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INFINITE SEQUENCES WITH SIGN-SYMMETRIC LIOUVILLE DISTRIBUTIONS

BY

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Abstract. We define and study the sign-symmetric Dirichlet-type and Liouville-type distributions on n-dimensional space. We give a complete stochastic representation for infinite sequences of random variables $\{X_n: n \in N\}$ with the property that for every $n \in N$ the random vector $(X_1, \ldots, X_n)$ has a sign-symmetric Liouville-type distribution.

1. Introduction. In this paper we study a family of multivariate distributions which contains as special cases many well-known classes of distributions, including the spherically symmetric distributions [3]. The distributions treated here may be obtained by the following construction. Let $Z_1, \ldots, Z_n$ be mutually independent, real-valued, random variables, where the probability density function of Z_i is

$$(1.1) \qquad \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |z_i|^{\beta_i - 1} \exp(-|z_i|^{\alpha_i})$$

for $z_i \in \boldsymbol{R}$ and positive parameters α_i and β_i, $i = 1, \ldots, n$. Further, define

$$(1.2) \qquad U_i = \frac{Z_i}{(\sum_{j=1}^{n} |Z_j|^{\alpha_j})^{1/\alpha_i}}$$

for $i = 1, \ldots, n$. This construction is similar to a well-known procedure for constructing the Dirichlet distributions using gamma random variables [11]. The distribution of the random vector $(U_1, \ldots, U_n)$ is called the *sign-symmetric Dirichlet-type distribution.*

We shall also study the distribution of the random vector

$$(1.3) \qquad (X_1, \ldots, X_n) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_n \Theta^{1/\alpha_n}),$$

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where Θ is a positive random variable independently distributed of $(U_1, \ldots, U_n)$. We call the distribution of the random vector $(X_1, \ldots, X_n)$ the *sign-symmetric Liouville-type distribution.* This construction is similar to a well-known procedure for constructing the multivariate Liouville distributions using Dirichlet random variables (cf. [5], [7]–[10], [18]).

When $\alpha_i \equiv 2$, the distribution of $(X_1, \ldots, X_n)$ has been considered in [4]; in addition, when $\beta_i \equiv 1$, this class reduces further to the spherically symmetric (or rotationally invariant) distributions, a family that is widely utilized in probability and statistics both in finite-dimensional settings (cf. [3], [5], [12], [17]) and infinite-dimensional applications (cf. [13], [14], [17]).

When $\alpha_i \equiv \alpha$ and $\beta_i \equiv 1$, the distribution of $(X_1, \ldots, X_n)$ reduces to the l_α-isotropic distributions (cf. [1], [2], [15]), so-called because they have the nice property that the level curves of their densities are spheres in the l_α-norm on $\boldsymbol{R}^n$. It has been shown recently [16] that, within Bayesian analysis, the l_α-isotropic distributions satisfy certain robustness properties known to hold for independent, identically distributed data.

To explain our choice of terminology, note that the distribution of the random vector $(X_1, \ldots, X_n)$ is sign-symmetric if it has the same distribution as $(r_1 X_1, \ldots, r_n X_n)$, where $r_1, \ldots, r_n$ is the Rademacher sequence of independent (and independent of $(X_1, \ldots, X_n)$) random variables with $P(r_i = 1) = P(r_i = -1) = 1/2$. We use the terms "sign-symmetric Dirichlet-type distribution" and "sign-symmetric Liouville-type distribution" to underline that our distributions differ from the Dirichlet and Liouville distributions not only by their sign-symmetry but that they also differ by the shape of their supports.

In Section 2 we derive certain preliminary results for the sign-symmetric Dirichlet-type distribution. In Section 3 we give some basic properties of the sign-symmetric Liouville-type distribution, including marginal and conditional distributions, and formulas for moments and correlations. We also deduce the extent to which a sign-symmetric Liouville-type random vector $(X_1, \ldots, X_n)$ has a uniqe representation of the form (1.3) for some sign-symmetric Dirichlet-type random vector $(U_1, \ldots, U_n)$ and positive random variable Θ.

