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ON SZASZ'S COMPACTNESS THEOREM AND APPLICATIONS TO GEOMETRIC STABILITY ON GROUPS

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Abstract. Within the rapidly developing theory of random limit theory for real-valued random variables the concepts of geometric convolution and geometric stability play a fundamental role. In several recent investigations it was pointed out that there is a one-to-one correspondence between "classical" limit theorems and stability concepts and their geometric counterparts (cf. [2], [3], [5], [11], [14]–[16]).

We are going to prove analogous results for randomized products of random variables taking values in a simply connected nilpotent Lie group G. This class of groups is natural in this setup since classical stability concepts were generalized to nilpotent groups (cf. [6] and [17]).

0. In the following, G is a locally compact second countable topological group (especially, a simply connected nilpotent Lie group), and $\mathrm{Aut}(G)$ is the group of topological automorphisms of G. Let us put

$$\boldsymbol{R}_+ = [0, \infty), \quad \boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}, \quad \boldsymbol{N} = \{1, 2, \ldots\}.$$

$\mathcal{M}^1(G)$ and $\mathcal{M}^1(\boldsymbol{R}_+)$ are the sets of all Borel probability measures on G and $\boldsymbol{R}_+$, respectively. Convergence of probability measures is always understood as weak convergence $\sigma(\mathcal{M}^1(G), C^b(G))$, where $C^b(G)$ is the space of bounded continuous complex-valued functions on G. In this case $\mathcal{M}^1(G)$, supplied with convolution product $*$, is a topological semigroup with identity ε_e, where e is the identity in G, and ε_x is the probability measure degenerated at the point $x \in G$. Let ν^n denote the n-th convolution power of $\nu \in \mathcal{M}^1(G)$, $\nu^0 := \varepsilon_e$. In the sequel, $(\mu_t)_{t \geq 0}$ usually denotes a continuous convolution semigroup (c.c.s.) in $\mathcal{M}^1(G)$, i.e.

$$\mu_t * \mu_s = \mu_{t+s}, \; t, s \geq 0; \quad \mu_t \to \varepsilon_e, \; t \to 0.$$

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If $\nu \in \mathcal{M}^1(G)$, $(\mu_t)_{t \geq 0}$ is a c.c.s., and $\xi, \varrho \in \mathcal{M}^1(\boldsymbol{R}_+)$, then

$$\nu^{\xi} := \int_0^{\infty} \nu^{[t]} \xi(dt) = \sum_{k=0}^{\infty} \nu^k \xi([k, k+1)), \qquad \mu_\varrho := \int_0^{\infty} \mu_t \varrho(dt).$$

The mappings $\xi \mapsto \nu^{\xi}$ and $\varrho \mapsto \mu_\varrho$ are continuous semigroup homomorphisms from $\mathcal{M}^1(\boldsymbol{Z}_+)$, respectively $\mathcal{M}^1(\boldsymbol{R}_+)$, into $\mathcal{M}^1(G)$ (see [7]). Moreover,

$$(\nu^{\xi_1})^{\xi_2} = \nu^{(\xi_1^{\xi_2})}, \qquad \xi_1, \xi_2 \in \mathcal{M}^1(\boldsymbol{Z}_+).$$

If τ_c: $\alpha \mapsto \alpha c$, $\alpha, c \in \boldsymbol{R}_+$, and $a \in \mathrm{Aut}(G)$, then $\tau_c(\xi)$ and $a(\nu)$ are the images of $\xi \in \mathcal{M}^1(\boldsymbol{R}_+)$ and $\nu \in \mathcal{M}^1(G)$ under τ_c and a, respectively.

If X is a random variable with values in G (respectively $\boldsymbol{R}_+$), we denote by $X(P)$ the distribution of X.

Let (X_{nk}), $n, k \geq 1$, be an array of rowwise i.i.d. G-valued random variables with distributions $X_{nk}(P) =: \nu_n \in \mathcal{M}^1(G)$, $k \geq 1$. Define $X_{n0} := e$, $n \geq 1$. Let T_n be an $\boldsymbol{R}_+$-valued random variable ("random time") independent of $(X_{nk})_{k \geq 1}$ with distribution $T_n(P) =: \xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$. Then a random product

$$Z_n := \prod_{k=0}^{T_n} X_{nk}$$

has the distribution $Z_n(P) = \nu_n^{\xi_n}$.

Let $(Y_t)_{t \geq 0}$ be a G-valued random process with stationary independent increments, the distributions of which are a c.c.s. $(\mu_t)_{t \geq 0}$. Let T be an $\boldsymbol{R}_+$-valued random variable ("random time") independent of $(Y_t)_{t \geq 0}$ with distribution $T(P) =: \varrho \in \mathcal{M}^1(\boldsymbol{R}_+)$. Then $U := Y_T$ has the distribution $Y_T(P) = \mu_\varrho$. (Cf. [7] for a survey on limit theorems for randomized products of group-valued random variables.)

Now we formulate several results which are the essential tools for the following investigations.

0.1. THEOREM (transfer theorem of Gnedenko and Fahim [4]). *Let $\nu_n \in \mathcal{M}^1(G)$, $k_n \in \boldsymbol{Z}_+$, $k_n \nearrow \infty$, and $\xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$. Assume that $\nu_n^{[k_n t]}$, the distributions of the deterministic products $\prod_{k=0}^{[k_n t]} X_{nk}$, converge to μ_t, the distributions of Y_t, $t \geq 0$. Assume further that $\tau_{1/k_n}(\xi_n)$, the distributions of normalized random times T_n/k_n, converge to ϱ, the distribution of T. Then $\nu_n^{\xi_n}$, the distributions of the randomized products $\prod_{k=0}^{T_n} X_{nk}$, converge to μ_ϱ, the distribution of Y_T.* (See [7].)

