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INFINITE DIVISIBILITY OF SOME FUNCTIONALS ON STOCHASTIC PROCESSES

BY

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Abstract. The paper deals with non-negative stochastic processes $X(t, \omega)$ $(t \geqslant 0)$ with stationary and independent increments, continuous on the right sample functions, non-degenerate to 0 and fulfilling the initial condition $X(0, \omega) = 0$. The main aim is to study the probability distribution of the functional $\int_0^\infty e^{-uX(t,\omega)}\,dt$ for $u > 0$. In particular, the multiplicative infinite divisibility of such functionals is discussed and a description of corresponding spectral measures is established.

1. Preliminaries and notation. We denote by $\mathcal{M}$ the set of all non-negative bounded measures defined on Borel subsets of the half-line $R_+ = [0, \infty)$. By $\mathcal{M}_+$ and $\mathcal{P}$ we denote the subsets of $\mathcal{M}$ consisting of measures M with $M(R_+) > 0$ and $M(R_+) = 1$, respectively. Further, by $\mathcal{M}_1$ we denote the subset of $\mathcal{M}$ consisting of measures M fulfilling the condition $\int_0^\infty e^{ax} M(dx) < \infty$ for all $a < 1$. Given $M \in \mathcal{M}$ we denote by $\hat{M}$ and $\langle M \rangle$ the Laplace transformation and the Bernstein transformation of M, respectively, i.e.

$$\hat{M}(z) = \int_0^\infty e^{-zx} M(dx) \quad \text{and} \quad \langle M \rangle(z) = \int_0^\infty \frac{1-e^{-zx}}{1-e^{-x}} M(dx)$$

for $z \geqslant 0$. For $x = 0$ the last integrand is assumed to be z. It is clear that

$$\hat{M}(z) + \langle M \rangle(z) = \langle M \rangle(z+1) \tag{1.1}$$

for $z \geqslant 0$. Moreover, it is well known ([4], Chapter XIII,7) that infinitely divisible measures from $\mathcal{P}$ are of the form $e_+(M)$ with $M \in \mathcal{M}$ and

$$e_+(M)^\wedge(z) = \exp\big(-\langle M \rangle(z)\big). \tag{1.2}$$

The uniquely determined measure M is called the *spectral measure* of $e_+(M)$. It is easy to verify that the following statements are true.

PROPOSITION 1.1. *$M \in \mathcal{M}_1$ if and only if the function $\langle M \rangle$ has an analytic extension in the half-plane* $\operatorname{Re} z > -1$.

PROPOSITION 1.2. *$M \in \mathcal{M}_1$ if and only if $e_+(M) \in \mathcal{M}_1$.*

PROPOSITION 1.3. *For every $M \in \mathcal{M}$ and $z \geqslant 1$ the inequality $\langle M \rangle (z) \leqslant \leqslant M(R_+) z$ is fulfilled.*

In the sequel we shall frequently use the measure Π defined by the formula

$$\Pi(dx) = x^{-1}(1-e^{-x})e^{-x}dx$$

on R_+. It is evident that $\Pi \in \mathcal{M}_1$,

$$\langle \Pi \rangle (z) = \log(1+z) \tag{1.3}$$

and

$$e_+(\Pi)(dx) = e^{-x}dx. \tag{1.4}$$

By δ_c we denote the probability measure concentrated at the point c. For $M, N \in \mathcal{M}$ we write $M \leqslant N$ whenever $M(A) \leqslant N(A)$ for all Borel subsets A of R_+. Further, by $M * N$ we denote the convolution of M and N.

For $a \geqslant 0$ and $b > 0$ we define two families of transformations $T(a, b)$ and $U(a, b)$ by setting

$$\big(T(a, b)M\big)(dx) = (1-e^{-x})(1-e^{-x/b})^{-1}\exp(b^{-1}-ab^{-1}-1)xM(dx/b) \tag{1.5}$$

and

$$\big(U(a, b)M\big)(dx) = (1-e^{-x})(1-e^{-x/b})^{-1}e^{-ax/b}M(dx/b) \tag{1.6}$$
$$+ab^{-1}(1-e^{-x})e^{-ax/b}\int_{x/b}^{\infty}(1-e^{-y})^{-1}M(dy)\,dx,$$

respectively. It is easy to check the inclusions

$$T(a, b)\mathcal{M}_1 \subset \mathcal{M}_1 \quad \text{for } a \geqslant 0 \text{ and } b > 0, \tag{1.7}$$

$$T(a, b)\mathcal{M}_+ \subset \mathcal{M}_+ \tag{1.8}$$

whenever $a \geqslant 0$, $b > 0$ and $a+b \geqslant 1$,

$$U(a, b)\mathcal{M}_+ \subset \mathcal{M}_+ \quad \text{for } a \geqslant 0 \text{ and } b > 0. \tag{1.9}$$

The relation $M \leqslant N$ is invariant under both transformations $T(a, b)$ and $U(a, b)$. Moreover, the families $T(a, b)$ and $U(a, b)$ $(a \geqslant 0, b > 0)$ are semigroups under the composition on $\mathcal{M}_1$ and $\mathcal{M}_+$, respectively, and

$$T(a, b)T(c, d) = T(ad+c, bd), \quad U(a, b)U(c, d) = U(ad+c, bd).$$

From (1.5) and (1.6) by standard calculation we get

$$\langle T(a, b)M \rangle (z) = \langle M \rangle (bz+a+b-1) - \langle M \rangle (a+b-1), \tag{1.10}$$

$$e_+\big(T(a, b)M\big)(dx) \tag{1.11}$$
$$= \exp\big(\langle M \rangle (a+b-1) + (b^{-1}-ab^{-1}-1)x\big)e_+(M)(dx/b)$$

and

$$(1.12)\qquad \langle U(a,b)M\rangle(z) = bz(bz+a)^{-1}\langle M\rangle(bz+a).$$

We define a binary operation $\circ$ on $\mathcal{M}_+$ by setting

$$(1.13)\qquad (M\circ N)(dx) = (1-e^{-x})\int_0^\infty (1-e^{-t})^{-1} e_+(tN)(dx)\, M(dt).$$

Then

$$(1.14)\quad \langle M\circ N\rangle(z) = \int_0^\infty (1-e^{-t})^{-1}(1-e^{-t\langle N\rangle(z)})\, M(dt) = \langle M\rangle(\langle N\rangle(z)),$$

which shows that the set $\mathcal{M}_+$ is closed under the operation $\circ$. Since $\langle M\rangle(1) = M(R_+) = 1$ for $M\in\mathcal{P}$, we conclude that also the set $\mathcal{P}$ is closed under the operation $\circ$. Moreover, for $a_1\geqslant 0$, $a_2\geqslant 0$, $M_1, M_2, N\in\mathcal{M}_+$ we have the formulae

$$(1.15)\qquad (a_1M_1+a_2M_2)\circ N = a_1(M_1\circ N)+a_2(M_2\circ N),$$

$$(1.16)\qquad M\circ\delta_0 = \delta_0\circ M = M$$

and

$$(1.17)\qquad M_1\circ N\leqslant M_2\circ N \quad \text{provided } M_1\leqslant M_2.$$

Given $a>0$ and $M, N\in\mathcal{M}_+$, by (1.12) and (1.14) we have

$$\langle M\circ(aN)\rangle(z) = \langle M\rangle(a\langle N\rangle(z)) = \langle U(0,a)M\rangle(\langle N\rangle(z)) = \langle (U(0,a)M)\circ N\rangle(z),$$

which yields

$$(1.18)\qquad M\circ(aN) = (U(0,a)M)\circ N.$$

Suppose that $a\geqslant 0$, $b>0$, $a+b\geqslant 1$ and $M, N\in\mathcal{M}_+$. Then, by (1.8), (1.10) and (1.14), we have the formula

$$(1.19)\qquad (T(c,1)M)\circ(T(a,b)N) = T(a,b)(M\circ N)$$

with $c = \langle N\rangle(a+b-1)$.