Section 4 contains the main results of this paper. We prove that an infinite sequence of sign-symmetric Dirichlet-type random variables exists if and only if $\sum_{i=1}^{\infty} \beta_i/\alpha_i < \infty$, and in that case we construct the sequence. In both cases, $\sum_{i=1}^{\infty} \beta_i/\alpha_i < \infty$ or $\sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty$, we derive a stochastic representation for infinite sequences of sign-symmetric Liouville-type random variables. It turns out that in the former case, the sequence of sign-symmetric Liouville-type random variables is a scale mixture of a related sign-symmetric Dirichlet-type sequence; while in the latter case, it is a scale mixture of a sequence of independent random variables each having density (1.1). This situation is completely unlike that of the exchangeable l_α-isotropic sequences [1], [2], [15]; in that context it is not possible to obtain more than one type of stochastic representation for exchangeable l_α-isotropic sequences of random variables, for then we have $\sum_{i=1}^{n} \beta_i/\alpha_i \to \infty$ as $n \to \infty$.

Finally, it should be noted that all the techniques used in this paper also apply to the corresponding distributions on the positive orthant; that is, the distributions of the random vectors $(U_1, \ldots, U_n)$ defined as in (1.2) but where the random variables Z_i have densities

$$\frac{\alpha_i}{\Gamma(\beta_i/\alpha_i)} z_i^{\beta_i - 1} \exp(-z_i^{\alpha_i})$$

for $z_i > 0$, $i = 1, \ldots, n$. Therefore all our results have natural analogues for the classical Dirichlet and Liouville random vectors as well as for the distributions considered in [19].

2. The sign-symmetric Dirichlet-type distributions. Throughout $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are positive parameters. We use the notation $p_i = \sum_{j=1}^{i} \beta_j/\alpha_j$ for $i = 1, \ldots, n$. We also let $(a)_+ = a$ or 0 according as $a > 0$ or $a \leqslant 0$, respectively.

2.1. LEMMA. *We have*

$$\int_{R^{n-1}} \ldots \int \Big(1 - \sum_{i=1}^{n-1} |u_i|^{\alpha_i}\Big)_+^{(\beta_n/\alpha_n)-1} \prod_{i=1}^{n-1} |u_i|^{\beta_i - 1}\, du_i = 2^{n-1} \frac{\Gamma(\beta_n/\alpha_n)}{\Gamma(p_n)} \prod_{i=1}^{n-1} \frac{\Gamma(\beta_i/\alpha_i)}{\alpha_i}.$$

Proof. To prove this result, we use the symmetry of the integrand to reduce the domain of integration from R^{n-1} to R_+^{n-1} (where $R_+ = [0, \infty)$); replace each u_i by u_i^{1/α_i}; and then apply the classical Dirichlet integral [5], [7]. ∎

2.2. DEFINITION. A random vector $(U_1, \ldots, U_n)$ is said to have a *sign-symmetric Dirichlet-type distribution* with parameters $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ if

(i) U_n is a symmetric random variable;

(ii) $\sum_{i=1}^{n} |U_i|^{\alpha_i} = 1$ almost surely; and

(iii) the joint density function of $(U_1, \ldots, U_{n-1})$ is

$$\frac{\Gamma(p_n)}{\Gamma(\beta_n/\alpha_n)} \prod_{i=1}^{n-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |u_i|^{\beta_i - 1} \Big(1 - \sum_{i=1}^{n-1} |u_i|^{\alpha_i}\Big)_+^{(\beta_n/\alpha_n)-1}. \tag{2.1}$$

Whenever Definition 2.2 holds we write

$$(U_1, \ldots, U_n) \sim \mathscr{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).$$

If $\alpha_i \equiv 1$, then the distribution of $(U_1, \ldots, U_n)$ may be called the *sign-symmetric Dirichlet distribution.* If $\alpha_i \equiv 2$ and $\beta_i \equiv 1$, then the random vector $(U_1, \ldots, U_n)$ is uniformly distributed on the unit sphere $\{(x_1, \ldots, x_n)\colon x_1^2 + \ldots + x_n^2 = 1\}$ in R^n (see [3] and [12]). If $\alpha_i \equiv \alpha$ and $\beta \equiv 1$, then $(U_1, \ldots, U_n)$ is l_α-isotropic [1], [2], [15].

