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AZÉMA MARTINGALES WITH DRIFT

BY

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Abstract. Pursuing the earlier work of Emery [1], Meyer [2], [3] and the author [4] it is shown that Azéma martingales starting from a varying initial point on the line constitute an Evans–Hudson flow in the framework of quantum stochastic calculus. A new transformation property of Azéma martingales is established. It turns out, rather remarkably, that the Azéma martingales and a class of their Weyl perturbations satisfy the same Itô's formula but, in the vacuum state, have very different asymptotic statistical properties as time increases to infinity.

1. Introduction. Inspired by the work of M. Emery [1] on structure equations for Azéma martingales and several discussions with P. A. Meyer we introduced in [4] the quantum stochastic differential equation (qsde)

$$dX = (c-1)\,X d\Lambda + dA + dA^{\dagger}, \qquad X(0) = 0, \tag{1.1}$$

in the boson Fock space $\Gamma\big(L^2(\boldsymbol{R}_+)\big)$ over $L^2(\boldsymbol{R}_+)$, where $A^{\dagger}$, Λ, A are the creation, conservation and annihilation processes, respectively, of quantum stochastic calculus [5], and c is a real scalar satisfying $-1 \leqslant c < 1$. It was shown in [4] that there exists a unique commutative, bounded and selfadjoint operator-valued solution $\{X_c(t),\, t \geqslant 0\}$ for (1.1) which satisfies the following properties:

(i) $\|X_c(t)\| \leqslant \big(2t/(1-c)\big)^{1/2}$.

(ii) In the Fock vacuum state Ω, $X_c(t)$ is a martingale as well as a Markov process with stationary transition probability for which $t^{-1/2} X_c(t)$ has a symmetric distribution ν_c independent of t.

(iii) For any polynomial p, $X_c(t)$ satisfies the Itô formula

$$dp\big(X_c(t)\big) = (\theta_0^1 p)\big(X_c(t)\big)\, dX_c(t) + (\theta_0^0 p)\big(X_c(t)\big)\, dt, \tag{1.2}$$

where θ_0^1 and θ_0^0 are maps on the algebra $\mathscr{A}$ of all polynomials given by

$$(\theta_0^1 p)(x) = \frac{p(cx) - p(x)}{(c-1)x}, \tag{1.3}$$

(1.4)
$$(\theta_0^0 p)(x) = \frac{p(cx)-p(x)-(c-1)xp'(x)}{(c-1)^2x^2}.$$

(iv) The distribution ν_c in (ii) has the moment sequence $\{m_n\}$ given by

(1.5)
$$m_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \left(\frac{n}{2}!\right)^{-1} \prod_{j=1}^{n/2} \frac{c^{2j}-1-2j(c-1)}{(c-1)^2} & \text{if } n \text{ is even.} \end{cases}$$

If $c \neq -1$, ν_c is absolutely continuous, and as $c \to 1$, ν_c converges weakly to the standard normal distribution. If $c = 0$, $(2t)^{-1/2} X_0(t)$ has the density function $\pi^{-1}(1-x^2)^{-1/2}$ in the interval $(-1, 1)$. In particular, $(2t)^{-1/2}|X_0(t)|$ has Lévy's arcsin law. If $c = -1$, ν_c is the Bernoulli distribution with probability 1/2 for each of the values ± 1. (Rather remarkably, $\{X_{-1}(t)\}$ is a multiplicative component of the fermionic Brownian motion.)

(v) $\{X_c(t)\}$ has the chaotic representation property [1].

Properties (i)–(iii) and (v) of the Azéma martingales $X_c(t)$ show that they share all the features of standard Brownian motion except for the lack of continuous trajectories. They deserve to be considered as legitimate and reasonable models of cumulative white noise. As $c \to 1$, (1.1) shows that $X_c(t)$ does indeed become the standard Brownian motion.

The aim of the present note is to make some comments on Azéma martingales starting from a given initial position x and also illustrate the fruitfulness of quantum stochastic methods in constructing a wide variety of new classical Markov processes. Indeed, one of the great advantages of realising a classical stochastic process as a pair $(X(\cdot), \varrho)$, where $\{X(t)\}$ is a commuting family of selfadjoint operators in a Hilbert space and ϱ is a vector state (or a density matrix), is that one can alter the state ϱ and obtain a variety of deformations of the original process. Another way of doing the same thing is to fix the state ϱ and consider the pair $(U^* X(\cdot) U, \varrho)$, where U is a unitary operator. It should be interesting to investigate the stability as well as changes in the statistical properties of a process under such deformations. The present note will provide an illustration of this idea by examining the behaviour of the Azéma martingales under deformation by a parametric family of Weyl operators in $\Gamma(L^2(\mathbf{R}_+))$.