We define a mapping $M\to\bar{M}$ from $\mathcal{M}_+$ onto $\mathcal{P}$ by setting $\bar{M} = M(R_+)^{-1}M$.

LEMMA 1.1. *For $M, N\in\mathcal{M}_+$ and $a = N(R_+)$ the formula*

$$\overline{M\circ N} = \overline{(U(0,a)M)}\circ\bar{N}$$

is true.

Proof. Introducing the notation $b = U(0,a)M(R_+)$ we have

$$N = a\bar{N} \quad\text{and}\quad U(0,a)M = b\,\overline{U(0,a)M}.$$

Consequently, by (1.8),

$$M \circ N = M \circ (a\bar{N}) = \big(U(0, a)M\big) \circ \bar{N} = b\overline{U(0, a)M} \circ \bar{N}.$$

Since the set $\mathscr{P}$ is closed under the operation $\circ$, we conclude that $(M \circ N)(R_+) = b$, which yields the assertion of the lemma.

Suppose that $M \in \mathscr{M}_+$. Then, by (1.8), $T(1, 1)\bar{M} \in \mathscr{M}_+$. Put

$$SM = \Pi \circ \big(T(1, 1)\bar{M}\big). \tag{1.20}$$

In the sequel the mapping S from $\mathscr{M}_+$ will play a crucial role. By (1.3), (1.10) and (1.14) we have the formula

$$\langle SM \rangle (z) = \log \langle \bar{M} \rangle (z+1) \tag{1.21}$$

which, by Proposition 1.1, yields the inclusion

$$S\mathscr{M}_+ \subset \mathscr{M}_1. \tag{1.22}$$

Moreover, by (1.5) and (1.16),

$$S\delta_0 = \Pi \tag{1.23}$$

and

$$SM = SN \quad \text{provided } \bar{M} = \bar{N}. \tag{1.24}$$

Since $\langle \bar{M} \rangle (1) = 1$, by (1.11) we have

$$e_+\big(tT(1, 1)\bar{M}\big)(dx) = e^{t-x} e_+(t\bar{M})(dx),$$

which, by (1.13) and (1.20), yields

$$(SM)(dx) = (1-e^{-x})e^{-x} \int_0^\infty y^{-1} e_+(y\bar{M})(dx)\, dy.$$

Substituting $y = M(R_+)t$ we get the formula

$$(SM)(dx) = (1-e^{-x})e^{-x} \int_0^\infty t^{-1} e_+(tM)(dx)\, dt. \tag{1.25}$$

EXAMPLE 1.1. Suppose that $M(dx) = c(1-e^{-x})P(dx)$, where $c > 0$, $P \in \mathscr{P}$ and $P(\{0\}) = 0$. Then

$$e_+(tM) = e^{-ct}\left(\delta_0 + \sum_{k=1}^{\infty} \frac{(ct)^k}{k!} P^{*k}\right),$$

where the power P^{*k} is taken in the sense of convolution. By the equality $(1-e^{-x})\delta_0(dx) = 0$, we have the formula

$$(1-e^{-x}) \int_0^\infty t^{-1} e_+(tM)(dx)\, dt = \sum_{k=1}^{\infty} \frac{P^{*k}(dx)}{k}.$$

Consequently, by (1.25),

$$(SM)(dx) = (1-e^{-x})e^{-x} \sum_{k=1}^{\infty} \frac{P^{*k}(dx)}{k}.$$

PROPOSITION 1.4. *For $a \geqslant 0$, $b > 0$ and $M \in \mathcal{M}_+$ the formula*

$$S(U(a, b)M) = \Pi - T(a, b)\Pi + T(a, b)(SM)$$

is true.

Proof. From (1.12) we get

$$\langle\overline{U(a, b)M}\rangle(z+1) = \frac{(z+1)(a+b)\langle M\rangle(bz+a+b)}{(bz+a+b)\langle M\rangle(a+b)},$$

which together with (1.21) yields

$$\langle S(U(a, b)M)\rangle(z) = \log(1+z) - \log(bz+a+b) + \log(a+b) + \log\langle M\rangle(bz+a+b) - \log\langle M\rangle(a+b).$$

On the other hand, by (1.3), (1.10) and (1.21),

$$\langle T(a, b)(SM)\rangle(z) = \log\langle M\rangle(bz+a+b) - \log\langle M\rangle(a+b)$$

and $\langle T(a, b)\Pi\rangle(z) = \log(bz+a+b) - \log(a+b)$. Thus

$$\langle S(U(a, b)M)\rangle(z) = \langle\Pi\rangle(z) - \langle T(a, b)\Pi\rangle(z) + \langle T(a, b)(SM)\rangle(z),$$

which yields our assertion.

PROPOSITION 1.5. *For $M, N \in \mathcal{M}_+$ and $a = N(R_+)$ the formula*

$$S(M \circ N) = S(U(0, a)M) \circ T(1, 1)N$$

is true.

Proof. Since $\langle\bar{N}\rangle(1) = 1$. We have, by (1.19) and Lemma 1.1,

$$(T(1, 1)\overline{U(0, a)M}) \circ (T(1, 1)\bar{N}) = T(1, 1)(\overline{U(0, a)M \circ \bar{N}}) = T(1, 1)(\overline{M \circ N}).$$

Consequently, by (1.20),

$$\begin{aligned} S(M \circ N) &= \Pi \circ (T(1, 1)\overline{M \circ N}) = (\Pi \circ T(1, 1)\overline{U(0, a)M}) \circ T(1, 1)\bar{N} \\ &= S(U(0, a)M) \circ T(1, 1)\bar{N}, \end{aligned}$$

which completes the proof.

2. The sets $\mathcal{M}_\infty$ and $\mathcal{H}_\infty$. We denote by $\mathcal{M}_\infty$ the subset of $\mathcal{M}_+$ consisting of measures M fulfilling the condition $SM \leqslant \Pi$.

THEOREM 2.1. *The set $\mathcal{M}_\infty$ is invariant under the semigroup $U(a, b)$ $(a \geqslant 0, b > 0)$.*

Proof. Let $M \in \mathcal{M}_\infty$. As we have mentioned, $T(a, b)(SM) \leqslant T(a, b)\Pi$, which, by Proposition 1.4, yields the inequality $S(U(a, b)M) \leqslant \Pi$. The theorem is thus proved.

THEOREM 2.2. *The set $\mathscr{M}_\infty$ is closed under the operation* $\circ$.

Proof. Let $M, N \in \mathscr{M}_\infty$. Setting $a = N(R_+)$ we have, by Theorem 2.1, $U(0, a)M \in \mathscr{M}_\infty$ and, consequently, $S\big(U(0, a)M\big) \leqslant \Pi$. Taking into account (1.20) and Proposition 1.5 we have the inequality

$$S(M \circ N) \leqslant \Pi \circ \big(T(1, 1)N\big) = SN \leqslant \Pi,$$

which completes the proof.