We now collect some basic properties of the distributions $\mathscr{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$. We will first deal with the properties of the marginal distributions.

2.3. PROPOSITION. *Suppose that* $(U_1, \ldots, U_n) \sim \mathscr{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ *and* $1 \leqslant k < n$.

(i) *The marginal density function of* $(U_1, \ldots, U_k)$ *is*

$$\frac{\Gamma(p_n)}{\Gamma(p_n - p_k)} \prod_{i=1}^{k} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |u_i|^{\beta_i - 1} \Big(1 - \sum_{i=1}^{k} |u_i|^{\alpha_i}\Big)_+^{p_n - p_k - 1}. \tag{2.2}$$

(ii) *Let* ε *be a random variable taking values* ± 1 *each with probability* 1/2, *and independent of* $(U_1, \ldots, U_n)$. *Then for any* $\alpha > 0$

$$\Big(U_1, \ldots, U_k, \varepsilon\big(\sum_{i=k+1}^{n} |U_i|^{\alpha_i}\big)^{1/\alpha}\Big) \sim \mathscr{SD}(\alpha_1, \ldots, \alpha_k, \alpha; \beta_1, \ldots, \beta_k, (p_n - p_k)\alpha).$$

Proof. (i) The marginal density function of $(U_1, \ldots, U_k)$ is

$$\frac{\Gamma(p_n)}{\Gamma(\beta_n/\alpha_n)} \prod_{i=1}^{n-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} \prod_{i=1}^{k} |u_i|^{\beta_i - 1} \int_{R^{n-k-1}} \ldots \int \Big(1 - \sum_{i=1}^{n-1} |u_i|^{\alpha_i}\Big)_+^{(\beta_n/\alpha_n) - 1} \prod_{i=k+1}^{n-1} |u_i|^{\beta_i - 1} du_i.$$

The desired result can be obtained by substituting

$$u_j = \Big(1 - \sum_{i=1}^{k} |u_i|^{\alpha_i}\Big)^{1/\alpha_j} y_j$$

for $j = k+1, \ldots, n-1$, and then using Lemma 2.1.

(ii) It is not difficult to see that the conditions (i) and (ii) of Definition 2.2 are satisfied. As for the third condition, this follows from the formula obtained above for the marginal distribution of $(U_1, \ldots, U_k)$. ∎

2.4. Remark. As a simple application of (2.2) we obtain the following result. Suppose that α and β are positive numbers. Also suppose that the $(k+1)$-dimensional random vector $(V_1, \ldots, V_{k+1}) \sim \mathscr{SD}(\alpha_1, \ldots, \alpha_k, \alpha; \beta_1, \ldots, \beta_k, \beta)$. If

$$\beta/\alpha = p_n - p_k,$$

then it follows from (2.2) that the vectors $(U_1, \ldots, U_k)$ and $(V_1, \ldots, V_k)$ have the same distribution.

2.5. PROPOSITION. *Suppose that* $(U_1, \ldots, U_n) \sim \mathscr{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ *and* $1 < k < n$. *Then the conditional density function of* $(U_1, \ldots, U_{k-1})$ *given* $\{U_{k+1} = u_{k+1}, \ldots, U_n = u_n\}$ *is*

$$\frac{\Gamma(p_k)}{\Gamma(\beta_k/\alpha_k)} \prod_{i=1}^{k-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |u_i|^{\beta_i - 1} \Big(1 - \sum_{\substack{i=1 \\ i \neq k}}^{n} |u_i|^{\alpha_i}\Big)_+^{(\beta_k/\alpha_k) - 1} \Big(1 - \sum_{i=k+1}^{n} |u_i|^{\alpha_i}\Big)_+^{1 - p_k}. \tag{2.3}$$