2. Azéma martingales starting from a given point. For every $-1 \leqslant c < 1$, let $\{\Gamma_c(t), t \geqslant 0\}$ be the contraction operator-valued process in $\Gamma(L^2(\mathbf{R}_+))$ defined by

(2.1)
$$\Gamma_c(t)e(v) = e(\{cI_{[0,t]} + I_{(t,\infty)}\}v), \quad t \geqslant 0, \qquad \text{for all } v \in L^2(\mathbf{R}_+),$$

where I_E is the indicator of $E \subset \mathbf{R}_+$, and $e(v)$ denotes the exponential vector

$$1 \oplus v \oplus \frac{v^{\otimes 2}}{\sqrt{2!}} \oplus \ldots \oplus \frac{v^{\otimes n}}{\sqrt{n!}} \oplus \ldots$$

Define

$$(2.2)\qquad Y_{c,x}(t) = x\Gamma_c(t) + X_c(t) \quad \text{for all } x \in \mathbf{R},$$

where X_c is the Azéma martingale which is the unique solution of (1.1). It follows from property (i) in Section 1 that $Y_{c,x}(t)$ is a bounded selfadjoint operator satisfying

$$(2.3)\qquad \|Y_{c,x}(t)\| \leqslant |x| + \sqrt{\frac{2t}{1-c}}.$$

Since

$$d\Gamma_c(t) = (c-1)\Gamma_c(t)\,d\Lambda, \qquad \Gamma_c(0) = I,$$

equations (1.1) and (2.2) imply that $Y(t) = Y_{c,x}(t)$ satisfies the qsde

$$(2.4)\qquad dY = (c-1)\,Y d\Lambda + d\Lambda + d\Lambda^\dagger, \qquad Y(0) = x.$$

Thus Y satisfies the same qsde in (1.1) but with initial value x. It now follows exactly by the same arguments as in [4], (2.3) and (2.4), that $\{Y(t)\}$ is a commutative process of bounded selfadjoint operators satisfying the Itô formula given by (1.2)–(1.4) with X_c replaced by $Y = Y_{c,x}$. Using (2.4) and quantum Itô's formula one obtains by induction

$$(2.5)\qquad dY^n = \lambda_n(c-1)\,Y^n d\Lambda + \lambda_n Y^{n-1}(d\Lambda^\dagger + d\Lambda) + \mu_n Y^{n-2}\,dt,$$

where

$$(2.6)\qquad \lambda_n = \frac{c^n-1}{c-1}, \quad \mu_n = \frac{c^n-1-n(c-1)}{(c-1)^2}, \quad n \geqslant 1.$$

On the algebra $\mathscr{A}$ of all polynomials in one variable define the homomorphism $\{j_t,\, t \geqslant 0\}$ by

$$(2.7)\qquad j_t(p) = p\big(Y(t)\big), \quad p \in \mathscr{A}.$$

Then (2.5) and (2.6) yield

$$(2.8)\quad dj_t(p) = j_t\big(\theta_0^1(p)\big)\,d\Lambda^\dagger + j_t\big(\theta_1^1(p)\big)\,d\Lambda + j_t\big(\theta_1^0(p)\big)\,d\Lambda + j_t\big(\theta_0^0(p)\big)\,dt, \quad j_0(p) = p,$$

where θ_0^1 and θ_0^0 are defined by (1.3) and (1.4), $\theta_1^0 = \theta_0^1$ and θ_1^1 is defined by

$$(2.9)\qquad \theta_1^1(p)(y) = p(cy) - p(y), \quad p \in \mathscr{A}.$$

Thus $\{j_t\}$ given by (2.7) is an abelian Evans–Hudson flow [5], with structure maps θ_j^i, $i, j \in \{0, 1\}$. Furthermore, (2.4) and (2.8) imply that $\{Y(t)\}$ satisfies the same classical Itô's formula (1.2)–(1.4) with X_c replaced by Y.

