PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 15 (1995), pp. 449–460

OPERATOR-STABLE PROCESSES AND OPERATOR FRACTIONAL STABLE MOTIONS

BY

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Abstract. A new notion of operator-stable processes is introduced and operator fractional stable motions are discussed as examples of operator-stable processes.

1. Introduction. In the previous paper [7], we have introduced a new notion of $\boldsymbol{R}^d$-valued *operator-stable process*, defined several operator fractional stable motions, and proved some limit theorems only in the sense of the convergence of all finite-dimensional distributions. In this paper, we shall redefine the operator-stable processes in a more natural way and prove the limit theorems in the sense of the weak convergence. Marginal processes of $\boldsymbol{R}^d$-valued stochastic processes will also be discussed.

2. Operator-stable processes. A full probability measure μ on $\boldsymbol{R}^d$ is said to be *strictly operator-stable* (or simply *operator-stable* in this paper) if there exists an invertible linear operator B on $\boldsymbol{R}^d$ such that the characteristic function φ of μ satisfies, for every $t > 0$,

$$\varphi(\theta)^t = \varphi(t^{B^*}\theta), \qquad \theta \in \boldsymbol{R}^d,$$

where B^* denotes the adjoint operator of B. An $\boldsymbol{R}^d$-valued random vector ξ is *symmetric* if $\xi \stackrel{d}{=} -\xi$. Let λ_B and Λ_B be the minimum and the maximum of the real parts of the eigenvalues of B, respectively.

Remark 2.1. Sharpe [8] proved that necessary and sufficient conditions for an operator B to be an exponent of some operator-stable distribution are (i) $\lambda_B \geqslant \frac{1}{2}$ and (ii) every eigenvalue of B having the real part equal to $\frac{1}{2}$ is a simple root of the minimal polynomial of B.

Remark 2.2. A full operator-stable measure μ can be classified as follows:

(i) *μ is Gaussian.* In this case, $B = \frac{1}{2}I$ is always taken as an exponent of μ. So, whenever we consider a full Gaussian operator-stable measure, we always assume $B = \frac{1}{2}I$.

(ii) μ *is purely non-Gaussian.* In this case, $\lambda_B > \frac{1}{2}$. When μ is a d-dimensional α-stable measure, we can take $B = \alpha^{-1} I$.

(iii) μ *is general.* Theorem 1 in [3] allows us to consider the Gaussian component and the non-Gaussian component separately. We do so in this paper.

If $\{X(t), t \in \boldsymbol{R}\}$ is an $\boldsymbol{R}^d$-valued Lévy process (namely, it has independent and stationary increments), is continuous in probability, $X(0) = 0$ a.s., and $X(1)$ has a symmetric operator-stable distribution with exponent B, then $\{X(t)\}$ is called a *B-operator-stable motion*, and will be denoted by $\{Z_B(t), t \in \boldsymbol{R}\}$ in this paper. Take any k distinct time points $t_1, \ldots, t_k$ and consider a $(d \times k)$-dimensional random vector

$$\bar{Z} = (Z_B(t_1), \ldots, Z_B(t_k)).$$

Then $\bar{Z}$ is again operator-stable in $\boldsymbol{R}^{d \times k}$ with exponent Q, where

$$Q = \begin{pmatrix} B & 0 & \ldots & 0 \\ 0 & B & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B \end{pmatrix}. \tag{2.1}$$

This fact is a special case of Theorem 2.2 below. Motivated by this fact, we introduce the following new definition of *operator-stable processes.*

Definition 2.1. Let $\{X(t)\}$ be an $\boldsymbol{R}^d$-valued stochastic process. If there exists an invertible linear operator B on $\boldsymbol{R}^d$ such that for any k distinct time points $t_1, \ldots, t_k$ the $(d \times k)$-dimensional random vector

$$\bar{X} = (X(t_1), \ldots, X(t_k))$$

is operator-stable in $\boldsymbol{R}^{d \times k}$ with exponent Q defined by (2.1), then $\{X(t)\}$ is called an *operator-stable process* with exponent B.