0.2. THEOREM (see Nobel [17]). *Let G be aperiodic and strongly root--compact (e.g., a simply connected nilpotent Lie group) and let $\nu_n, \mu \in \mathcal{M}^1(G)$, $k_n \in \boldsymbol{Z}_+$, $k_n \nearrow \infty$, $n \geq 1$. Assume $\nu_n^{k_n} \to \mu$. Then there exists a c.c.s. $(\mu_t)_{t \geq 0}$ satisfying $\mu_1 = \mu$ and there exists a subsequence (n') such that*

$$\nu_n^{[k_n t]} \xrightarrow[(n')]{} \mu_t \quad \textit{for every } t \geq 0.$$

0.3. COROLLARY. *Let* $\nu_n \in \mathcal{M}^1(G)$, $k_n \in \boldsymbol{Z}_+$, $k_n \nearrow \infty$, *and let* $(\nu_n^{k_n})_{n\geq 1}$ *be relatively compact. Assume* $\xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$ *and let* $\left(\tau_{1/k_n}(\xi_n)\right)_{n\geq 1}$ *be relatively compact. Then* $(\nu_n^{\xi_n})_{n\geq 1}$ *is relatively compact.*

In Section 1 we prove an inverse result to Theorem 0.1, which is due to Szasz for real-valued random variables (see [19]), namely that under natural assumptions the convergence of the distributions of the randomized products $\nu_n^{\xi_n} \to \kappa$ yields the relative compactness of the normalized random time distributions $\tau_{1/k_n}(\xi_n)$ and the existence of deterministic limits $\nu_n^{[k_n t]} \to \mu_t$, $t \geq 0$, at least for a subsequence (n').

The investigations in Sections 2 and 3 are based on Szasz's theorem and on the transfer theorem, as well as on the limit behaviour of deterministic products (see [10], [13], and [17]). In Section 2 the limit behaviour of geometric convolutions is considered. The limit of normalized geometric distributions is the exponential distribution E. Hence limit measures of geometric convolutions take the form

$$\mu_E = \int_0^\infty \mu_t e^{-t} dt.$$

In Section 3 we introduce the notions of geometric stability, semistability and the domains of attraction (cf. [14] for $G = \boldsymbol{R}^1$). We will see that these measures have representations $\kappa = \mu_E$ as exponential mixtures of stable (respectively semistable) c.c.s.

Note that we have only weak limit laws in mind, therefore throughout we consider only the distributions $\nu_n^{k_n}$, $\nu_n^{\xi_n}$, ξ_n, μ_t, μ_ϱ instead of the corresponding random variables or processes.

1. The inverse transfer theorem. For real-valued random variables an inverse transfer theorem was first proved in [19]. We prove an analogous result for simply connected nilpotent Lie groups under slightly stronger conditions on the random times.

1.1. THEOREM. *Let* G *be a simply connected nilpotent Lie group. Let* $\nu_n \in \mathcal{M}^1(G)$, *and* $\xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$. *Furthermore assume that*

(1) *there exist* $h_n \in \boldsymbol{N}$, $h_n \nearrow \infty$, *such that* $\{\tau_{1/h_n}(\xi_n)\}_{n\geq 1}$ *is relatively compact and* ε_0 *is not an accumulation point;*

(2) $\nu_n^{\xi_n} \to \kappa \in \mathcal{M}^1(G)$.

Then:

(a) $\{\nu_n^{h_n}\}_{n\geq 1}$ *is relatively compact;*

(b) *there exist a c.c.s.* $(\mu_t, t \geq 0)$ *in* $\mathcal{M}^1(G)$, $\varrho \in \mathcal{M}^1(\boldsymbol{R}_+)$ *and a subsequence* (n') *such that*

$$\nu_n^{[h_n t]} \to \mu_t,\ t \geq 0, \qquad \tau_{1/h_n}(\xi_n) \to \varrho,\ n \in (n'),$$

and hence $\kappa = \mu_\varrho$.

The proof is based on several steps which are of independent interest, and therefore are formulated in a more general setup of locally compact groups.

1.2. Definition. Let $K \subseteq G$ be compact, $\nu \in \mathcal{M}^1(G)$. The *concentration function* of ν is defined as

$$Q_K(\nu) := \sup_{x \in G} \nu(Kx).$$

Note that $m \mapsto Q_K(\nu^m)$ is non-increasing. Furthermore, a subset $\mathcal{A} \subseteq \mathcal{M}^1(G)$ is relatively shift-compact iff for every $\varepsilon > 0$ there exists a compact K_ε such that $Q_{K_\varepsilon}(\nu) > 1-\varepsilon$ for every $\nu \in \mathcal{A}$.

1.3. Proposition. *Let $\nu_n \in \mathcal{M}^1(G)$, $k_n \nearrow \infty$. Let K be a compact subset of G and $\xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$. Then*

$$Q_K(\nu_n^{\xi_n}) \leqslant \xi_n([0, k_n)) + Q_K(\nu_n^{k_n})\, \xi_n([k_n, \infty)).$$

The inequality follows immediately from the representation

$$\nu_n^{\xi_n}(Kx) = \Big(\sum_{k<k_n} + \sum_{k \geqslant k_n}\Big)\, \nu_n^k(Kx)\, \xi_n([k, k+1)).$$

1.4. Corollary. *Let $(\nu_n^{\xi_n})_{n \geqslant 1}$ be relatively shift-compact and assume that*

$$\alpha_0 := \liminf_n \xi_n([k_n, \infty)) > 0.$$

Then $(\nu_n^{k_n})_{n \geqslant 1}$ is relatively shift-compact.

Proof. Assume that

$$\inf_n Q_K(\nu_n^{k_n}) \leqslant 1-\beta_0 < 1$$

for all compact K. Then

$$\liminf_n Q_K(\nu_n^{\xi_n}) \leqslant \liminf_n \big[1-(1-Q_K(\nu_n^{k_n}))\, \xi_n([k_n, \infty))\big] \leqslant 1-\beta_0 \cdot \alpha_0$$

for all K. But this contradicts the relative shift-compactness of $(\nu_n^{\xi_n})_{n \geqslant 1}$. ∎

1.5. Proposition. *Assume* (1) *and* (2) *of Theorem* 1.1 *hold true. Then there exists some $c_0 \in (0, 1)$ such that $(\nu_n^{k_n})_{n \geqslant 1}$ is relatively shift-compact, where $k_n = [c_0 h_n]$, and* (1) *is fulfilled (if h_n is replaced by k_n, $n \geqslant 1$).*

Proof. 1. First we show that there exists $c_0 \in (0, 1)$ such that

$$\alpha_0 := \liminf_n \xi_n([c_0 h_n, \infty)) > 0.$$

Indeed, assume that $\xi_n([ch_n, \infty)) \to 0$, $n \in (n')$, as $n \to \infty$ for all $c > 0$. Then

$$\tau_{1/h_n} \xi_n([c, \infty)) \to 0 \quad \text{for all } c > 0,$$

and hence $\tau_{1/h_n} \xi_n \to \varepsilon_0$. But this contradicts (1).