Given $M \in \mathscr{M}_+$ we put

$$\gamma_M(dx) = \bar{M}(\{0\})\,\delta_0(dx) + e^{-x}\int_x^\infty (1-e^{-y})^{-1}\,\bar{M}(dy)\,dx. \tag{2.1}$$

Integrating by parts we conclude that $\gamma_M \in \mathscr{P}$. Moreover,

$$\hat{\gamma}_M(z) = \frac{\langle \bar{M} \rangle (z+1)}{z+1}. \tag{2.2}$$

It is evident that the measure γ_M and the number $M(R_+)$ determine the measure M uniquely.

PROPOSITION 2.1. *For every $M \in \mathscr{M}_+$ the equality*

$$\gamma_M * e_+(SM) = e_+(\Pi)$$

holds.

Proof. By (1.2) and (1.21) we have the equality

$$e_+(SM)^\wedge(z) = \exp\big(-\langle SM \rangle (z)\big) = \big(\langle \bar{M} \rangle (z+1)\big)^{-1}.$$

Comparing this with (1.4) and (2.2) we get

$$\hat{\gamma}_M(z)\, e_+(SM)^\wedge(z) = (1+z)^{-1} = e_+(\Pi)^\wedge(z),$$

which yields our assertion.

THEOREM 2.3. *$M \in \mathscr{M}_\infty$ if and only if the probability measure γ_M is infinitely divisible. In the affirmative case we have the formula*

$$\gamma_M = e_+(\Pi - SM). \tag{2.3}$$

Proof. First assume that $M \in \mathscr{M}_\infty$ and, consequently, $SM \leqslant \Pi$. Setting $H = \Pi - SM$ we get a measure belonging to $\mathscr{M}$ with $e_+(H)$ fulfilling the equation $e_+(H) * e_+(SM) = e_+(\Pi)$. Comparing this with Proposition 2.1 and observing that the measures on R_+ fulfil the cancellation law for convolution we conclude that $\gamma_M = e_+(H)$. Thus the measure γ_M is infinitely divisible and formula (2.3) is true.

Conversely, suppose that the measure γ_M is infinitely divisible and, consequently, $\gamma_M = e_+(H)$ for some $H \in \mathscr{M}$. Then, by Proposition 2.1, $H + SM = \Pi$, which yields the inequality $SM \leqslant \Pi$. Thus $M \in \mathscr{M}_\infty$, which completes the proof.

THEOREM 2.4. *The measures $c\delta_0$ with $c > 0$ are the only measures with a bounded support belonging to $\mathcal{M}_\infty$.*

Proof. Suppose that $M \in \mathcal{M}_\infty$ and the support of M is bounded. From (2.1) it follows immediately that the measure γ_M has also a bounded support. By Theorem 2.3, γ_M is infinitely divisible. Applying the Chatterjee–Pakshirajan Theorem from [3] we conclude that the measure γ_M is concentrated at a single point, i.e., $\gamma_M = \delta_a$ for some $a \geqslant 0$. Consequently, $\hat{\gamma}_M(z) = e^{-az}$ and, by (2.2),

$$\langle \bar{M} \rangle (z) = ze^a e^{-az}.$$

Since the derivative of the Bernstein transformation $\langle \bar{M} \rangle$ is non-negative, we infer that $a = 0$. Thus $\langle \bar{M} \rangle (z) = z$ or, equivalently, $\bar{M} = \delta_0$. Hence we get the formula $M = c\delta_0$ with $c = M(R_+)$. By (1.23) we infer that the measures of this form belong to $\mathcal{M}_\infty$, which completes the proof.

THEOREM 2.5. *Let f be a function completely monotone on* $(0, \infty)$ *which fulfils the condition*

$$0 < \int_0^\infty (1-e^{-x}) f(x)\, dx < \infty. \tag{2.4}$$

Then the measure $M(dx) = (1-e^{-x}) f(x)\, dx$ belongs to $\mathcal{M}_\infty$.

Proof. It is clear that $M \in \mathcal{M}_+$. By the Bernstein Theorem on integral representation of completely monotone functions ([2], Theorem 9.3) the function f can be written in the form

$$f(x) = \int_0^\infty e^{-xy} y(1+y) N(dy),$$

where, by (2.4), $N \in \mathcal{M}_+$. Hence

$$\langle M \rangle (z) = z \int_0^\infty \frac{1+y}{z+y} N(dy),$$

which, by (2.2), yields

$$\hat{\gamma}_M(z) = \int_0^\infty \frac{1+y}{z+1+y} \bar{N}(dy). \tag{2.5}$$

Consider the family $\lambda_a(dx) = ae^{-ax}\, dx$ $(a > 0)$ of the exponential probability distributions on R_+. Since $\hat{\lambda}_a(z) = a/(z+a)$, formula (2.5) shows that the measure γ_M is a mixture of exponential distributions

$$\gamma_M = \int_0^\infty \lambda_{1+y} \bar{N}(dy).$$

Thus, by the Steutel Theorem ([4], Chapter XIII,7), γ_M is infinitely divisible. Applying Theorem 2.3 we conclude that $M \in \mathcal{M}_\infty$, which completes the proof.

Denote by $\mathcal{H}_\infty$ the set of spectral measures for γ_M with $M \in \mathcal{M}_\infty$. By Theorem 2.3, $H \in \mathcal{H}_\infty$ if and only if $H = \Pi - SM$ for some $M \in \mathcal{M}_\infty$.

Hence we get the inequality $H \leqslant \Pi$ for $H \in \mathscr{H}_\infty$. Consequently, by (1.22), we have the inclusion

$$\mathscr{H}_\infty \subset \mathscr{M}_1. \tag{2.6}$$

THEOREM 2.6. *The set $\mathscr{H}_\infty$ is invariant under the semigroup $T(a, b)$ $(a \geqslant 0, b > 0)$.*

Proof. Suppose that $H \in \mathscr{H}_\infty$ and, consequently, $H = \Pi - SM$ for some $M \in \mathscr{M}_\infty$. Applying Theorem 2.1 we conclude that $U(a, b)M \in \mathscr{M}_\infty$. Consequently, by Theorem 2.3,

$$H_1 = \Pi - S\big(U(a, b)M\big) \in \mathscr{H}_\infty.$$

On the other hand, by Proposition 1.4, $H_1 = T(a, b)H$, which completes the proof.

A measure M from $\mathscr{M}$ is said to be *unimodal about* 0 if

$$M(dx) = c\delta_0(dx) + m(x)\,dx, \tag{2.7}$$

where $c \geqslant 0$ and the function m is non-increasing on R_+. A measure M from $\mathscr{M}$ is said to be *exponentially unimodal about* 0 if formula (2.7) is true and the function $e^x m(x)$ is non-increasing on R_+.

It is clear that measures exponentially unimodal about 0 belong to $\mathscr{M}_1$. Moreover, by (2.1), for every $M \in \mathscr{M}_+$ the measure γ_M is exponentially unimodal about 0.