This definition extends the real-valued stable process in the following sense.

A real-valued stochastic process $\{X(t)\}$ is said to be *α-stable* if, for any $t_1, \ldots, t_k$, $(X(t_1), \ldots, X(t_k))$ is α-stable. If we reread this definition in terms of operator-stability, a real-valued stochastic process $\{X(t)\}$ is said to be *α-stable* if, for any $t_1, \ldots, t_k$, $(X(t_1), \ldots, X(t_k))$ is a k-dimensional operator-stable random vector with exponent Q:

$$Q = \begin{pmatrix} \alpha^{-1} & 0 & \ldots & 0 \\ 0 & \alpha^{-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha^{-1} \end{pmatrix}.$$

As we have mentioned, the operator-stable motion $\{Z_B(t)\}$ is operator-stable in the sense of Definition 2.1. More generally, the operator-stable integral processes which will be defined below are operator-stable and not necessarily have independent and stationary increments. (In [7], the operator-stable integral processes are introduced as operator-stable processes.)

THEOREM 2.1 (Maejima and Mason [7]). *Let $\{Z_B(t)\}$ be an $\mathbf{R}^d$-valued operator stable motion with exponent B. Let $\{A(u),\, u\in\mathbf{R}\}$ be a set of linear operators on $\mathbf{R}^d$. Define*

$$\mathrm{Com}(B) = \{A\colon A \text{ is a linear operator on } \mathbf{R}^d \text{ and commutes with } B\}$$

and suppose, for each $u\in\mathbf{R}$, $A(u)\in\mathrm{Com}(B)$. Then if all components of $A(u)$ are measurable as functions of u, and

$$\int_{-\infty}^{\infty} \|A(u)\|^2\, du < \infty$$

when $Z_B(1)$ is Gaussian, or

$$\int_{-\infty}^{\infty} \left(\|A(u)\|^{1/\lambda_B+\varepsilon} + \|A(u)\|^{1/\Lambda_B-\varepsilon}\right) du < \infty$$

for some ε with $0<\varepsilon<\min\{2-1/\lambda_B,\, 1/\Lambda_B\}$ when $Z_B(1)$ is purely non-Gaussian, then the stochastic integral

$$\int_{-\infty}^{\infty} A(u)\, dZ_B(u),$$

called the operator-stable integral, is well defined.

These are Remark 3.2 and Theorem 5.3 of [7].

THEOREM 2.2. *Suppose that, for each t and u, $A_t(u)\in\mathrm{Com}(B)$ and that the $\mathbf{R}^d$-valued operator-stable integral*

$$X(t) = \int_{-\infty}^{\infty} A_t(u)\, dZ_B(u)$$

is well defined. Then $\{X(t)\}$ is operator-stable in the sense of Definition 2.1.

Proof. Suppose

$$\varphi(\theta) := \mathrm{E}\left[\exp\{i\langle\theta, Z_B(1)\rangle\}\right], \qquad \theta\in\mathbf{R}^d,$$

where $\langle\,,\,\rangle$ represents the inner product. For simplicity, let us write $I(A) = \int A(u)\, dZ_B(u)$. Take k distinct time points $t_1, \ldots, t_k$. It is enough to show that the $(d\times k)$-dimensional random vector

$$\bar{X} = \left(I(A_{t_1}), \ldots, I(A_{t_k})\right)$$

is operator-stable on $\mathbf{R}^{d\times k}$ and its exponent Q is given by (2.1). To this end, we shall show that for

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_k \end{pmatrix} \in \boldsymbol{R}^{d \times k}, \qquad \theta_j \in \boldsymbol{R}^d,$$

the characteristic function

$$(2.2) \qquad \psi(\theta) := \mathrm{E}\left[\exp\left\{i\left\langle \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_k \end{pmatrix}, \begin{pmatrix} I(A_{t_1}) \\ \vdots \\ I(A_{t_k}) \end{pmatrix} \right\rangle\right\}\right]$$

satisfies for every $t > 0$

$$(2.3) \qquad \psi(\theta)^t = \psi(t^{Q^*}\theta).$$

Note that using (2.2) we obtain

$$\psi(\theta) = \mathrm{E}\left[\exp\{i \sum_{j=1}^{k} \langle \theta_j, I(A_{t_j})\rangle\}\right].$$