2. Put $k_n = [c_0 h_n]$. By Corollary 1.4, $(\nu_n^{k_n})_{n\geq 1}$ is relatively shift-compact.

3. On the other hand,

$$\tau_{1/k_n}\xi_n = \tau_{c_n}\tau_{1/h_n}\xi_n, \qquad \text{where } c_n = h_n/[c_0 h_n] \to 1/c_0 \in (1, \infty).$$

Hence $(\tau_{1/k_n}\xi_n)_{n\geq 1}$ inherits the properties of $(\tau_{1/h_n}\xi_n)_{n\geq 1}$. ∎

In the next step we use for the first time the special structure of G.

1.6. Lemma. *Let G be a simply connected nilpotent Lie group, and $N \approx \boldsymbol{R}^1$ be a central subgroup. Let $\nu_n \in \mathcal{M}^1(G)$, $k_n \in \boldsymbol{N}$, $k_n \nearrow \infty$. Assume* (1) *(where h_n is replaced by k_n) and* (2) *of Theorem* 1.1 *hold true. Furthermore, assume the existence of $x_n \in N$ such that $(\nu_n^{k_n} * \varepsilon_{x_n})_{n\geq 1}$ is relatively compact. Then $(\nu_n^{k_n})_{n\geq 1}$ is relatively compact.*

Proof. Let $\phi\colon \boldsymbol{R} \to N$ be an (algebraic and topological) isomorphism. For $t \in \boldsymbol{R}$ and $x \in N$ define $tx := \phi(t\cdot\phi^{-1}(x))$. Hence, especially, x_n/k_n, $n \geq 1$, is well defined.

1. Since N is central, we obtain

$$\nu_n^{k_n} * \varepsilon_{x_n} = (\nu_n * \varepsilon_{x_n/k_n})^{k_n} =: \lambda_n^{k_n}.$$

Hence $(\lambda_n^{k_n})_{n\geq 1}$ is relatively compact. According to Corollary 0.3, $(\lambda_n^{\xi_n})_{n\geq 1}$ is relatively compact.

2. Let $\{X_{nk}\}$ be an array of rowwise i.i.d. random variables with distribution $X_{nk}(P) = \nu_n$. Let T_n be random times independent of the row X_{nk}, $k \geq 1$, with distributions $T_n(P) = \xi_n$. Put

$$Z_n := \prod_0^{T_n} X_{nk}, \qquad W_n := \prod_0^{T_n} (X_{nk}\cdot(x_n/k_n)) = Z_n\cdot((T_n/k_n)\, x_n).$$

Then we have $Z_n(P) = \nu_n^{\xi_n}$ and $W_n(P) = \lambda_n^{\xi_n}$, respectively.

3. The group G is topologically isomorphic to a vector space $\boldsymbol{R}^d$. Analogously, since $N \cong \boldsymbol{R}^1$, $G/N \cong \boldsymbol{R}^{d-1}$, we obtain a decomposition $G \cong \boldsymbol{R}^{d-1} \times \boldsymbol{R}^1$. For any vector $Y \in \boldsymbol{R}^d$ let $Y = (Y^{(1)}, Y^{(2)})$ be the corresponding decomposition with $Y^{(1)} \in \boldsymbol{R}^{d-1}$, $Y^{(2)} \in \boldsymbol{R}^1$. Hence, if we apply this decomposition to Z_n and W_n, we obtain

$$Z_n^{(1)} = W_n^{(1)}, \qquad W_n^{(2)} - Z_n^{(2)} = (T_n/k_n)\, x_n.$$

(Here we identify $x_n \in N$ with the ϕ^{-1} picture in $\boldsymbol{R}^1$.)

4. The sets of distributions of $Z_n^{(2)}$ and $W_n^{(2)}$ are relatively compact. Then $(x_n)_{n\geq 1}$ is also relatively compact since ε_0 is not a limit point of $\{(T_n/k_n)(P) = \tau_{1/k_n}(\xi_n)\}$. Therefore $(\nu_n^{k_n} = \lambda_n^{k_n} * \varepsilon_{x_n^{-1}})_{n\geq 1}$ is relatively compact. ∎

1.7. Corollary (cf. [19]). *Let $G = \boldsymbol{R}$. Then conditions* (1) *and* (2) *of Theorem* 1.1 *imply that $(\nu_n^{k_n})_{n\geq 1}$ is relatively compact.*

In [19] this result is proved with (1) replaced by the condition

(1') $T_n \to \infty$ stochastically.

Proof of Theorem 1.1. (a) Assume (1) for some sequence $\{h_n\}$ and (2) hold true. Choose $k_n := [c\,h_n]$ for some $c > 0$ according to Proposition 1.5. Hence especially (1) holds for k_n, and $(\nu_n^{k_n})_{n\geqslant 1}$ is relatively shift-compact.

1. We have to prove that $(\nu_n^{k_n})_{n\geqslant 1}$ is relatively compact. We prove this by induction on the dimension $d = \dim(G)$.