LEMMA 2.1. *Suppose that $H \in \mathscr{M}$ and $e_+(H)$ is exponentially unimodal about 0. Then for every $a \in (0, 1)$ there exists a measure $M_a \in \mathscr{M}$ such that*

$$\langle M_a \rangle (z) = z \exp\big(-\langle H \rangle (z-a)\big).$$

Proof. Suppose that $H \in \mathscr{M}$ and

$$e_+(H)(dx) = c\delta_0(dx) + g(x)\,dx, \tag{2.8}$$

where $c \geqslant 0$ and the function $e^x g(x)$ is non-increasing on R_+. As we have mentioned, the measure $e_+(H)$ belongs to $\mathscr{M}_1$. Consequently, by Propositions 1.1 and 1.2, $H \in \mathscr{M}_1$ and the function $\langle H \rangle$ is analytic in the half-plane $\operatorname{Re} z > -1$. Given $a \in (0, 1)$ we put $H_a(dx) = e^{ax} H(dx)$. Of course, $H_a \in \mathscr{M}$ and

$$T(a, 1)H_a = H, \tag{2.9}$$

which, by (1.10), yields

$$\langle H \rangle (z) = \langle H_a \rangle (z+a) - \langle H_a \rangle (a)$$

for $\operatorname{Re} z > -1$. Thus

$$\langle H_a \rangle (z) = \langle H \rangle (z-a) - \langle H \rangle (-a) \tag{2.10}$$

for $z \geqslant 0$. Further, by (1.11) and (2.9),

$$e_+(H)(dx) = b_a e^{-ax} e_+(H_a)(dx)$$

with some positive constant b_a. Comparing this with (2.8) we get the formula

$$e_+(H_a)^\wedge(z) = \frac{1}{z}\int_0^z \hat{\mu}_a(y)\,dy \tag{2.11}$$

for some $\mu_a \in \mathscr{P}$. Put

$$F_a(z) = q_a \int_0^z \hat{\mu}_a(y)\,dy,$$

where $q_a = \exp(-\langle H\rangle(-a))$. The function F_a is continuous on R_+, its derivative is completely monotone on the open half-line $(0, \infty)$ and $F_a(0) = 0$. Consequently, by Theorem 9.8 in [2], the function F_a has a representation $F_a(z) = \langle M_a\rangle(z)$ with some $M_a \in \mathscr{M}_+$. On the other hand, by (2.10) and (2.11),

$$F_a(z) = q_a z e_+(H_a)^\wedge(z) = q_a z \exp(-\langle H_a\rangle(z)) = z\exp(-\langle H\rangle(z-a)),$$

which completes the proof.

THEOREM 2.7. *$H \in \mathscr{H}_\infty$ if and only if $H \in \mathscr{M}$, $e_+(H)$ is exponentially unimodal about* 0 *and*

$$\log z + \int_0^\infty e^{(1-z)x} H(dx) \to -\infty \tag{2.12}$$

as $z \to 0+$.

Proof. Necessity. Let us suppose that $H \in \mathscr{H}_\infty$ and $e_+(H) = \gamma_M$ for some $M \in \mathscr{M}_\infty$. We know that the measure γ_M is exponentially unimodal about 0 and, consequently, belongs to $\mathscr{M}_1$. Thus, by Propositions 1.1 and 1.2, the function $\langle H\rangle$ is analytic for $\operatorname{Re} z > -1$, which, by (1.1), yields the equality

$$\int_0^\infty e^{(1-z)x} H(dx) = \langle H\rangle(z) - \langle H\rangle(z-1) \tag{2.13}$$

for $z > 0$. Further, by (1.2) and (2.2),

$$\hat{\gamma}_M(z) = \exp(-\langle H\rangle(z)) = \frac{\langle \bar{M}\rangle(z+1)}{z+1}.$$

Thus $\langle H\rangle(z-1) = \log z - \log\langle \bar{M}\rangle(z)$. Comparing this with (2.13) we get the formula

$$\log z + \int_0^\infty e^{(1-z)x} H(dx) = \langle H\rangle(z) + \log\langle \bar{M}\rangle(z)$$

which together with the equality $\langle H\rangle(0) = \langle \bar{M}\rangle(0) = 0$ yields condition (2.12). This completes the proof of the necessity of conditions in question.

Sufficiency. Suppose now that $H \in \mathcal{M}$, the measure $e_+(H)$ is exponentially unimodal about 0 and condition (2.12) is fulfilled. Then, by Lemma 2.1, for every $a \in (0, 1)$ there exists a measure $M_a \in \mathcal{M}$ with the property

$$\langle M_a \rangle (z) = F_a(z) = z \exp\big(-\langle H \rangle (z-a)\big). \tag{2.14}$$

Put

$$F_1(z) = z \exp\big(-\langle H \rangle (z-1)\big) \tag{2.15}$$

for $z > 0$ and $F_1(0) = 0$. It is clear that the function F_1 is continuous on the open half-line $(0, \infty)$ and

$$\lim_{a \to 1-} F_a(z) = F_1(z) \tag{2.16}$$

for $z \geqslant 0$. First we shall prove that the function F_1 is continuous at 0. Observe that, by (1.1),

$$\log F_1(z) = \log z - \langle H \rangle (z-1) = \log z + \int_0^\infty e^{(1-z)x} H(dx) - \langle H \rangle (z)$$

for $z > 0$, which, by (2.12), yields $\log F_1(z) \to -\infty$ or, equivalently, $F_1(z) \to 0$ as $z \to 0+$. Thus the function F_1 is continuous on R_+. Now taking into account (2.14), (2.16) and applying Theorem 9.6 from [2] we infer that $F_1(z) = \langle M \rangle (z)$ for some $M \in \mathcal{M}$. Since $F_1(1) = \langle M \rangle (1) = M(R_+) = 1$, we have $\bar{M} = M$ and, by (2.2) and (2.15),

$$\hat{\gamma}_M(z) = \exp\big(-\langle H \rangle (z)\big) = e_+(H)^\wedge(z).$$

In other words, $\gamma_M = e_+(H)$, which shows that $M \in \mathcal{M}_\infty$ and $H \in \mathcal{H}_\infty$. The theorem is thus proved.

We illustrate Theorem 2.7 by a simple example.

EXAMPLE 2.1. Put $H_p = p\Pi$ for $p \geqslant 0$. By a standard calculation we have the formulae $e_+(H_0) = \delta_0$,

$$e_+(H_p)(dx) = \frac{e^{-x} x^{p-1}}{\Gamma(p)} dx \qquad \text{for } p > 0$$

and

$$\log z + \int_0^\infty e^{(1-z)x} H_p(dx) = (1-p)\log z + p \log(1+z).$$

Applying Theorem 2.7 we conclude that $H_p \in \mathcal{H}_\infty$ if and only if $p \in [0, 1)$.

THEOREM 2.8. *If $H \in \mathcal{H}_\infty$ and $\int_0^\infty (1-e^{-x})^{-2} H(dx) < \infty$, then $H = 0$ identically.*

Proof. We may assume that $e_+(H) = \gamma_M$ for some $M \in \mathcal{P}$. On the other hand, by (2.1), we have the formula

(2.17) $$\gamma_M(dx) = M(\{0\})\,\delta_0(dx) + g(x)\,dx,$$

where

(2.18) $$g(x) = e^{-x} \int_x^\infty (1-e^{-y})^{-1} M(dy).$$

Setting $N(dx) = (1-e^{-x})^{-1} H(dx)$ and $Q(dx) = (1-e^{-x})^{-2} H(dx)$, we have $N, Q \in M$ and

$$e_+(H)^\wedge(z) = \exp \int_0^\infty (e^{-zx}-1)\,N(dx),$$

which shows that $e_+(H)$ is a compound Poisson distribution

(2.19) $$e_+(H) = e^{-b}\left(\delta_0 + \sum_{k=1}^{\infty} \frac{N^{*k}}{k!}\right),$$

where $b = N(R_+)$. Moreover, by induction, we get the inequality

$$N^{*k}(dx) \leqslant (1-e^{-x})^k Q^{*k}(dx) \qquad (k = 1, 2, \ldots)$$

which, by (2.17) and (2.19), yields

$$g(x)\,dx \leqslant e^{-b} \sum_{k=1}^{\infty} \frac{(1-e^{-x})^k}{k!} Q^{*k}(dx).$$

By (2.18) the function g is non-increasing. Setting $q = Q(R_+)$, from the last inequality we get

$$g(u) \leqslant \frac{1}{u}\int_0^u g(x)\,dx \leqslant \frac{1-e^{-u}}{u} Q\big((0, u]\big) + \frac{(1-e^{-u})^2}{u} \sum_{k=2}^{\infty} \frac{q^k}{k!}$$

for $u > 0$. Letting $u \to 0+$, we obtain $g(0+) = 0$, which, by (2.18), yields $M\big((0, \infty)\big) = 0$. Since $M \in \mathscr{P}$, we have $M = \delta_0$ and, consequently, $e_+(H) = \gamma_M = \delta_0$. Hence it follows that $H = 0$ identically, which completes the proof.