Let $\{A_{t_j}(u)\}$ be simple functions, namely

$$A_{t_j}(u) = \sum_{p=1}^{M} A_{t_j}^{(p)} I_{(u_{p-1}, u_p]}(u), \qquad A_{t_j}^{(p)} \in \mathrm{Com}(B).$$

Here $u_0 < u_1 < \ldots < u_M$ are a common decomposition for all $A_{t_j}(u)$'s, which is possible. Then

$$\begin{aligned}
&\mathrm{E}\left[\exp\{i \sum_{j=1}^{k} \langle \theta_j, I(A_{t_j})\rangle\}\right] \\
&= \mathrm{E}\left[\exp\{i \sum_{j=1}^{k} \langle \theta_j, \sum_{p=1}^{M} A_{t_j}^{(p)}(Z_B(u_p) - Z_B(u_{p-1}))\rangle\}\right] \\
&= \mathrm{E}\left[\exp\{i \sum_{p=1}^{M} \langle \sum_{j=1}^{k} A_{t_j}^{(p)*}\theta_j, Z_B(u_p) - Z_B(u_{p-1})\rangle\}\right] \\
&= \prod_{p=1}^{M} \mathrm{E}\left[\exp\{i \langle \sum_{j=1}^{k} A_{t_j}^{(p)*}\theta_j, Z_B(u_p) - Z_B(u_{p-1})\rangle\}\right] \\
&= \prod_{p=1}^{M} \varphi\left(\sum_{j=1}^{k} A_{t_j}^{(p)*}\theta_j\right)^{u_p - u_{p-1}} \\
&= \prod_{p=1}^{M} \exp\{(u_p - u_{p-1}) \log \varphi\left(\sum_{j=1}^{k} A_{t_j}^{(p)*}\theta_j\right)\} \\
&= \exp\{\sum_{p=1}^{M} (u_p - u_{p-1}) \log \varphi\left(\sum_{j=1}^{k} A_{t_j}^{(p)*}\theta_j\right)\} \\
&= \exp\{\int_{-\infty}^{\infty} \log \varphi\left(\sum_{j=1}^{k} A_{t_j}^{*}(u)\theta_j\right) du\}.
\end{aligned}$$

For a general $A_{t_j}(u)$, if we take a sequence of simple functions $\{A_{t_j}^{(n)}(u)\}$, the standard argument gives us the same relationship

$$(2.4)\qquad \mathrm{E}\big[\exp\{i\sum_{j=1}^{k}\langle\theta_j, I(A_{t_j})\rangle\}\big] = \exp\{\int_{-\infty}^{\infty}\log\varphi(\sum_{j=1}^{k}A_{t_j}^*(u)\theta_j)\,du\}.$$

Hence, noticing $\varphi(\theta)^t = \varphi(t^{B^*}\theta)$, we have

$$\psi(\theta)^t = \exp\{t\int_{-\infty}^{\infty}\log\varphi(\sum_{j=1}^{k}A_{t_j}^*(u)\theta_j)\,du\} = \exp\{\int_{-\infty}^{\infty}\log[\varphi(\sum_{j=1}^{k}A_{t_j}^*(u)\theta_j)]^t\,du\}$$

$$= \exp\{\int_{-\infty}^{\infty}\log\varphi(t^{B^*}\sum_{j=1}^{k}A_{t_j}^*(u)\theta_j)\,du\} = \exp\{\int_{-\infty}^{\infty}\log\varphi(\sum_{j=1}^{k}A_{t_j}^*(u)t^{B^*}\theta_j)\,du\},$$

where we have used the assumption that $A_{t_j}(u)\in\mathrm{Com}(B)$. By using (2.4) again, we obtain

