T(s))_{s \leqslant \tau}$-process, $\{\xi_{lk}\colon 1 \leqslant l \leqslant n, l \leqslant k \leqslant m\}$ are independent random variables and v_{lk} are constants. Thus using the Lindeberg–Feller theorem for triangular arrays (it is easy to verify the Lyapunov condition for $\delta = 2$; see, e.g., [74], p. 331), we see that, conditioned on $\{T(t)\colon t \leqslant \tau\}$, the finite-dimensional distribution of the process $(X_t)_{t \geqslant 0}$ converges to the corresponding distribution of a Gaussian process with independent increments. The mean function and variance function of the Gaussian process are respectively given by

$$\lim_{n,m\to\infty}\sum_{l,k}\mathbf{E}([\xi_{lk} \mid T(s):\ s\leqslant\tau])$$

and

$$\lim_{n,m\to\infty}\sum_{l,k}\mathbf{E}\big[[\xi_{lk}-\mathbf{E}(\xi_{lk} \mid T(s):\ s\leqslant\tau)]^2 \mid T(s):\ s\leqslant\tau\big],$$

where the summation is over l, k: $\tau\big((l-1)/n+(k-1)/nm\big)\leqslant t$.

Explicit computations show that these two limits are $\int_0^t\mu(s)dT(s)$ and $T(t)$, respectively. Thus, as n, $m\to\infty$, the conditional distribution of $(X_{t_1},\ldots,X_{t_p})$ given $\big(T(s)\big)_{s\leqslant\tau}$ converges to the conditional distribution of $\{Z(t_1),\ldots,Z(t_p)\}$ given $\big(T(s)\big)_{s\leqslant t}$. Under the conditional measure, tightness of the laws of $\{\big(X_t(n,m)\big)_{t\geqslant 0}:\ n,m\geqslant 1\}$ can be proved in the same fashion as the Donsker invariance principle. Thus, we can conclude that, conditional on $\big(T(s)_{s\leqslant\tau}\big)$, the process $(X_t)_{t\geqslant 0}$ converges to $(Z_t)_{t\geqslant 0}$. From this it follows that, as $(n,m)\to\infty$, the processes $\big(X_t(n,m)\big)_{t\geqslant 0}$ converge weakly to $(Z_t)_{t\geqslant 0}$. Thus, $\big(S_t(n,m)\big)_{t\geqslant 0}$ $\xrightarrow{w}(S_t)_{t\geqslant 0}$. Now, the price x^*_{nm} of the option on $S_t(n,m)$ is (using (2.10))

$$x^*_{nm}=R_\tau^{-1}\mathbf{E}\big(S^*_\tau(n,m)-K\big)^+, \tag{3.5}$$

where $R_t=\exp\big(\int_0^t r(s)ds\big)$. In deducing (3.4) we have assumed that the interest rate over a period

$$\left[\tau\left(\frac{i-1}{n}+\frac{j-1}{nm}\right),\ \tau\left(\frac{i-1}{n}+\frac{j}{nm}\right)\right]$$

is

$$\exp\left[\int_0^{\tau/(nm)} r\left[\left(\frac{i-1}{n}+\frac{j-1}{nm}\right)\tau+s\right]ds\right]-1$$

and that portfolio can be changed only at multiples of $\tau/(nm)$. Also that at time $t=\big((i-1)/n+(j-1)/nm\big)\tau$ one has observed $\big(T(s)\big)_{s\leqslant t}$, and that this information can be used to decide the investment strategy.