3. Some functionals on stochastic processes. In the sequel $L(\xi)$ will denote the probability distribution of a random variable ξ. Let $\mathscr{X}$ be the class of non-negative stochastic processes $X = \{X(t, \omega)\colon t \geqslant 0\}$ with stationary and independent increments, continuous on the right sample functions, non-degenerate to 0, and fulfilling the initial condition $X(0, \omega) = 0$. It is well known that to every process X from $\mathscr{X}$ there corresponds a measure $M \in \mathscr{M}_+$ satisfying the condition

(3.1) $$L\big(X(t, \omega)\big) = e_+(tM)$$

or, equivalently,

(3.2) $$\hat{L}\big(X(t, \omega)\big) = \exp\big(-t\,\langle M \rangle(z)\big)$$

for $t \geqslant 0$. This uniquely determined measure M is called the *representing measure* for X.

A stochastic process X from $\mathscr{X}$ is said to be *deterministic* if $X(t, \omega) = ct$ for a positive constant c with probability 1 or, equivalently, $c\delta_0$ is the representing measure for X.

It is evident that, for any $a > 0$, $aX \in \mathscr{X}$ provided $X \in \mathscr{X}$ and, by (1.12) and (3.2),

$$\hat{L}(aX(t, \omega))(z) = \exp(-t\langle M\rangle(az)) = \exp(-t\langle U(0, a)M\rangle(z)).$$

This yields the following proposition.

PROPOSITION 3.1. *If M is the representing measure for X and $a > 0$, then $U(0, a)M$ is the representing measure for aX.*

Two processes X and Y from $\mathscr{X}$ are said to be *independent* if for all finite collections $t_1, t_2, \ldots, t_n$ and $u_1, u_2, \ldots, u_n$ of non-negative numbers the vector-valued random variables

$$(X(t_1, \omega), X(t_2, \omega), \ldots, X(t_n, \omega)) \quad \text{and} \quad (Y(u_1, \omega), Y(u_2, \omega), \ldots, Y(u_n, \omega))$$

are independent. One can easily check that for independent processes X and Y from $\mathscr{X}$ the composition $Y(X(t, \omega), \omega)$ also belongs to $\mathscr{X}$. Moreover, if M and N are the representing measures for X and Y, respectively, then

$$\hat{L}(Y(X(t, \omega), \omega))(z) = \exp(-t\langle M\rangle(\langle N\rangle(z))).$$

Comparing this with formula (1.14) we get the following proposition.

PROPOSITION 3.2. *Let X and Y be independent processes from $\mathscr{X}$ with representing measures M and N, respectively. Then $M \circ N$ is the representing measure for the composition $Y(X(t, \omega), \omega)$.*

It was proved in [8], Example 3.4, that for every process X from $\mathscr{X}$ and $u > 0$ the integral functional

$$I(u, \omega) = \int_0^\infty e^{-uX(t,\omega)}\, dt \tag{3.3}$$

is finite and positive with probability 1. The aim of this section is to study the multiplicative infinite divisibility of $L(I(u, \omega))$, i.e., the infinite divisibility of $L(\log I(u, \omega))$. By formula (3.15) in [8], all moments of $I(u, \omega)$ are finite and

$$\mathrm{E}I^n(u, \omega) = n! \prod_{k=1}^{n} \langle M\rangle(ku)^{-1}, \tag{3.4}$$

where M stands for the representing measure for X.

Given a probability measure λ on the real line $(-\infty, \infty)$, by $\tilde{\lambda}$ we denote its characteristic function, i.e.,

$$\tilde{\lambda}(s) = \int_{-\infty}^{\infty} e^{isx}\lambda(dx) \quad (-\infty < s < \infty).$$

By the Lévy–Khinchin Representation Theorem ([4], Chapter XVII,2), infinitely divisible probability measures on $(-\infty, \infty)$ are of the form $e(a, N)$, where a is a real number, N is a bounded non-negative Borel measure on $(-\infty, \infty)$, and

$$\tilde{e}(a, N)(s) = \exp\left[ias + \int_{-\infty}^{\infty}\left(e^{isx}-1-\frac{isx}{1+x^2}\right)\frac{N(dx)}{(1-e^{-|x|})^2}\right], \tag{3.5}$$

where for $x = 0$ the integrand is assumed to be $-s^2/2$.

It is easy to verify that the following statement is true.

PROPOSITION 3.3. *Suppose that the support of N is contained in R_+ and $N \in \mathcal{M}_1$. Then for every real number a the characteristic function $\tilde{e}(a, N)$ is analytic in the half-plane* $\operatorname{Im} s > -1$.

Given $N \in \mathcal{M}$ we shall use the notation

$$l(N) = \int_0^{\infty}\left(e^{-x}-1+\frac{x}{1+x^2}\right)\frac{N(dx)}{(1-e^{-x})^2}, \tag{3.6}$$

where for $x = 0$ the integrand is assumed to be $1/2$. In the sequel we shall frequently use the measure $e\big(l(\Pi), \Pi\big)$. Applying Malmsten's formula ([1], 1.9) we have

$$\tilde{e}\big(l(\Pi), \Pi\big)(s) = \exp\int_0^{\infty}\left(\frac{e^{isx}-1}{1-e^{-x}}-is\right)\frac{e^{-x}}{x}dx = \Gamma(1-is). \tag{3.7}$$

On the other hand,

$$\Gamma(1-is) = \int_{-\infty}^{\infty} e^{isx-x}\exp(-e^{-x})\,dx,$$

which yields the formula

$$e\big(l(\Pi), \Pi\big)(dx) = \exp(-x-e^{-x})\,dx \tag{3.8}$$

for $x \in (-\infty, \infty)$.

THEOREM 3.1. *Let X be a process from $\mathcal{X}$ with the representing measure M. Then the probability distribution*

$$\mu_M = L\Big(-\log\int_0^{\infty} e^{-X(t,\omega)}\,dt\Big) \tag{3.9}$$

fulfils the equation

$$\mu_M * e\big(-\log M(R_+)+l(SM), SM\big) = e\big(l(\Pi), \Pi\big). \tag{3.10}$$

Proof. Consider a triplet ξ, η and ζ of random variables with the probability distributions μ_M, $e\big(-\log M(R_+)+l(SM), SM\big)$ and $e\big(l(\Pi), \Pi\big)$, respectively. Moreover, assume that the random variables ξ and η are independent.