$$\psi(\theta)^t = \mathrm{E}\big[\exp\{i\sum_{j=1}^{k}\langle t^{B^*}\theta_j, I(A_{t_j})\rangle\}\big]$$

$$= \mathrm{E}\left[\exp\left\{i\left\langle\begin{pmatrix}t^{B^*}\theta_1\\ \vdots\\ t^{B^*}\theta_k\end{pmatrix}, \begin{pmatrix}I(A_{t_1})\\ \vdots\\ I(A_{t_k})\end{pmatrix}\right\rangle\right\}\right]$$

$$= \mathrm{E}\left[\exp\left\{i\left\langle t^{Q^*}\begin{pmatrix}\theta_1\\ \vdots\\ \theta_k\end{pmatrix}, \begin{pmatrix}I(A_{t_1})\\ \vdots\\ I(A_{t_k})\end{pmatrix}\right\rangle\right\}\right],$$

which is the right-hand side of (2.3), completing the proof. ∎

3. Operator fractional stable motions. The following operator fractional stable motions have been introduced in [7] as examples of operator-self-similar process.

DEFINITION 3.1. Let $\{Z_B(t), t\in\boldsymbol{R}\}$ be an operator-stable motion with exponent B, and D be an invertible linear operator in $\mathrm{Com}(B)$. If

$$(3.1)\qquad \Delta_{D,B}(t) = \int_{-\infty}^{\infty}(|t-u|^{D-B}-|u|^{D-B})\,dZ_B(u)$$

is well defined, the process $\{\Delta_{D,B}(t)\}$ is called the *operator fractional stable motion.*

THEOREM 3.1 (Maejima and Mason [7]). *Suppose $D\neq B$ and $D\in\mathrm{Com}(B)$. If*

$$(3.2)\qquad \lambda_{D-B}+\lambda_B > 0 \quad\textit{and}\quad \Lambda_{D-B-I}+\Lambda_B < 0,$$

then the stochastic integral (3.1) *can be defined.*

These are Theorems 3.1, 4.3 and 5.4 in [7].

Remark 3.1. When $B = \alpha^{-1}I$, $0 < \alpha \leqslant 2$, the condition (3.2) is simplified to that $0 < \lambda_D, \Lambda_D < 1$.

Remark 3.2. If $Z_B(1)$ is Gaussian, then $\{\Delta_{D,B}(t)\}$ is a Gaussian operator--stable process. If $Z_B(1)$ is purely non-Gaussian, then $\{\Delta_{D,B}(t)\}$ is a purely non-Gaussian operator-stable process.

As to the continuous versions of the process $\{\Delta_{D,B}(t)\}$, we have the following

THEOREM 3.2. (i) *If $Z_B(1)$ is Gaussian, then for any $T > 0$ the process $\{\Delta_{D,B}(t), 0 \leqslant t \leqslant T\}$ has a continuous version.*

(ii) *If $Z_B(1)$ is purely non-Gaussian and $\lambda_{D-B} > 0$, then for any $T > 0$ the process $\{\Delta_{D,B}(t), 0 \leqslant t \leqslant T\}$ has a continuous version.*

These facts will be shown as direct consequences of Theorems 4.2 and 4.3 in the next section.

4. Weak convergence to operator fractional stable motions. In Theorem 6.2 of [7], we have proved the following limit theorem about the finite-dimensional convergence:

THEOREM 4.1. *Let $\{Z_B(t), t \in \mathbf{R}\}$ be a symmetric operator-stable motion with an exponent B such that*

$$\mathrm{E}[\exp\{i\langle\theta, Z_B(1)\rangle\}] = \varphi(\theta), \quad \theta \in \mathbf{R}^d.$$