For $d = 1$, $G \approx \mathbf{R}^1$, see Corollary 1.7. Assume the assertion holds for $\dim(G) \leqslant d$. Let $\dim(G) = d+1$. The group G, being nilpotent and simply connected, has a central subgroup $N \approx \mathbf{R}^1$. Let $\pi\colon G \to G/N$ be the canonical projection. Since $\pi(\nu_n)$, ξ_n, and k_n fulfil the assumption of the induction hypothesis, $(\pi(\nu_n)^{k_n})_{n\geqslant 1}$ is relatively compact in $\mathcal{M}^1(G/N)$. On the other hand, $(\nu^{k_n})_{n\geqslant 1}$ is relatively shift-compact. Hence there exist $x_n \in N$ such that $(\nu_n^{k_n} * \varepsilon_{x_n})_{n\geqslant 1}$ is relatively compact. Now Lemma 1.6 is applied and yields the relative compactness of $(\nu_n^{k_n})_{n\geqslant 1}$.

2. Now we have to prove that $\{\nu_n^{h_n}\}_{n\geqslant 1}$ is relatively compact.

This is obvious, since $\nu_n^{h_n} = \nu_n^{[k_n \cdot c_n]}$, where $c_n \in \mathbf{R}_+$, $0 \leqslant c_n = h_n/k_n \leqslant [1/c]+1$. Hence (a) is proved.

(b) follows from the transfer theorem 0.1. Indeed, for any subsequence (n') there exist another subsequence (n'') and a c.c.s. $(\mu_t,\, t \geqslant 0)$ such that

$$\nu_n^{[h_n t]} \xrightarrow[(n'')]{} \mu_t,\ t \geqslant 0, \quad \text{and} \quad \tau_{1/h_n}(\xi_n) \xrightarrow[(n'')]{} \varrho.$$

(See Theorem 0.2 and (1).) According to the transfer theorem we obtain $\kappa = \mu_\varrho$. The theorem is proved. ∎

1.8. Remarks. (a) Obviously, if $P(T_n = 0) \to 0$, then (1) implies (1′).

(b) Assume conditions (1′) and (2) with $\kappa = \varepsilon_x$, $x \in G$, are satisfied. Then there exists some sequence $k_n \nearrow \infty$ such that $\nu_n^{k_n} \to \varepsilon_x$ and $(\tau_{1/k_n}(\xi_n))_{n\geqslant 1}$ is relatively compact (cf. [19]).

To see this, let K_n be compact, $K_n \downarrow \{x\}$, $\varepsilon_n \downarrow 0$, without loss of generality $\varepsilon_1^2 < 1/2$, and choose a sequence $N \mapsto n_0(N) \in \mathbf{N}$ such that

$$\varepsilon_{N+1} \leqslant \nu_n^{\xi_n}(\complement K_n) \leqslant \varepsilon_N, \quad n_0(N) \leqslant n < n_0(N+1).$$

(Note that $n_0(N) \uparrow \infty$.) Let

$$m_n := \min\{k \in \mathbf{Z}_+ \colon \xi_n([k, \infty)) \leqslant 1 - \sqrt{\varepsilon_N}\}$$

be the $(1-\sqrt{\varepsilon_N})$-quantile. We have

$$\varepsilon_N \geqslant \nu_n^{\xi_n}(\complement K_n) \geqslant \sum_{k \geqslant m_n} \nu_n^k(\complement K_n)\,\xi_n([k, k+1)).$$

Hence there exist $k_n \geqslant m_n$, $n_0(N) \leqslant n < n_0(N+1)$, such that

$$\nu_n^{k_n}(\complement K_n) \leqslant \sqrt{\varepsilon_N}\,.$$

Then $\nu_n^{k_n} \to \varepsilon_x$ as $n \to \infty$. On the other hand,

$$\tau_{1/k_n} \xi_n([0, 1]) \geqslant \tau_{1/m_n} \xi_n([0, 1]) \geqslant 1 - \sqrt{\varepsilon_N} \to 1,$$

and hence $\{\tau_{1/k_n} \xi_n\}$ is relatively compact.

(c) For $G = \boldsymbol{R}^1$ it is sufficient to assume that (1′) holds if $\kappa \neq \varepsilon_e$ (see [19]). Then (1) is satisfied. This holds true for nilpotent groups if we assume that κ is a full measure.

Indeed, if κ is full (see [9]), then there exists a projection $\pi\colon G \to \boldsymbol{R}$ with $\pi(\kappa) \neq \varepsilon_0$. Hence Szasz's proof for $G = \boldsymbol{R}$ is applicable.

(d) Assume (1′) and (2) with $\kappa \neq \varepsilon_e$ are satisfied. Assume moreover that $(\nu_n^{k_n})_{n \geqslant 1}$ and $(\tau_{1/k_n}(\xi_n))_{n \geqslant 1}$ are relatively compact. Then (1) holds true.

To see this let for some subsequence (n')

$$\nu^{[k_n t]} \xrightarrow[(n')]{} \mu_t, \ t \geqslant 0, \quad \text{and} \quad \tau_{1/k_n} \xi_n \xrightarrow[(n')]{} \varrho.$$

Then, again by the transfer theorem, we have $\kappa = \mu_\varrho \neq \varepsilon_e$ by assumption. Hence we have proved that $\varrho \neq \varepsilon_0$ for any accumulation point of $(\tau_{1/k_n} \xi_n)$.

2. Geometric convolutions. Following the development for real-valued random variables (cf. [2], [3], [5], [14], and [15]) we assume now that the random times T_n have geometric distributions. We start with definitions and more or less well-known arithmetic properties of geometric and exponential distributions, which can easily be checked using generating functions.

2.1. Definitions. Let $0 < p < 1$, $q := 1 - p$, $\alpha > 0$. Define the *geometric distributions* as

$$\xi(p) := p \sum_{k=1}^{\infty} q^{k-1} \varepsilon_k, \quad \eta(p) := p \sum_{k=0}^{\infty} q^k \varepsilon_k.$$

Furthermore we define the Poisson distribution

$$\pi_\alpha := \sum_{k=0}^{\infty} e^{-\alpha} \frac{\alpha^k}{k!} \varepsilon_k,$$

and the exponential distribution $E = E_1$ with parameter 1.

2.2. Lemma. (a) $\xi(p) = \varepsilon_1 * \eta(p)$, $p \in (0, 1)$.

(b) $\xi(p_1)^{\xi(p_2)} = \xi(p_1 p_2)$, $p_1, p_2 \in (0, 1)$.