We shall now evaluate the moments of the Azéma martingale $Y(t) = Y_{c,x}(t)$ in the vacuum state Ω. Define

$$f_n(t, x) = \langle \Omega,\, Y(t)^n \Omega \rangle.$$

Then (2.5) implies that

$$f_n(t, x) = x^n + \mu_n \int_0^t f_{n-2}(s, x)\,ds \qquad \text{if } n \geqslant 2,$$

$$f_0(t, x) = 1, \qquad f_1(t, x) = x.$$

A simple calculation yields

$$f_{2n}(t, x) = x^{2n} + \mu_{2n} x^{2n-2} t + \mu_{2n}\mu_{2n-2} x^{2n-4} \frac{t^2}{2!} + \ldots + \mu_{2n}\mu_{2n-2} \ldots \mu_2 \frac{t^n}{n!}, \tag{2.10}$$

$$f_{2n+1}(t, x) = x^{2n+1} + \mu_{2n+1} x^{2n-1} t + \mu_{2n+1}\mu_{2n-1} x^{2n-3} \frac{t^2}{2!} + \ldots + \mu_{2n+1}\mu_{2n-1} \ldots \mu_3 x \frac{t^n}{n!}. \tag{2.11}$$

In particular, we have the identity

$$\frac{f_n(t, x)}{x^n} = f_n(tx^{-2}, 1) \qquad \text{if } x \neq 0.$$

This has the interpretation that the two Markov processes $\{x^{-1} Y_{x,c}(t)\}$ and $\{Y_{1,c}(t/x^2)\}$ are identical in law.

From (2.10) and (2.11) we also have

$$\lim_{t\to\infty} \frac{f_n(t, x)}{t^{n/2}} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mu_n \mu_{n-2} \mu_{n-4} \ldots \mu_2 & \text{if } n \text{ is even.} \end{cases}$$

In other words, as $t \to \infty$, the distribution of $t^{-1/2} Y_{x,c}(t)$ converges weakly to the distribution ν_c of $X_c(1)$ in the vacuum state.

3. Weyl perturbation of the Azéma martingale. In the boson Fock space $\Gamma(L^2(\boldsymbol{R}_+))$ consider the special Weyl operators $W_u(t)$, $t \geqslant 0$, $u \in \boldsymbol{C}$, defined by

$$W_u(t)\,e(f) = \exp\left[-\tfrac{1}{2}|u|^2 t - \bar{u} \int_0^t f(s)\,ds\right] e(f + uI_{[0,t]}) \qquad \text{for all } f \in L^2(\boldsymbol{R}_+), \tag{3.1}$$

where $e(f)$ is the exponential vector with exponent f. Then W_u is a unitary operator-valued process satisfying the qsde

$$dW_u = W_u(u\,dA^\dagger - \bar{u}\,dA - \tfrac{1}{2}|u|^2\,dt). \tag{3.2}$$

Let Y be the Azéma martingale $Y_{c,x}$ defined by (2.2).

Define

$$Z(t) = W_u^\dagger(t)\, Y(t)\, W_u(t). \tag{3.3}$$

A routine computation using quantum Itô's formula, (2.4) and (3.2) yields

(3.4) $$dZ = (c-1)Z dA + \big(1+(c-1)uZ\big) dA^{\dagger} + \big(1+(c-1)\bar{u}Z\big) dA + \{2\mathrm{Re}\,u + (c-1)|u|^2 Z\}\, dt$$

with $Z(0) = x$. Since the dt coefficient is not equal to 0, $\{Z(t)\}$ is no more a martingale. However, $\{Z(t)\}$ is a bounded selfadjoint operator-valued commutative process. Indeed, the factorisability property of the Weyl process implies that

$$Z(t) = W_u^{\dagger}(T)\, Y(t)\, W_u(T) \quad \text{for all } 0 \leqslant t \leqslant T,$$

and hence the commutativity of the family $\{Z(t)\}$. Define the homomorphisms $\tilde{j}_t$ on the polynomial algebra $\mathscr{A}$ by

$$\tilde{j}_t(p) = p\big(Z(t)\big), \quad t \geqslant 0.$$

Then

(3.5) $$\tilde{j}_t(p) = W_u^{\dagger}(t)\, j_t(p)\, W_u(t),$$

where $j_t(p)$ satisfies (2.7). Once again using quantum Itô's formula, (2.8) and (3.2) one obtains