Let D be a linear operator in Com(B) *such that $D \neq B$, $\lambda_{D-B} + \lambda_B > 0$, and $\Lambda_{D-B-I} + \Lambda_B < 0$. Let $\{X_j, j = 0, \pm 1, \pm 2, \ldots\}$ be i.i.d. symmetric $\mathbf{R}^d$-valued random vectors such that*

$$n^{-B} \sum_{j=1}^{n} X_j \xrightarrow{\mathrm{w}} Z_B(1), \tag{4.1}$$

and let a sequence of matrices $\{C_j\}$ be such that

$$C_j = \begin{cases} O & \text{if } j = 0, -1, \\ \int_j^{j+1} \operatorname{sgn}(s)|s|^{D-B-I}\, ds & \text{otherwise.} \end{cases}$$

Define a new sequence of $\mathbf{R}^d$-valued random vectors $\{Y_k\}$ by

$$Y_k = \sum_{j \in \mathbf{Z}} C_j X_{k-j}. \tag{4.2}$$

Then

$$(D-B) n^{-D} \sum_{k=1}^{[nt]} Y_k \overset{\text{f.d.}}{\Rightarrow} \Delta_{D,B}(t).$$

In this section, we show the weak convergence in $C([0, T], \mathbf{R}^d)$ when (i) $Z_B(1)$ is Gaussian or (ii) $Z_B(1)$ is purely non-Gaussian and $\lambda_{D-B} > 0$ (cf. Theorem 3.2).

LEMMA 4.1. *If* $E[\|X_j\|^{2p}] < \infty$, $p \geq 2$, *then for the generated random vector* Y_k *defined by* (4.2) *the following inequality holds*:

$$E[\|\sum_{k=1}^{n} Y_k\|^{2p}] \leq A\{\mathrm{Var}(\|\sum_{k=1}^{n} Y_k\|)\}^p.$$

The proof is the same as that of Lemma 4 of Davydov [1]. ∎

Let

$$A_m(t) := \sum_{j=1-m}^{[t]-m} C_j + (t-[t])\, C_{[t]+1-m}$$

and

$$W_t := (D-B)\, n^{-D}\Big(\sum_{k=1}^{[t]} Y_k + (t-[t])\, Y_{[t]+1}\Big) = (D-B)\, n^{-D} \sum_{m\in\mathbf{Z}} A_m(nt)\, X_m.$$

LEMMA 4.2. *We have*

$$\|(D-B)\, n^{-(D-B)}(A_m(t)-A_m(s))\| \leq C\, \|n^{-(D-B)}(|t-m|^{D-B}-|s-m|^{D-B})\|.$$

Proof. The lemma can be shown in exactly the same way as in Lemma 5 of [5]. ∎

The following is easy.

LEMMA 4.3. *Suppose* $\lambda_D > 0$, *and fix* $T > 0$. *For any* $\delta > 0$, *there exists* $C_1 > 0$ *such that* $\|u^D\| \leq C_1 u^{\lambda_D - \delta}$ *for all* $0 < u \leq T$.

THEOREM 4.2. *Suppose that* $Z_B(1)$ *is Gaussian and that* $E[\|X_j\|^{2p}] < \infty$ *for some* $2p > 1/\lambda_D$. *Then for any fixed* $T > 0$

$$W_{nt} \overset{\mathcal{L}}{\Rightarrow} \Delta_{D,B}(t) \quad in\ C([0, T], \mathbf{R}^d).$$

Proof. We show the tightness of W_{nt}. Let $0 \leq s < t \leq T$. We have, by Lemma 4.1,

$$(4.3)\qquad E[\|W_{nt}-W_{ns}\|^{2p}] \leq A\{\mathrm{Var}(\|W_{nt}-W_{ns}\|)\}^p \leq C(E[\|W_{nt}-W_{ns}\|^2])^p,$$

where and in what follows C denotes an absolute positive constant. Then, by the use of Lemma 4.2 with $B = \frac{1}{2}I$ and Lemma 4.3,