(c) $\eta(p_1)^{\xi(p_2)} = \eta(p)$, *where* $p, p_1, p_2 \in (0, 1)$ *are related by*

$$p = p_1 p_2 / (1 - (1 - p_2) p_1) \quad \text{or} \quad p_1 = p / (p + p_2 (1 - p)).$$

(d) $\tau_p(E)^{\xi(p)} = E$, $p \in (0, 1)$.

(e) $(\pi_\alpha)_E = \eta(p)$, $\alpha > 0$, $p \in (0, 1)$, $p = 1/(1 + \alpha)$ *or* $\alpha = q/p$.

2.3. Corollary. *Let G be a locally compact group, $\nu \in \mathcal{M}^1(G)$. Then*

(a) $\nu^{\xi(p)} = \nu * \nu^{\eta(p)} = \nu^{\eta(p)} * \nu$;

(b) $\nu^{\eta(p)} = \lambda_E = \int_0^\infty \lambda_t e^{-t}\, dt$, *where*

$$\lambda_t = \exp\left(t\frac{q}{p}(\nu - \varepsilon_e)\right),$$

i.e. generalized geometric distributions $\nu^{\eta(p)}$ *are representable as exponential mixtures of compound Poisson distributions, and conversely*:

(c) *let* $\lambda_t = \exp(t\alpha(\nu - \varepsilon_e))$, $t \geqslant 0$, $\alpha > 0$; *then* $\lambda_E = \nu^{\eta(p)}$, *where* $p = 1/(1+\alpha)$.

2.4. Lemma. (a) $\tau_p(\xi(p)) \to E$ *as* $p \to 0$.

(b) $\tau_p(\eta(p)) \to E$ *as* $p \to 0$.

(c) *Therefore, for any sequence* $p_n \downarrow 0$, $k_n \in \mathbf{Z}_+$, $k_n \uparrow \infty$, *with* $k_n p_n \to 1$ *as* $n \to \infty$, *and* $\xi_n := \xi(p_n)$ *(respectively,* $\xi_n := \eta(p_n)$*), condition* (1) *from Theorem* 1.1 *is fulfilled.*

2.5. Lemma. *Let G be a locally compact group,* $(\mu_t, t \geqslant 0)$ *and* $(\lambda_t, t \geqslant 0)$ *be c.c.s. in* $\mathcal{M}^1(G)$. *Assume* $\mu_E = \lambda_E$. *Then* $\mu_t = \lambda_t$, $t \geqslant 0$.

Proof. For $\sigma \in \mathcal{M}^1(G)$ let $T_\sigma: f \mapsto \sigma * f$ be the convolution operator on the Banach space $C_0(G)$. Then $(T_{\mu_t}, t \geqslant 0)$ and $(T_{\lambda_t}, t \geqslant 0)$ are C_0-contraction semigroups, the resolvents of which coincide:

$$T_{\mu_E} = \int_0^\infty T_{\mu_t} e^{-t}\, dt = \int_0^\infty T_{\lambda_t} e^{-t}\, dt = T_{\lambda_E}.$$

Let M and L be the generators of (T_{μ_t}) and (T_{λ_t}). Then equality of resolvents $(M-I)^{-1} = (L-I)^{-1}$ implies $M = L$, and therefore $T_{\mu_t} = T_{\lambda_t}$, $t \geqslant 0$. ∎

2.6. Proposition. *Let G be a simply connected nilpotent Lie group. Let* $\nu_n, \kappa \in \mathcal{M}^1(G)$, $p_n \downarrow 0$. *Then* (i) $\nu_n^{\xi(p_n)} \to \kappa$ *iff* (ii) $\nu_n^{\eta(p_n)} \to \kappa$, *and in this case* $\nu_n \to \varepsilon_e$ *(infinitesimality).*

Proof. Choose $k_n \uparrow \infty$ such that $k_n p_n \to 1$. Then according to Lemma 2.4 (c) and Theorem 1.1 each of the conditions (i) and (ii) implies that $(\nu_n^{k_n})$ is relatively compact. Hence infinitesimality follows from Proposition 1 of [17]. Now Corollary 2.3 (a) shows the equivalence of (i) and (ii). ∎

We give by analogy with the case of real-valued random variables (cf. [14]) the following

2.7. Definition. $\kappa \in \mathcal{M}^1(G)$ is called *geometrically infinitely divisible* if for any $p \in (0, 1)$ there exists $\kappa_p \in \mathcal{M}^1(G)$ such that $(\kappa_p)^{\xi(p)} = \kappa$. (See [14] for $G = \mathbf{R}^1$.)

The "compound geometric" distributions $\nu^{\eta(p)}$ play the role of compound Poisson distributions within the class of infinitely divisible distributions.

2.8. Proposition. *Let* $\nu \in \mathcal{M}^1(G)$, $p \in (0, 1)$. *Then* $\kappa := \nu^{\eta(p)}$ *is geometrically infinitely divisible.*

The proposition follows immediately from Lemma 2.2 (c). Indeed, let $p_2 \in (0, 1)$, $p_1 = p/(p + p_2(1-p))$, and put $\kappa_{p_2} := \nu^{\eta(p_1)}$. Then $(\kappa_{p_2})^{\xi(p_2)} = \kappa$.

More generally, we have

2.9. PROPOSITION. *Let $(\lambda_t)_{t\geqslant 0}$ be a c.c.s. in $\mathcal{M}^1(G)$. Then the exponential mixture $\kappa := \lambda_E$ is geometrically infinitely divisible.*

Indeed, applying Lemma 2.2 (d) we have $(\tau_p(E))^{\xi(p)} = E$. Hence for

$$\kappa_p := \lambda_{(\tau_p(E))} = \int_0^\infty \lambda_{t/p} e^{-t} dt$$

we obtain

$$\kappa_p^{\xi(p)} = \lambda_E = \kappa.$$

The following theorem will enable us to describe completely the structure of geometrically infinitely divisible laws on simply connected nilpotent Lie groups.