(3.6) $$d\tilde{j}_t(p) = \tilde{j}_t\big(\phi_0^1(p)\big) dA^{\dagger} + \tilde{j}_t\big(\phi_1^1(p)\big) dA + \tilde{j}_t\big(\phi_1^0(p)\big) dA + \tilde{j}_t\big(\phi_0^0(p)\big) dt,$$

where $\phi_1^1 = \theta_1^1$ (see (2.9)),

(3.7) $$\phi_0^1(p)(y) = \frac{p(cy)-p(y)}{(c-1)y} + u\big(p(cy)-p(y)\big),$$

(3.8) $$\phi_1^0(p)(y) = \frac{p(cy)-p(y)}{(c-1)y} + \bar{u}\big(p(cy)-p(y)\big),$$

(3.9) $$\phi_0^0(p) = \frac{p(cy)-p(y)-(c-1)yp'(y)}{(c-1)^2 y^2} + 2(\mathrm{Re}\,u)\frac{p(cy)-p(y)}{(c-1)y} + |u|^2\big(p(cy)-p(y)\big).$$

Thus the process $\{Z(t)\}$ starting from x at $t = 0$ yields an Abelian Evans–Hudson flow over the initial algebra $\mathscr{A}$ with structure maps ϕ_j^i, $i, j \in \{0, 1\}$. Once again (3.4) and (3.6) imply that

$$dp\big(Z(t)\big) = \theta_0^1(p)\big(Z(t)\big) dZ(t) + \theta_0^0(p)\big(Z(t)\big) dt,$$

where θ_0^1 and θ_0^0 are given by (1.3) and (1.4). In other words, in the vacuum state $\{Z(t)\}$ is a Markov process starting from x and satisfying the same classical Itô's formula as $Y(t)$. Indeed, it is one of the interesting problems in the theory of Markov processes to describe all the Markov processes $\xi(t)$ satisfying a given Itô's formula of the form

$$dp(\xi) = (Lp)(\xi)\,d\xi + (Mp)(\xi)\,dt,$$

where p varies over an algebra $\mathscr{C}$ of functions, and L and M are two fixed linear operators on the linear space $\mathscr{C}$.

We call $\{Z(t)\}$ defined by (3.3) a *Weyl perturbation* of the Azéma martingale $\{Y(t)\}$. In analogy with the case of the Brownian motion $\{Z(t)\}$ can also be called an *Azéma martingale with drift*. But it should be emphasised that $\{Z(t)\}$ is not a martingale. Even though Itô's formula remains stable under the Weyl perturbation, the fact that there is a nontrivial dt coefficient in (3.4) implies a dramatic change in the asymptotic behaviour of $Z(t)$ as $t \to \infty$. We shall now describe this phenomenon in more detail. To this end we examine the moments of $Z(t)$ in the vacuum state. Define

$$g_n(t, x) = \langle \Omega, Z(t)^n \Omega \rangle.$$

Then from (3.5)–(3.8) we have

$$\frac{dg_n}{dt} = \frac{c^n - 1 - n(c-1)}{(c-1)^2} g_{n-2} + 2(\mathrm{Re}\,u)\frac{c^n - 1}{c-1} g_{n-1} + |u|^2 (c^n - 1) g_n(t) \quad \text{if } n \geqslant 2 \tag{3.10}$$

with

$$g_n(0, x) = x^n, \qquad g_0(t, x) = 1,$$

$$\frac{dg_1}{dt} = 2\mathrm{Re}\,u + |u|^2 (c-1) g_1. \tag{3.11}$$

Thus

$$g_1(t, x) = x \exp[|u|^2 (c-1)t] + 2(\mathrm{Re}\,u)\frac{\exp[|u|^2 (c-1)t] - 1}{|u|^2 (c-1)}$$

and, in particular,

$$\lim_{t\to\infty} g_1(t, x) = \frac{2\mathrm{Re}\,u}{(1-c)|u|^2} \quad \text{for } -1 \leqslant c < 1. \tag{3.12}$$

From (3.9) we have

$$g_n(t, x) = x^n \exp[|u|^2 (c^n - 1)t] + \int_0^t \exp[|u|^2 (c^n - 1)(t-s)] \times \left\{ \frac{c^n - 1 - n(c-1)}{(c-1)^2} g_{n-2}(s, x) + 2(\mathrm{Re}\,u)\frac{c^n - 1}{c-1} g_{n-1}(s, x) \right\} ds. \tag{3.13}$$