$$(4.4)\qquad \begin{aligned} E[\|W_{nt}-W_{ns}\|^2] &= E\Big[\Big\|(D-\tfrac{1}{2}I)\sum_m n^{-D}(A_m(nt)-A_m(ns))\, X_m\Big\|^2\Big] \\ &\leq C\sum_m \big\|(D-\tfrac{1}{2}I)\, n^{-D}(A_m(nt)-A_m(ns))\big\|^2 E[\|X_m\|] \\ &\leq \frac{C}{n}\int_{-\infty}^{\infty} \|n^{-(D-(1/2)I)}(|nt-x|^{D-(1/2)I}-|ns-x|^{D-(1/2)I})\|^2\, dx \end{aligned}$$

$$\leqslant \frac{C}{n} \int_{-\infty}^{\infty} \|n^{-(D-(1/2)I)}\big(|n(t-s)-n(t-s)x|^{D-(1/2)I}$$

$$-|n(t-s)x|^{D-(1/2)I}\big)\|^2 n(t-s)\,dx$$

$$\leqslant C\|(t-s)^{D-(1/2)I}\|^2 (t-s) \int_{-\infty}^{\infty} \| |1-x|^{D-(1/2)I} - |x|^{D-(1/2)I} \|^2 dx$$

$$\leqslant C(t-s)^{2(\lambda_D-\delta)}.$$

Since we are assuming $2p\lambda_D > 1$, we can find a $\delta > 0$ such that $2(\lambda_D-\delta)p > 1$. Thus we infer from (4.3) and (4.4) that

$$\mathrm{E}[\|W_{nt}-W_{ns}\|^{2p}] \leqslant C(t-s)^{1+a},$$

where $a := 2(\lambda_D-\delta)p-1 > 0$. Thus the tightness of $\{W_{nt}\}$ follows from the Kolmogorov criterion. ∎

Next suppose $Z_B(1)$ is purely non-Gaussian. For some $a > 0$, define

$$\bar{X}_m = X_m I[\|n^{-B}X_m\| \leqslant a],$$

where $I[A]$ is the indicator function of a set A, and

$$\bar{Y}_k = \sum_{j\in\mathbf{Z}} C_j \bar{X}_{k-j}.$$

Note that $\mathrm{E}[\bar{X}_m] = 0$ since X_m is symmetric. Let

$$\bar{W}_t = (D-B)n^{-D}\Big(\sum_{k=1}^{[t]} \bar{Y}_k + (t-[t])\bar{Y}_{[t]+1}\Big) = (D-B)n^{-D}\sum_{m\in\mathbf{Z}} A_m(nt)\bar{X}_m.$$

LEMMA 4.4. *Under* (4.1) *we have*

$$\sup_n \mathrm{E}[\|n^{-B}\bar{X}_1\|^2] < \infty.$$

Proof. By Lemma 9 of [6],

$$\sup_n n\int_0^a xP\{\|n^{-B}\bar{X}_1\| > x\}\,dx < \infty,$$

which concludes the lemma. ∎

LEMMA 4.5. *Under* (4.1), *for any* $p < 1/\lambda_B$,

$$\sup_n n\mathrm{E}[\|n^{-B}(X_1-\bar{X}_1)\|^p] < \infty.$$

Proof. We have

$$n\mathrm{E}[\|n^{-B}(X_1-\bar{X}_1)\|^p] = n\mathrm{E}[\|n^{-B}X_1\|^p I[\|n^{-B}X_1\| > a]]$$

$$\leqslant Cn\int_a^{\infty} x^{p-1}P\{\|n^{-B}X_1\| > x\}\,dx.$$

Let $\varepsilon > 0$ and choose a so large that

$$2P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > a\} < \varepsilon \quad \text{for all } n,$$

which is possible by tightness (the convergence (4.1)). Thus

$$2P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\} < \varepsilon \quad \text{for all } x \geqslant a \text{ and for all } n.$$

Since $\{X_j\}$ are symmetric, we have

$$P\{\max_{1\leqslant j\leqslant n} \|n^{-B}X_j\| > x\} \leqslant 2P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\}.$$