2.10. THEOREM. *Let $p_n \downarrow 0$, $k_n \in \mathbf{Z}_+$, $k_n \uparrow \infty$, and $k_n p_n \to 1$. Let ν_n, $\kappa \in \mathcal{M}^1(G)$. Then the following assertions are equivalent:*

(i) $\nu_n^{\xi(p_n)} \to \kappa$;

(ii) $\nu_n^{\eta(p_n)} \to \kappa$;

(iii) $\nu_n^{[k_n t]} \to \mu_t$, $t \geqslant 0$, *a c.c.s., and* $\kappa = \mu_E$.

Proof. For (i)$\Leftrightarrow$(ii) see Proposition 2.6; (iii)$\Rightarrow$(i) follows from Theorem 0.1 and Lemma 2.4 (a). To prove (ii)$\Rightarrow$(iii) we consider the representation

$$\nu_n^{\eta(p_n)} = \int_0^\infty \exp(tB_n) e^{-t} dt,$$

where B_n is the Poisson generator $B_n := q_n p_n^{-1}(\nu_n - \varepsilon_e)$ (cf. Corollary 2.3 (b)). Let A_n be the Poisson generator $A_n := k_n(\nu_n - \varepsilon_e)$. By the choice of k_n, obviously, $\exp(tB_n) \to \mu_t$, $t \geqslant 0$, iff $\exp(tA_n) \to \mu_t$, $t \geqslant 0$ (where (μ_t) is a c.c.s.). Furthermore, we have $\nu_n^{[k_n t]} \to \mu_t$, $t \geqslant 0$, iff $\exp(tA_n) \to \mu_t$, $t \geqslant 0$ (cf., e.g., [17], Remark 2; [12], Section IX, § 2; see also [18]).

Now we apply Theorem 1.1 to $(\nu_n^{\eta(p_n)})_{n\geqslant 1}$. Note that $\tau_{1/k_n}(\eta(p_n)) \to E$ implies (1). Hence $(\nu_n^{k_n})_{n\geqslant 1}$ is relatively compact. Let (n') be any subsequence of N. There exist a subsequence $(n'') \subset (n')$ and a c.c.s. $(\mu_t, t \geqslant 0)$ such that

$$\nu_n^{[k_n t]} \xrightarrow[(n'')]{} \mu_t, \qquad t \geqslant 0,$$

and $\kappa = \mu_E$. Therefore, by the considerations above, we obtain the convergence of the resolvents of convolution semigroups

$$\nu_n^{\eta(p_n)} = \int_0^\infty \exp(tB_n) e^{-t} dt \xrightarrow[(n'')]{} \int_0^\infty \mu_t e^{-t} dt = \kappa.$$

Moreover, according to Lemma 2.5, (μ_t) is uniquely determined. Hence we have

$$\int_0^\infty \exp(tB_n) e^{-t} dt \xrightarrow[n\to\infty]{} \int_0^\infty \mu_t e^{-t} dt.$$

But the convergence of resolvents is equivalent to the convergence of convolution semigroups (see [12], [1], [8]). Hence we obtain

$$\exp(tB_n) \to \mu_t, \qquad t \geqslant 0.$$

The theorem is proved. ∎

2.11. Remark. The assertions (i)–(iii) of Theorem 2.10 are further equivalent to

(iv) $\nu_n^{\xi_n} \to \kappa$, where $\xi_n \in \mathcal{M}^1(\boldsymbol{R}_+)$ fulfil the "geometric law of large numbers" $\tau_{p_n}(\xi_n) \to E$ (equivalently, $\tau_{1/k_n}(\xi_n) \to E$).

Indeed, (iii)$\Rightarrow$(iv) follows from Theorem 0.1. Assume (iv) holds. By Theorem 1.1, $(\nu_n^{k_n})_{n \geqslant 1}$ is relatively compact. Hence, for any $T > 0$, $\{\nu_n^{[k_n t]}, 0 \leqslant t \leqslant T, n \geqslant 1\}$ is uniformly tight. The assertion follows since $\tau_{1/k_n}(\eta(p_n) - \xi_n) \to 0$, $n \to \infty$.

Now we are ready to characterize completely the set of geometrically infinitely divisible distributions on simply connected nilpotent Lie groups.

2.12. Theorem. *The following assertions are equivalent for* $\kappa \in \mathcal{M}^1(G)$:

(i) κ *is geometrically infinitely divisible*;

(ii) *there exist sequences* $p_n \downarrow 0$, $\nu_n \in \mathcal{M}^1(G)$ *such that* $\nu_n^{\xi(p_n)} = \kappa$;

(iii) *there exist sequences* $p_n \downarrow 0$, $\nu_n \in \mathcal{M}^1(G)$ *such that* $\nu_n^{\xi(p_n)} \to \kappa$;

(iv) *there exist sequences* $p_n \downarrow 0$, $\nu_n \in \mathcal{M}^1(G)$ *such that* $\nu_n^{\eta(p_n)} \to \kappa$;

(v) *there exists a c.c.s* $(\mu_t)_{t \geqslant 0}$ *in* $\mathcal{M}^1(G)$ *such that* $\kappa = \mu_E$.

Obviously, (i) $\Rightarrow$(ii)$\Rightarrow$(iii). The equivalence (iii)$\Leftrightarrow$(iv) holds according to Proposition 2.6. (iv)$\Rightarrow$(v) by Theorem 2.10, and (v)$\Rightarrow$(i) by Proposition 2.9.

2.13. Corollary. *The set* $\mathcal{G} := \{\kappa \in \mathcal{M}^1(G)$: κ *is geometrically infinitely divisible*$\}$ *is closed in* $\mathcal{M}^1(G)$.

The corollary follows immediately from the equivalence (i)$\Leftrightarrow$(v) in 2.12 and from the closedness of the set of exponential mixtures of c.c.s.

3. Geometrically stable and semistable measures. In this section we apply the previous considerations in order to obtain a complete description of geometrically (semi-) stable distributions. These concepts were introduced for real-valued random variables in [14].

Note that the underlying groups are in general non-Abelian, therefore throughout our concepts are generalizations of strict (semi-) stability.