It now follows by induction and (3.10)–(3.13) that for $-1 < c < 1$ the limit $\lim_{t\to\infty} g_n(t, x) = \varrho_n$ exists for every n, ϱ_n is independent of x

and

$$\varrho_n = \{|u|^2(1-c^n)\}^{-1}\left\{2(\mathrm{Re}\,u)\frac{c^n-1}{c-1}\varrho_{n-1}+\frac{c^n-1-n(c-1)}{(c-1)^2}\varrho_{n-2}\right\} \quad \text{for } n \geqslant 2,$$

$$\varrho_0 = 1, \quad \varrho_1 = \frac{2\,\mathrm{Re}\,u}{(1-c)\,|u|^2}. \tag{3.14}$$

Since $\{\varrho_n\}$ is the limit of a moment sequence, it is clear that there exists a probability distribution with $\{\varrho_n\}$ as its sequence of moments. We shall now prove that this probability distribution is unique. For $|c| < 1$ it is clear from (3.14) that there exist positive constants α and β (depending on c) such that

$$|\varrho_n| \leqslant \alpha\,|\varrho_{n-1}|+n\beta\,|p_{n-2}| \quad \text{for all } n \geqslant 2. \tag{3.15}$$

Choose and fix a constant $\gamma > 0$ satisfying

$$\alpha\gamma+\beta \leqslant \gamma^2, \quad |\varrho_1| \leqslant \gamma, \quad |\varrho_2| \leqslant 2\gamma^2.$$

We now claim that $|\varrho_j| \leqslant j!\,\gamma^j$ for all j. By choice this holds for $j = 1, 2$. Suppose that the claim holds for all $j < n$. Then (3.15) implies

$$|\varrho_n| \leqslant \alpha(n-1)!\,\gamma^{n-1}+(n-2)!\,n\beta\gamma^{n-2} \leqslant n!\,\gamma^{n-2}(\alpha\gamma+\beta) \leqslant n!\,\gamma^n.$$

Thus the claim holds for all j and

$$\sum_{j=0}^{\infty} |\varrho_j|\frac{|t|^j}{j!} < \infty \quad \text{for } |t| < \frac{1}{\gamma}.$$

In other words, $\{\varrho_n\}$ is the moment sequence of a unique probability measure λ_c on the line with the moment generating function $\sum_{j=0}^{\infty}\varrho_j(t^j/j!)$ defined in the interval $|t| < 1/\gamma$. This shows that the transition probability measure $P(t, x, \cdot)$ of the Markov process $\{Z(t)\}$ converges weakly to the probability measure λ_c as $t \to \infty$ for $-1 < c < 1$.

We can now summarise the discussions of Sections 2 and 3 in the form of a theorem.

THEOREM. *Let $\{Y_{c,x}(t)\}$ be the Azéma martingale starting from x with parameter $-1 \leqslant c < 1$, defined by* (2.2). *Let*

$$Z_{c,x}(t) = W_u(t)^{\dagger}\, Y_{c,x}(t)\, W_u(t), \quad u \in C,\ u \neq 0. \tag{3.16}$$

In the vacuum state Ω, both $\{Y_{c,x}(t)\}$ and $\{Z_{c,x}(t)\}$ are Markov processes with stationary transition probabilities satisfying the same Itô's formula (1.2)–(1.4). *As $t \to \infty$, the distribution of $t^{-1/2}\,Y_{c,x}(t)$ in the state Ω converges weakly to the distribution ν_c of $t^{-1/2}\,Y_{c,0}(t)$ which is independent of t. For $-1 < c < 1$, the distribution of $Z_{c,x}(t)$ in the state Ω converges weakly as $t \to \infty$ to a limiting distribution λ_c independent of x.*

Remark 1. Suppose $c = -1$. Then, in the vacuum state, Z is a Markov process starting from x at time 0 and having stationary transition probability

$P(t, y, \cdot)$ with support at the two points $\pm\sqrt{y^2+t}$ given by

$$P(t, y, \{\pm\sqrt{y^2+t}\}) = \tfrac{1}{2}(1 \pm \alpha(t, y)),$$

where $\alpha(t, y) = (y^2+t)^{-1/2}\{y\exp[-2|u|^2 t] + |u|^{-2}\operatorname{Re} u(1-\exp[-2|u|^2 t])\}$.

Remark 2. In the Theorem, if we write

$$Z(c, x, t, u) = W_u^\dagger(t)\, Y_{c,x}(t)\, W_u(t),$$

then the differential equations (3.10) and (3.11) imply that for every $t \geqslant 0$ the observables $Z(c, x, t, u)$ and $xZ(c, 1, tx^{-2}, ux)$ have the same moment sequence in the vacuum state, and hence the corresponding Markov processes generated by them are identical in law for each $x \neq 0$.

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