Thus

$$\begin{aligned}[P\{\|n^{-B}X_1\| \leqslant x\}]^n &= P\{\max_{1\leqslant j\leqslant n} \|n^{-B}X_j\| \leqslant x\} \\ &= 1-P\{\max_{1\leqslant j\leqslant n} \|n^{-B}X_j\| > x\} \\ &\geqslant 1-2P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\}\end{aligned}$$

so that

$$\begin{aligned}nP\{\|n^{-B}X_1\| > x\} &\leqslant n\{1-[1-2P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\}]^{1/n}\} \\ &\leqslant \frac{2}{1-\varepsilon} P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\}.\end{aligned}$$

Hence

$$\begin{aligned}\sup_n n\int_a^\infty x^{p-1} P\{\|n^{-B}X_1\| > x\}\,dx &\leqslant \frac{2}{1-\varepsilon}\sup_n \int_a^\infty x^{p-1}P\{\|n^{-B}\sum_{j=1}^{n} X_j\| > x\}\,dx \\ &\leqslant \frac{2}{1-\varepsilon}\sup_n \mathrm{E}[\|n^{-B}\sum_{j=1}^{n} X_j\|^p].\end{aligned}$$

By Theorem 3 of [4], for every $p < 1/\Lambda_B$

$$\mathrm{E}[\|n^{-B}\sum_{j=1}^{n} X_j\|^p] \to \mathrm{E}[\|Z_B\|^p] < \infty,$$

and hence

$$\sup_n n\int_a^\infty x^{p-1} P\{\|n^{-B}X_1\| > x\}\,dx < \infty,$$

concluding the lemma. ∎

THEOREM 4.3. *If* $\lambda_{D-B} > 0$ *and* $\Lambda_{D-B-I} + \Lambda_B < 0$, *then*

$$W_{nt} \overset{\mathscr{L}}{\Rightarrow} \Delta_{D,B}(t) \quad in\ C([0, T], \mathbf{R}^d).$$

Proof. We show the tightness of $\{\bar{W}_{nt}\}$ and $\{W_{nt} - \bar{W}_{nt}\}$ separately. Let $0 \leqslant s < t \leqslant T$.

(i) The tightness of $\{\bar{W}_{nt}\}$: We have

$$J_1 := \mathrm{E}[\|\bar{W}_{nt} - \bar{W}_{ns}\|^2] = \mathrm{E}\Big[\Big\|(D-B)\sum_m n^{-D}(A_m(nt) - A_m(ns))\bar{X}_m\Big\|^2\Big]$$

$$= \mathrm{E}\Big[\Big\|(D-B)\sum_m n^{-(D-B)}(A_m(nt) - A_m(ns))\,n^{-B}\bar{X}_m\Big\|^2\Big]$$

(since $D \in \mathrm{Com}(B)$)

$$\leqslant C\sum_m \big\|(D-B)\,n^{-D-B}(A_m(nt) - A_m(ns))\big\|^2\,\mathrm{E}[\|n^{-B}\bar{X}_m\|^2].$$

By Lemmas 4.2 and 4.4, we have

$$J_1 \leqslant \frac{C}{n}\int_{-\infty}^{\infty} \|n^{D-B}(|nt-x|^{D-B} - |ns-x|^{D-B})\|^2\,dx$$

$$= \frac{C}{n}\int_{-\infty}^{\infty} \big\|n^{D-B}(|n(t-s) - n(t-s)x|^{D-B} - |n(t-s)x|^{D-B})\big\|^2\,n(t-s)\,dx$$

$$\leqslant C\,\|(t-s)^{D-B}\|^2 (t-s)\int_{-\infty}^{\infty} \||1-x|^{D-B} - |x|^{D-B}\|^2\,dx.$$

Note that since $\lambda_{D-B} > 0$ and $\Lambda_{D-B+I} + \Lambda_B < 0$, we obtain

$$\int_{-\infty}^{\infty} \||1-x|^{D-B} - |x|^{D-B}\|^2\,dx < \infty.$$

Thus

$$J_1 \leqslant C\,\|(t-s)^{D-B}\|^2 (t-s).$$

However, by Lemma 4.3, for any $\delta > 0$

$$\|(t-s)^{D-B}\| \leqslant C(t-s)^{\lambda_{D-B}-\delta}, \qquad t, s \leqslant T.$$

Since $\lambda_{D-B} > 0$, we can take $\delta > 0$ so small that $\lambda_{D-B} - \delta > 0$, and thus

$$J_1 \leqslant C(t-s)^{1+a}, \qquad \text{where } a := 2(\lambda_{D-B} - \delta) > 0.$$

The tightness of $\{\bar{W}_{nt}\}$ thus follows from the Kolmogorov criterion.