3.1. Definition. $\kappa \in \mathcal{M}^1(G)$ is called *geometrically* ((a, p)-) *semistable* if for some $a \in \operatorname{Aut}(G)$, $p \in (0, 1)$, the relation

$$a(\kappa^{\xi(p)}) = \kappa$$

holds.

κ is *geometrically stable* if for any $p \in (0, 1)$ there exists $a_p \in \mathrm{Aut}(G)$ such that

$$a_p(\kappa^{\xi(p)}) = \kappa$$

holds.

If $T = (b_t, t > 0) \subseteq \mathrm{Aut}(G)$ is a continuous one-parameter group, we call κ *geometrically T-stable* if we can choose $a_p \in T$, $p \in (0, 1)$.

Stability concepts in the usual sense (cf. [17]) are closely related to domains of attraction. Hence we have

3.2. DEFINITION. Let $\nu, \kappa \in \mathcal{M}^1(G)$. ν belongs to the *geometric domain of partial attraction* of the measure κ if there exist $p_n \in (0, 1)$, $p_n \downarrow 0$, $a_n \in \mathrm{Aut}(G)$ such that

$$a_n(\nu^{\xi(p_n)}) \to \kappa.$$

ν is in the *domain of geometric p-semistable attraction* of κ, denoted by $\nu \in \mathrm{DGSSA}(\kappa, p)$, if ν is in the geometric domain of partial attraction and $p_{n+1}/p_n \to p \in (0, 1]$.

ν is in the *domain of geometric stable attraction* of κ, denoted by $\nu \in \mathrm{DGSA}(\kappa)$, if there exist $a_p \in \mathrm{Aut}(G)$, $p \in (0, 1)$, such that

$$a_p(\nu^{\xi(p)}) \to \kappa, \qquad p \to 0.$$

(Obviously, $\mathrm{DGSA}(\kappa) \subset \bigcap_{0<p<1} \mathrm{DGSSA}(\kappa, p)$.)

Recall that a c.c.s. (μ_t) is called *(a, c)-semistable* for $a \in \mathrm{Aut}(G)$, $c \in (0, 1)$, if $a(\mu_t) = \mu_{ct}$, $t \geq 0$. Let $T = (a_t)_{t>0}$ be a continuous one-parameter group of automorphisms of G. Then (μ_t) is called *T-stable* if $a_t(\mu_s) = \mu_{ts}$, $t > 0$, $s \geq 0$.

We say that ν belongs to the *domain of partial attraction* of (μ_t) if

(A) there exist sequences $a_n \in \mathrm{Aut}(G)$, $k_n \in \mathbf{Z}_+$, $k_n \nearrow \infty$, such that $a_n \nu^{[k_n t]} \to \mu_t$, $t \geq 0$.

ν belongs to the *domain of semistable attraction* of (μ_t) if (A) is satisfied with $k_n/k_{n+1} \to c \in (0, 1]$.

ν belongs to the *domain of stable attraction* of (μ_t) if (A) is satisfied with $k_n/k_{n+1} \to 1$. (Cf. [17] and [6].)

3.3. PROPOSITION. *Let $\nu, \kappa \in \mathcal{M}^1(G)$, $(\mu_t)_{t \geq 0}$ be a c.c.s. Furthermore, let $p_n \downarrow 0$, $k_n \in \mathbf{Z}_+$, $k_n p_n \to 1$, $a_n \in \mathrm{Aut}(G)$.*

(a) *Assume that ν belongs to the domain of partial attraction of (μ_t), i.e. $a_n(\nu^{[k_n t]}) \to \mu_t$, $t \geq 0$. Then ν belongs to the geometric domain of partial attraction of $\kappa := \mu_E$, precisely*

$$a_n(\nu^{\xi(p_n)}) \to \kappa.$$

(b) *Conversely, let ν belong to the geometric domain of partial attraction of κ, i.e. $a_n(\nu^{\xi(p_n)}) \to \kappa$. Then there exists a c.c.s. (μ_t) such that $a_n(\nu^{[k_n t]}) \to \mu_t$, $t \geq 0$, and $\kappa = \mu_E$.*

The proposition is a reformulation of Theorem 2.10 with $\nu_n := a_n(\nu)$.

3.4. Corollary. *Let κ be geometrically (a, p)-semistable (respectively, geometrically stable with respect to $T = (a_t) \subseteq$ Aut (G)). Then there exists an (a, p)-semistable (respectively, a T-stable) c.c.s. with $\kappa = \mu_E$.*

Proof. The existence of $(\mu_t = \lim_n a^n(\kappa^{[p^{-n}t]}))_{t \geqslant 0}$ follows from Proposition 3.3 since by assumption $a(\kappa^{\xi(p)}) = \kappa$. Hence $a^n(\kappa^{\xi(p^n)}) = \kappa$. It is easily seen then that $a(\mu_t) = \mu_{pt}$, $t \geqslant 0$. The proof for the stable case is analogous. ∎

3.5. Corollary. *Conversely, let the c.c.s. $(\mu_t)_{t \geqslant 0}$ be (a, p)-semistable (respectively, T-stable). Then $\mu_E = \kappa$ is geometrically (a, p)-semistable (respectively, T-stable).*

Proof. We use Lemma 2.2 (d): $\tau_p(E)^{\xi(p)} = E$. Let $\lambda_t := a^{-1}(\mu_t) = \mu_{t/p}$, $t \geqslant 0$. Then

$$(\lambda_E)^{\xi(p)} = (\mu_{\tau_p(E)})^{\xi(p)} = \mu_{(\tau_p(E))^{\xi(p)}} = \mu_E = \kappa.$$

The proof for the stable case is analogous. ∎

3.6. Corollary. *ν belongs to the geometric domain of semistable (respectively, stable) attraction of $\kappa \in \mathcal{M}^1(G)$ iff there exists a c.c.s. $(\mu_t)_{t \geqslant 0}$ with $\kappa = \mu_E$ such that ν belongs to the domain of semistable (respectively, stable) attraction of $(\mu_t)_{t \geqslant 0}$.*