It remains to evaluate the limit of x^*_{nm}. Let us recall that $S^*_t(n,m)$ is given by the same expression as (3.4) with ξ_{lk} replaced by ξ^*_{lk}; here

$$P\big(\xi^*_{lk}(n,m)=1 \mid T\big((\tau i)/n\big):\ 0\leqslant i\leqslant n\big)=p_{lk}(n,m)$$

and again $\{\xi^*_{lk}\}$ are independent given $\{T\big((\tau i)/n\big):\ i\leqslant n\}$. Using the choice of U's and D's, it can be verified that

$$p_{lk}(n,m)=1/2. \tag{3.6}$$

Now taking $\mu = 1/2$ in the previous convergence argument, we see that $(S_t^*(n, m)) \xrightarrow{w} (S_t^*)$. By (3.6), we easily obtain $\mathbf{E}S_t^*(n, m) = S_0 R_t$ for all t, n, m. Similarly, using conditioning with respect to $\{T(s)\colon\ s \leqslant \tau\}$, we get $\mathbf{E}S_t^* = S_0 R_t$. Thus, $S_\tau^*(n, m) \to S_\tau^*$ weakly and $\mathbf{E}S_\tau^*(n, m) \to \mathbf{E}S_\tau^*$. Hence $\mathbf{E}g(S_\tau^*(n, m)) \to \mathbf{E}g(S_\tau^*)$ for all continuous functions g on $\boldsymbol{R}_+$ with $\limsup_{x\to\infty} g(x)/x < \infty$. Taking $g(x) = (x-K)^+$, we get

$$\mathbf{E}\big(S_\tau^*(n, m) - K\big)^+ \to \mathbf{E}(S_\tau^* - K)^+ \quad \text{or} \quad x_{nm}^* \to x^* = \mathbf{E}\big(R_\tau^{-1}(S_\tau^* - K)^+\big).$$

Thus the price of the option on (S_t) with terminal time τ and striking price K is

$$x^* = \mathbf{E}\big(R_\tau^{-1}(S_\tau^* - K)^+\big). \blacksquare$$

The interpretation of Theorem 2 is not that of existence of a hedging strategy, but that with an initial endowment x^* one can choose a strategy, adapted to the observations $(S(t), T(t))$ at time t, which yields "on the average" the same yield as the option, as long as the stock movement confirms to the model (3.1) irrespective of the drift $(\mu(s))$.

It is important to note that we have used arguments involving conditioning on $\{T(t)\colon\ 0 \leqslant t \leqslant \tau\}$ only in the convergence proofs. When it came to define a strategy in the discretized version, at time t, only $\{T(s)\colon\ s \leqslant t\}$ were assumed to have been observed.

We now obtain a simpler expression for x^*. For $a > 0$, $\sigma > 0$, let

$$\psi(a, \sigma^2) = \mathbf{E}[\exp(\sigma Z - \tfrac{1}{2}\sigma^2) - a]^+,$$

where Z is a standard normal. It is easy to see that

$$\psi(a, \sigma^2) = \Phi\left(-\frac{\log a}{\sigma} + \frac{1}{2}\sigma\right) - a\Phi\left(-\frac{\log a}{\sigma} - \frac{1}{2}\sigma\right),$$

where $\Phi(u)$ is the standard normal distribution. Now for the option value x^* we obtain

$$\begin{aligned} x^* &= R_\tau^{-1}\mathbf{E}(S_\tau^* - K)^+ \\ &= S_0 \mathbf{E}\big(\exp\big[W(T(\tau)) - \tfrac{1}{2}T(\tau)\big] - KR_\tau^{-1}S_0^{-1}\big)^+ = S_0 \mathbf{E}v(T(\tau)), \end{aligned}$$

where $v(\sigma) = \psi(KR_\tau^{-1}S_0^{-1}, \sigma^2)$. Here $v(\sigma)$ can be interpreted as the value of the option if the stock price was modelled as $\hat{S}_t^\sigma = \sigma W(t) + \int_0^t r_s ds$. Thus to compute x^*, we need to:

(i) estimate parameters of an $(\alpha/2)$-stable r.v. $T(\tau)$ (see (1.2)), since K, R_τ, and S_0 are given,

(ii) simulate values of $T(\tau)$, say $Y_j \stackrel{d}{=} T(\tau)$, $j = 1, \ldots, N$, and

(iii) evaluate numerically $v(\sigma)$.

Then

$$x^* \cong N^{-1} S_0 \sum_{j=1}^{N} v(Y_j);$$

for more details we refer to [56], [57], and [37].

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