(ii) The tightness of $\{W_{nt} - \bar{W}_{nt}\}$: Note that there exists $p < 1/\Lambda_B$ such that

$$\int_{-\infty}^{\infty} \||1-x|^{D-B} - |x|\|^p\,dx < \infty.$$

(See the proof of Theorem 5.4 of [7].) Then we have

$$J_2 := \mathrm{E}[\|(W_{nt}-\bar{W}_{nt})-(W_{ns}-\bar{W}_{ns})\|^p]$$
$$= \mathrm{E}\Big[\Big\|(D-B)\sum_m n^{-(D-B)}(A_m(nt)-A_m(ns))\,n^{-B}(X_m-\bar{X}_m)\Big\|^p\Big]$$
$$\leqslant C\sum_m \big\|(D-B)\,n^{-(D-B)}(A_m(nt)-A_m(ns))\big\|^p\,\mathrm{E}[\|n^{-B}(X_m-\bar{X}_m)\|^p],$$

where we have applied the Marcinkiewicz–Zygmund inequality to $\{X_m-\bar{X}_m, m\in \boldsymbol{Z}\}$, a sequence of i.i.d. random vectors with $\mathrm{E}[X_m-\bar{X}_m]=0$.

By Lemmas 4.2 and 4.5, we have

$$J_2 \leqslant \frac{C}{n}\int_{-\infty}^{\infty} \|n^{D-B}(|nt-x|^{D-B}-|ns-x|^{D-B})\|^p\,dx$$
$$= \frac{C}{n}\int_{-\infty}^{\infty} \big\|n^{D-B}(|n(t-s)-n(t-s)x|^{D-B}-|n(t-s)x|^{D-B})\big\|^p\,n(t-s)\,dx$$
$$\leqslant C\,\|(t-s)^{D-B}\|^p(t-s)\int_{-\infty}^{\infty} \|\,|1-x|^{D-B}-|x|^{D-B}\|^p\,dx$$
$$\leqslant C\,\|(t-s)^{D-B}\|^p(t-s).$$

Since $\lambda_{D-B}>0$, by Lemma 4.3, we can find $a>0$ such that

$$J_2 \leqslant C(t-s)^{1+a}.$$

The tightness of $\{W_{ns}-\bar{W}_{ns}\}$ thus follows from the Kolmogorov criterion. ∎

5. Operator-self-similarity and self-similar marginals.

THEOREM 5.1. *Let $X=\{X(t), t\geqslant 0\}$ be an $\boldsymbol{R}^d$-valued operator-self-similar process with exponent D. Assume that the distribution of $X(1)$ is absolutely continuous and*

$$|\mathrm{E}[e^{i\langle\theta, X(1)\rangle}]| \neq 0 \quad \textit{for any } \theta\in\boldsymbol{R}^d. \tag{5.1}$$

If X has a complete set of univariate self-similar marginals, then D is semisimple. Furthermore, the self-similar parameters of the univariate self-similar marginals are the real parts of the eigenvalues of D.

The proof is almost the same as that of Theorem 2 of [2].

Remark 5.1. If $X(1)$ is infinitely divisible, the condition (5.1) is automatically satisfied.

THEOREM 5.2. *Let X be an $\boldsymbol{R}^d$-valued operator-self-similar process with exponent D. If D is diagonalizable over $\boldsymbol{R}$, then X has a complete set of univariate marginals.*

The proof is also almost the same as that of Theorem 3 of [2].

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Received on 8.2.1994;
revised version on 14.8.1994