Note that, in Corollary 3.6, $(\mu_t)_{t \geqslant 0}$ need not be semistable (respectively, stable). To prove sharper results we need by analogy with the classical situation the notion of full measures in order to obtain the convergence-of-types-theorem. Recall that a probability measure $\lambda \in \mathcal{M}^1(G)$ is called *full* if it is not concentrated on a proper closed connected normal subgroup of G (cf. [9] and [17]). Obviously, an exponential mixture $\kappa = \mu_E$ is full iff μ_t is full, $t > 0$. Hence we obtain

3.7. Proposition. *Let κ be full. Then ν belongs to the domain of geometric semistable (respectively, stable) attraction of κ iff there exists a semistable (respectively, stable) c.c.s. $(\mu_t)_{t \geqslant 0}$ with $\kappa = \mu_E$, and ν is in the domain of semistable (respectively, stable) attraction of $(\mu_t)_{t \geqslant 0}$.*

Indeed, according to Corollary 3.6 we have to show that for full κ the convolution semigroup $(\mu_t)_{t \geqslant 0}$ in 3.6 is semistable (respectively, stable). But $(\mu_t)_{t \geqslant 0}$ is full and the domain of (semi-) stable attraction is nonempty. The assertion follows from Corollary 4 in [17].

As an immediate consequence we obtain

3.8. Corollary. *Let κ be geometrically semistable (respectively, stable). Then the domain of geometric semistable (respectively, stable) attraction is nonempty. Conversely, let κ be full and assume the domain of geometric semistable (respectively, stable) attraction is nonempty. Then κ is geometrically semistable (respectively, stable).*

Proof. The first assertion is obvious, since we have $a^n(\kappa^{\xi(p^n)}) = \kappa$, $n \geq 1$, and hence $\kappa \in \mathrm{DGSSA}(\kappa, p)$ (respectively, $\kappa \in \mathrm{DGSA}(\kappa)$). Conversely, let κ be full and $\nu \in \mathrm{DGSSA}(\kappa, p)$. Then, according to Propositions 3.3 and 3.7 there exists a semistable c.c.s. $(\mu_t)_{t \geq 0}$ with $\kappa = \mu_E$. Hence Corollary 3.5 yields that κ is geometrically semistable (respectively, stable). ■

In the group case we have to distinguish between domains of attraction of convolution semigroups ("functional attraction" [10]), i.e., $a_n(\nu^{[k_n t]}) \to \mu_t$, $t \geq 0$, and domains of attraction of a single measure, i.e., $a_n(\nu^{k_n}) \to \mu$. For full measures we can improve Proposition 3.7 in the following way:

3.9. PROPOSITION. *Let μ be a full measure in $\mathcal{M}^1(G)$. Let ν belong to the domain of semistable attraction of μ, i.e. for $a_n \in \mathrm{Aut}(G)$, $k_n \in N$, $k_n \nearrow \infty$, $k_n/k_{n+1} \to c \in (0, 1]$, we have $a_n(\nu^{k_n}) \to \mu$. Let $p_n \in (0, 1)$, $k_n p_n \to 1$. Then there exist $b_n \in I(\mu) := \{\alpha \in \mathrm{Aut}(G): \alpha(\mu) = \mu\}$ such that $b_n a_n(\nu^{\xi(p_n)}) \to \kappa$ and κ is geometrically semistable, i.e. ν belongs to the domain of geometrically semistable attraction of κ.*

Furthermore, if $c = 1$, then μ is stable and in this case we can choose $b_n = \mathrm{id}$, $n \geq 1$.

Proof. μ is embeddable into a semistable c.c.s. $(\mu_t)_{t \geq 0}$ such that $\mu_1 = \mu$ and $b_n a_n(\nu^{[k_n t]}) \to \mu_t$, $t \geq 0$ ([10], Theorem 4.2). Therefore, $b_n a_n(\nu^{\xi(p_n)}) \to \kappa = \mu_E$ and the assertion follows from Proposition 3.7. If $c = 1$, μ is stable, and then $(\mu_t)_{t \geq 0}$ is uniquely determined by $\mu_1 = \mu$ ([17], Proposition 6). Hence we can choose $b_n = \mathrm{id}$ in this case ([10], Corollary 4.2). ■

Semistable (respectively, stable) c.c.s. $(\mu_t)_{t \geq 0}$ on nilpotent simply connected Lie groups correspond in a one-to-one way to operator semistable (respectively, operator stable) c.c.s. $(\mathring{\gamma}_t)_{t \geq 0}$ on the tangent space $\mathcal{G}$ (see [10]). Now we are ready to show that this holds true for geometric (semi-) stability.

Let $\mathcal{G} \approx \mathbf{R}^d$ be the Lie algebra of G. We use the notations introduced in [10] to denote the correspondence between c.c.s. $(\mu_t)_{t \geq 0}$ on G and $(\mathring{\gamma}_t)_{t \geq 0}$ on $\mathcal{G}$ via the generating distributions.

3.10. PROPOSITION. *Let $\kappa \in \mathcal{M}^1(G)$ be geometrically semistable (respectively, stable). Then $\kappa = \mu_E$, where $(\mu_t)_{t \geq 0}$ is a semistable (respectively, stable) c.c.s. Let $(\mathring{\gamma}_t)$ be the corresponding operator semistable (respectively, stable) c.c.s. on the vector space $\mathcal{G}$ and define $\mathring{\sigma} := \mathring{\gamma}_E$. Then $\mathring{\sigma}$ is geometrically semistable (respectively, stable) on the vector space $\mathcal{G}$. Conversely, to any geometrically semistable (respectively, stable) $\mathring{\sigma}$ on $\mathcal{G}$ the corresponding geometrically semistable (respectively, stable) κ on G is uniquely determined.*

The assertion follows easily from Theorems 0.1 and 2.10 (see also [7]).

Note added in proof. The investigations in geometric divisibility and semistability are continued by the first-named author in: *On geometric convolutions of distributions of group-valued random variables*, in: *Probability Measures on Groups and Related Structures. XI* (Proc. Oberwolfach 1994), World Scientific, 1995, pp. 167–181.

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