By (3.4) and (3.7) we have the formulae

$$\mathrm{E}e^{-n\xi} = n! \prod_{k=1}^{n} \langle M\rangle (k)^{-1} \qquad (n = 1, 2, \ldots) \tag{3.11}$$

and

$$\mathrm{E}e^{-n\zeta} = \tilde{e}\big(l(\Pi), \Pi\big)(in) = n! \qquad (n = 1, 2, \ldots). \tag{3.12}$$

Since, by (1.22), $SM \in \mathscr{M}_1$, we conclude, by Proposition 3.3, that the characteristic function of $L(\eta)$ is analytic in the half-plane $\operatorname{Im} s > -1$. Thus

$$\mathrm{E}e^{-n\eta} = \tilde{e}\big(-\log M(R_+) + l(SM), SM\big)(in) \qquad (n = 1, 2, \ldots),$$

whence, by standard calculation, we get the formula

$$\mathrm{E}e^{-n\eta} = M(R_+)^n \prod_{k=0}^{n-1} \exp \langle SM\rangle (k) \qquad (n = 1, 2, \ldots).$$

Now applying formula (1.21) we have

$$\mathrm{E}e^{-n\eta} = M(R_+)^n \prod_{k=1}^{n} \langle \bar{M}\rangle (k) = \prod_{k=1}^{n} \langle M\rangle (k). \tag{3.13}$$

Taking into account the independence of ξ and η we get from (3.11)–(3.13) the equalities

$$\mathrm{E}e^{-n(\xi+\eta)} = \mathrm{E}e^{-n\xi}\,\mathrm{E}e^{-n\eta} = \mathrm{E}e^{-n\zeta} \qquad (n = 1, 2, \ldots). \tag{3.14}$$

Observe that, by (3.8), $L(e^{-\zeta})(dx) = e^{-x}dx$ for $x \geqslant 0$. Since the characteristic function of the exponential distribution is analytic in the circle $|s| < 1$, we conclude, by (3.14), that the moments determine the probability distributions $L(e^{-(\xi+\eta)})$ and $L(e^{-\zeta})$ uniquely and

$$L(e^{-(\xi+\eta)}) = L(e^{-\zeta}).$$

Hence we get the equality $L(\xi+\eta) = L(\zeta)$ which, by the independence of ξ and η, yields $L(\xi) * L(\eta) = L(\zeta)$. This completes the proof.

Observe that the characteristic function $\tilde{e}\big(-\log M(R_+) + l(SM), SM\big)$ does not vanish on the real line. Consequently, equation (3.10) determines the probability measure μ_M uniquely. Moreover, if $\mu_M = \mu_N$, then $SM = SN$ and $M(R_+) = N(R_+)$, which, by (1.21), yields $M = N$. Thus the following corollary is true.

COROLLARY 3.1. *Let X and Y be processes from $\mathscr{X}$ with the representing measures M and N, respectively. If*

$$L\Big(\int_0^\infty e^{-X(t,\omega)}\,dt\Big) = L\Big(\int_0^\infty e^{-Y(t,\omega)}\,dt\Big),$$

then $M = N$.

In the following limit theorem we consider the weak convergence of probability distributions.

THEOREM 3.2. *Let $X_1, X_2, \ldots$ be a sequence of independent non-deterministic processes from $\mathscr{X}$ with the same representing measure $M \in \mathscr{P}$. If*

$$Y_1(t, \omega) = X_1(t, \omega) \quad \text{and} \quad Y_{n+1}(t, \omega) = X_{n+1}\big(Y_n(t, \omega), \omega\big) \quad \text{for } n = 1, 2, \ldots,$$

then

$$\lim_{n\to\infty} L\Big(\int_0^\infty e^{-Y_n(t,\omega)}\,dt\Big)(dx) = e^{-x}\,dx.$$

Proof. Denote by M_n the representing measure for the process Y_n. By Proposition 3.2 we have the formula $M_n = M^{\circ n}$, where the power is taken in the sense of operation $\circ$. Since $\langle M\rangle(1) = M(R_+) = 1$, we have, by Proposition 1.3 and formula (1.14),

$$\langle M_{n+1}\rangle(z) \leqslant \langle M_n\rangle(z) \leqslant z \qquad \text{if } z \geqslant 1,$$

$\langle M_n\rangle(1) = M_n(R_+) = 1$ and, consequently, $\langle M_n\rangle(z) \geqslant 1$ for $z \geqslant 1$. Hence it follows that for $z \geqslant 1$ the limit

$$F(z) = \lim_{n\to\infty} \langle M_n\rangle(z)$$

exists and fulfils the conditions

$$1 \leqslant F(z) \leqslant z \tag{3.15}$$

and $F(z) = \langle M\rangle\big(F(z)\big)$ for $z \geqslant 1$. Moreover, by Theorem 9.6 in [2], the function F is continuous for $z > 1$. We shall prove that the function F is constant. Suppose the contrary. Then the equation $w = \langle M\rangle(w)$ holds for w belonging to an open non-empty interval contained in $[1, \infty)$. But this yields $\langle M\rangle(z) = z$ for $z > 0$ because the function $\langle M\rangle$ is analytic in the half-plane $\operatorname{Re} z > 0$. Hence we get the formula $M = \delta_0$ which shows that the processes X_n are deterministic. This contradiction proves that the function F is constant for $z > 1$. Taking into account (3.15) we conclude that $F(z) = 1$ for $z \geqslant 1$. Consequently, by (1.21),

$$\langle SM_n\rangle(z) = \log\langle M_n\rangle(z+1) \to 0 \qquad \text{as } n \to \infty.$$

This yields the relation $SM_n \to 0$ as $n \to \infty$ which together with the equality $M_n(R_+) = 1$ and Theorem 3.1 shows the convergence

$$\mu_{M_n} \to e\big(l(\Pi), \Pi\big) \qquad \text{as } n \to \infty.$$

Now our assertion is a direct consequence of formula (3.8).

Denote by $\mathscr{X}_\infty$ the set of all stochastic processes X from $\mathscr{X}$ for which the probability distribution $L\big(-\log\int_0^\infty e^{-X(t,\omega)}\,dt\big)$ is infinitely divisible. In other words, $X \in \mathscr{X}_\infty$ if and only if the probability distribution $L\left(\int_0^\infty e^{-X(t,\omega)}\,dt\right)$ is multiplicatively infinitely divisible. We are now in a position to prove a characterization of the class $\mathscr{X}_\infty$ in terms of the representing measures.

THEOREM 3.3. *Let X be a process from $\mathscr{X}$ with the representing measure M. Then $X \in \mathscr{X}_\infty$ if and only if $M \in \mathscr{M}_\infty$. In the affirmative case the formula*

$$L\left(-\log \int_0^\infty e^{-X(t,\omega)}\,dt\right) = e\left(\log M(R_+) + l(\Pi - SM),\ \Pi - SM\right) \tag{3.16}$$

is true.

Proof. Using notation (3.9) we infer, by Theorem 3.1, that the probability distribution μ_M is infinitely divisible if and only if $SM \leqslant \Pi$. In other words, $X \in \mathscr{X}_\infty$ if and only if $M \in \mathscr{M}_\infty$. Formula (3.16) is a straightforward consequence of (3.10). The theorem is thus proved.

Theorems 3.3 and 2.3 show that the set of all spectral measures H for which

$$e(a, H) = L\left(-\log \int_0^\infty e^{-X(t,\omega)}\,dt\right) \quad \text{for some } X \in \mathscr{X}_\infty \text{ and } a \in (-\infty, \infty)$$

coincides with $\mathscr{H}_\infty$.

As an immediate consequence of Theorem 2.1 and Proposition 3.1 we get the following theorem.

THEOREM 3.4. *If $X \in \mathscr{X}_\infty$ and $a > 0$, then $aX \in \mathscr{X}_\infty$.*

This theorem yields the following rather unexpected statement.

COROLLARY 3.2. *Let us assume that $X \in \mathscr{X}$. If the probability distribution $L\left(-\log \int_0^\infty e^{-X(t,\omega)}dt\right)$ is infinitely divisible, then for every $a > 0$ the probability distribution $L\left(-\log \int_0^\infty e^{-aX(t,\omega)}dt\right)$ is also infinitely divisible.*

Similarly, from Theorem 2.2 and Proposition 3.2 we get the following result.

THEOREM 3.5. *If X and Y are independent processes from $\mathscr{X}_\infty$, then the composition $Y(X(t, \omega), \omega)$ belongs to $\mathscr{X}_\infty$.*

THEOREM 3.6. *Let X be a non-deterministic process from $\mathscr{X}_\infty$. Then the probability distribution $L\left(\int_0^\infty e^{-X(t,\omega)}dt\right)$ is absolutely continuous with respect to the Lebesgue measure on R_+.*

Proof. Denote by M the representing measure for X. Since the process in question is non-deterministic, we have $\bar{M} \neq \delta_0$, which, by (1.23), yields $SM \neq \Pi$. Moreover, by Theorem 3.3, $M \in \mathscr{M}_\infty$. Consequently, the spectral measure $H = \Pi - SM$ appearing in (3.16) does not vanish identically and, by Theorem 2.3, belongs to $\mathscr{H}_\infty$. Applying Theorem 2.8 we conclude that

$$\int_0^\infty (1 - e^{-x})^{-2}\, H(dx) = \infty,$$

which, by the Tucker Theorem on absolute continuity of infinitely divisible measures in [7] and formula (3.16), shows that the probability distribution $L\left(-\log \int_0^\infty e^{-X(t,\omega)}dt\right)$ is absolutely continuous with respect to the Lebesgue measure on the real line. This yields the assertion of the theorem.

4. Examples. The results of the preceding section may serve for the determining of the probability distribution $L\left(-\log \int_0^\infty e^{-X(t,\omega)}dt\right)$ with $X \in \mathscr{X}_\infty$. We shall illustrate this by some examples. The right-hand side of formula (3.16) is determined by $M(R_+)$ and SM. Consequently, in what follows we shall calculate these parameters only.

EXAMPLE 4.1. *Stable processes.* For stable processes from $\mathscr{X}$ we have $\langle M\rangle(z) = cz^p$ with $c > 0$ and $p \in (0, 1]$. By (1.3) and (1.21) we get $\langle M\rangle(z) = p\log(1+z) = \langle p\Pi\rangle(z)$, which yields $SM = p\Pi$. Moreover, $\langle M\rangle(1) = M(R_+) = c$. It is evident that stable processes belong to $\mathscr{X}_\infty$.

EXAMPLE 4.2. *Bessel processes.* For Bessel processes X from $\mathscr{X}$ we have

$$L(X(t, \omega))(dt) = \frac{e^{-x}t}{x} I_t(x)\,dx = e_+(tM)(dx),$$

where t is positive and

$$M(dx) = \frac{(1-e^{-x})e^{-x}}{x} I_0(x)\,dx$$

([4], Chapter XIII,7). Here I_t denotes the modified Bessel function of the first kind. Moreover, we have the formula

$$\langle M\rangle(z) = \log\left(1+z+\sqrt{(1+z)^2-1}\right)$$

which yields $M(R_+) = \langle M\rangle(1) = \log(2+\sqrt{3})$. Setting

$$f(x) = \frac{e^{-x}}{x} I_0(x)$$

we have $M(dx) = (1-e^{-x})f(x)\,dx$ and, by the integral representation

$$I_0(x) = \frac{1}{\sqrt{\pi}} \int_{-1}^{1} (1-y^2)^{-1/2} e^{-xy}\,dy,$$

the function f is completely monotone on $(0, \infty)$ and fulfils condition (2.4). Applying Theorems 2.5 and 3.3 we infer that Bessel processes belong to $\mathscr{X}_\infty$ and, by (1.25),

$$(SM)(dx) = \frac{(1-e^{-x})e^{-2x}}{x} \int_0^\infty I_t(x)\,dt\,dx.$$

In the forthcoming examples we shall use the following notation. Given $s > 0$, we denote by P_s the gamma probability distribution on R_+:

$$P_s(dx) = \frac{e^{-x}x^{s-1}}{\Gamma(s)}\,dx.$$

It is clear that $P_s * P_t = P_{s+t}$ and

$$P_s = e_+(s\Pi). \tag{4.1}$$

Given $s > 0$ we denote by E_s the Mittag–Leffler function

$$E_s(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ks+1)}.$$

In particular, for n being positive integers we have the formula

$$E_n(x^n) = \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} = \frac{1}{n}\sum_{k=0}^{n-1} \exp\big(q(k, n)x\big), \tag{4.2}$$

where $q(k, n) = \exp(2\pi ki/n)$ $(k = 0, 1, \ldots, n-1)$.

Example 4.3. *The gamma process.* The gamma process X from $\mathscr{X}$ is defined by assuming $L\big(X(t, \omega)\big) = P_t$ for $t > 0$. By (4.1) we have $M = \Pi$ and, consequently, $M(R_+) = \log 2$. Setting $f(x) = e^{-x}/x$ we infer that $M(dx) = (1-e^{-x})f(x)\,dx$, the function f is completely monotone on $(0, \infty)$ and fulfils condition (2.4). Thus, by Theorems 2.5 and 3.3, $X \in \mathscr{X}_\infty$. Using formula (1.25) we get

$$(SM)(dx) = (1-e^{-x})e^{-2x}\int_0^\infty \frac{x^{t-1}}{\Gamma(t+1)}\,dt\,dx.$$

Example 4.4. Consider a process X from $\mathscr{X}$ with the representing measure

$$M(dx) = e^{-x}x^{-1-p}(1-e^{-x})\,dx, \qquad \text{where } p \in (0, 1).$$

Setting $f(x) = e^{-x}x^{-1-p}$ we infer that the function f is completely monotone on $(0, \infty)$, $M(dx) = (1-e^{-x})f(x)\,dx$, and condition (2.4) is fulfilled. Applying Theorems 2.5 and 2.3 we conclude that $X \in \mathscr{X}_\infty$. Moreover, by standard calculation,

$$\langle M\rangle(z) = p\Gamma(1-p)\big((1+z)^p - 1\big).$$

Thus $M(R_+) = \langle M\rangle(1) = p\Gamma(1-p)(2^p-1)$. Using formula (1.21) we get the equality

$$\langle SM\rangle(z) = \log\big((z+2)^p - 1\big) - \log(2^p - 1). \tag{4.3}$$

Put

$$N_p(dx) = \frac{(1-e^{-x})e^{-2x}}{x}\,pE_p(x^p)\,dx, \tag{4.4}$$

$$Q_s(dx) = (1-e^{-x})e^{-x}p_s(dx) \text{ for } s > 0 \quad \text{and} \quad Q_0(dx) = e^{-x}\Pi(dx).$$

By standard calculation we get the formulae

$$\langle Q_0\rangle(z) = \log(z+2)-\log 2, \quad \langle Q_s\rangle(z) = 2^{-s}-(z+2)^{-s} \text{ for } s>0$$

and

$$N_p = pQ_0 + \sum_{k=1}^{\infty} \frac{Q_{kp}}{k},$$

which yield

$$\langle N_p\rangle(z) = \log\big((z+2)^p-1\big)-\log(2^p-1).$$

Comparing this with (4.3) we get the equality $SM = N_p$. Since $M \in \mathcal{M}_\infty$, we have $N_p \leqslant \Pi$, which, by (4.4), yields the inequality

$$pE_p(x^p) \leqslant e^x \tag{4.5}$$

for $x \geqslant 0$ and $p \in (0, 1)$.

EXAMPLE 4.5. *Gamma Poisson processes.* Given $s>0$ we denote by X_s a process from $\mathscr{X}$ with the representing measure $M_s(dx) = (1-e^{-x})P_s(dx)$. Then we have

$$L(X_s(t, \omega)) = e^{-t}\Big(\delta_0 + \sum_{k=1}^{\infty} \frac{t^k}{k!} P_{sk}\Big).$$

Taking into account Example 1.1 we get the formula

$$(SM_s)(dx) = (1-e^{-x})e^{-x} \sum_{k=1}^{\infty} \frac{P_{sk}(dx)}{k} = \frac{(1-e^{-x})e^{-2x}s}{x}\big(E_s(x^s)-1\big)dx.$$

As an immediate consequence of the above formula we obtain the following lemma.

LEMMA 4.1. $M_s \in \mathcal{M}_\infty$ *if and only if* $sE_s(x^s) \leqslant s+e^x$ *for* $x \geqslant 0$.

Observe that, by (4.2), $4E_4(x^4) = e^x+e^{-x}+2\cos x \leqslant 3+e^x$ for $x \geqslant 0$. Consequently, by Lemma 4.1, we have the following corollary.

COROLLARY 4.1. $M_4 \in \mathcal{M}_\infty$.

LEMMA 4.2. *If* $M_s \in \mathcal{M}_\infty$, *then* $M_r \in \mathcal{M}_\infty$ *for every* $r \in (0, s)$.

Proof. Suppose that $M_s \in \mathcal{M}_\infty$. Then, by Lemma 4.1, the inequality

$$sE_s(x^s) \leqslant s+e^x \tag{4.6}$$

is fulfilled for $x \geqslant 0$. Given $r \in (0, s)$ we put $p = r/s$. Of course, $p \in (0, 1)$. Denote by g_p the density function of the stable probability distribution ν_p on R_+ with the Laplace transformation $\hat{\nu}_p(z) = e^{-z^p}$. Zolotarev proved in [9], Theorem 2.10.3, the formulae

$$E_r(x) = \int_0^\infty E_s(xy^{-r})g_p(y)\,dy, \quad E_p(x) = \int_0^\infty \exp(xy^{-p})g_p(y)\,dy.$$

The above formulae and inequality (4.6) yield

$$rE_r(x^r) \leqslant r+p\int_0^\infty \exp(x^p y^{-p})\, g_p(y)\, dy = r+pE_p(x^p).$$

Now, by (4.5), we get the inequality $rE_r(x^r) \leqslant r+e^x$ for $x \geqslant 0$ which, by Lemma 4.1, shows that $M_r \in \mathcal{M}_\infty$. This completes the proof.

LEMMA 4.3. *If* $M_s \in \mathcal{M}_\infty$, *then* $s \in (0, 4]$.

Proof. Denote by $\mathcal{F}$ the set of all complex-valued Borel functions f on R_+ fulfilling the condition $|f(x)| \leqslant ae^{bx}$ $(x \in R_+)$ with some positive constants a and b. Let $\mathcal{B}$ be the subset of $\mathcal{F}$ consisting of bounded functions. We define the operator K on $\mathcal{F}$ by setting for $f \in \mathcal{F}$ and $x \geqslant 0$

$$(Kf)(x) = \frac{1}{\sqrt{\pi x}} \int_0^\infty f(y) \exp(-y^2/4x)\, dy$$

and $(Kf)(0) = f(0)$. It is clear that $K\mathcal{F} \subset \mathcal{F}$ and $K\mathcal{B} \subset \mathcal{B}$. We define the mapping $w \to F(w)$ from the complex plane into $\mathcal{F}$ by setting $F(w)(x) = e^{wx}$ $(x \in R_+)$. By standard calculation we get the formula

$$KF(w) - 2F(w^2) \in \mathcal{B} \quad \text{provided } \operatorname{Re} w > 0 \text{ and } \operatorname{Re} w^2 > 0$$

and $KF(w) \in \mathcal{B}$ otherwise. Hence, by induction, we get for $r = 1, 2, \ldots$

$$K^r F(w) - 2^r F(w^{2^r}) \in \mathcal{B} \quad \text{provided } \operatorname{Re} w^{2^j} > 0 \text{ for } j = 0, 1, \ldots, r \tag{4.7}$$

and $K^r F(w) \in \mathcal{B}$ otherwise. Further, we define the mapping $s \to G(s)$ for $s > 0$ by setting

$$G(s)(x) = E_s(x^s), \tag{4.8}$$

where E_s is the Mittag-Leffler function. From (4.2) it follows that $G(n) \in \mathcal{F}$ $(n = 1, 2, \ldots)$. Moreover, by the Humbert formula from [6],

$$G(n/2^r) = K^r G(n) \quad (n, r = 1, 2, \ldots),$$

which, by (4.2) and (4.7), yields

$$\frac{n}{2^r} G(n/2^r) - \sum_{k \in Z(r,n)} F\big(q(k, n)^{2^r}\big) \in \mathcal{B}, \tag{4.9}$$

where $k \in Z(r, n)$ if and only if $\operatorname{Re} q(k, n)^{2^j} > 0$ for $j = 0, 1, \ldots, r$. It is easy to check that a non-negative integer k belongs to $Z(r, n)$ if and only if either $0 \leqslant k \leqslant 2^{-r-2} n$ or $n(1-2^{-r-2}) \leqslant k \leqslant n-1$. Hence $Z(r, 2^{r+2}+1) = \{0, 1, 2^{r+2}\}$. Setting

$$n = 2^{r+2}+1, \qquad s(r) = 4+r^{-r}, \qquad u(r) = \operatorname{Re} q(1, 2^{r+2}+1)^{2^r} = \cos\frac{2\pi}{s(r)},$$

$$v(r) = \operatorname{Im} q(1, 2^{r+2}+1)^{2^r} = \sin\frac{2\pi}{s(r)},$$

we infer, by (4.8) and (4.9), that the function $s(r)E_{s(r)}(x^{s(r)}) - e^x - 2e^{u(r)x}\cos v(r)x$ is bounded on R_+. Observe that $u(r) > 0$. Consequently, the relation

$$\overline{\lim_{x\to\infty}}\left(s(r)E_{s(r)}(x^{s(r)}) - e^x\right) = \infty$$

is true. Comparing this with Lemma 4.1 we conclude that $M_{s(r)} \notin \mathcal{M}_\infty$ $(r = 1, 2, \ldots)$. Of course, $s(r) \to 4$ as $r \to \infty$, which, by Lemma 4.2, yields the assertion of Lemma 4.3.

As an immediate consequence of Corollary 4.1, Lemmas 4.2 and 4.3 and Theorem 3.3 we get the following statement.

PROPOSITION 4.1. *$X_s \in \mathscr{X}_\infty$ if and only if $s \in (0, 4]$.*

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