Proof. This is obtained in the standard way by dividing the joint density function of $(U_1, \ldots, U_{k-1}, U_{k+1}, \ldots, U_n)$ by the marginal density function of $(U_{k+1}, \ldots, U_n)$. ∎

2.6. Remark. By rewriting the conditional density function of $(U_1, \ldots, U_{k-1})$, given $\{U_{k+1} = u_{k+1}, \ldots, U_n = u_n\}$ in (2.3), in the form

$$\frac{\Gamma(p_k)}{\Gamma(\beta_k/\alpha_k)} \prod_{i=1}^{k-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} \left|\frac{u_i}{R^{1/\alpha_i}}\right|^{\beta_i - 1} \left(1 - \sum_{\substack{i=1 \\ i \neq k}}^{n} \left|\frac{u_i}{R^{1/\alpha_i}}\right|^{\alpha_i}\right)_+^{(\beta_k/\alpha_k)-1} R^{A_i}, \tag{2.4}$$

where $R = 1 - \sum_{k+1}^{n} |U_i|^{\alpha_i}$, and $A_i = -\sum_{i=1}^{k-1} \alpha_i^{-1}$, we see that (2.4) or (2.3) is also the conditional density function of the random vector $(U_1 R^{1/\alpha_1}, \ldots, U_{k-1} R^{1/\alpha_{k-1}})$, given R, where

$$(U_1, \ldots, U_k) \sim \mathcal{SD}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k).$$

2.7. Proposition. (i) *For* $h_1, \ldots, h_n > 0$,

$$\mathrm{E}\left(\prod_{i=1}^{n} |U_i|^{h_i}\right) = \frac{\Gamma(p_n)}{\Gamma(p_n + \sum_{i=1}^{n} h_i/\alpha_i)} \prod_{i=1}^{n} \frac{\Gamma(\beta_i/\alpha_i + h_i/\alpha_i)}{\Gamma(\beta_i/\alpha_i)}.$$

(ii) *For* $i, j = 1, \ldots, n,\ i \neq j$,

$$\mathrm{E}(U_i) = \mathrm{Cov}(U_i, U_j) = 0 \quad \textit{and} \quad \mathrm{Var}(U_i) = \frac{\Gamma(p_n)\Gamma(\beta_i/\alpha_i + 2/\alpha_i)}{\Gamma(p_n + 2/\alpha_i)\Gamma(\beta_i/\alpha_i)}.$$

(iii) *For* $i = 1, \ldots, k$ *and* $h > 0$,

$$\mathrm{E}(|U_i|^h \mid U_{k+1} = u_{k+1}, \ldots, U_n = u_n) = \frac{\Gamma(p_k)\Gamma(\beta_i/\alpha_i + h/\alpha_i)}{\Gamma(p_k + h/\alpha_i)\Gamma(\beta_i/\alpha_i)} \left(1 - \sum_{j=k+1}^{n} |u_j|^{\alpha_j}\right)^{h/\alpha_i}.$$

Proof. These formulas follow immediately from Lemma 2.1. ∎

2.8. Remark. In the Introduction it was implied that if $Z_1, \ldots, Z_n$ are independent random variables with density function (1.1), and $U_1, \ldots, U_n$ are defined as in (1.2), then

$$(U_1, \ldots, U_n) \sim \mathcal{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).$$

This claim can be verified by observing that U_n is a symmetric random variable; that $\sum_{i=1}^{n} |U_i|^{\alpha_i} = 1$ with probability 1; and that the joint density function of $(U_1, \ldots, U_{n-1})$, which is calculated from the joint density of $Z_1, \ldots, Z_n$, is of the form (2.1).

3. The sign-symmetric Liouville-type distributions.

3.1. Definition. A random vector $(X_1, \ldots, X_n)$ is said to have a *sign-symmetric Liouville-type distribution* if there exist a sign-symmetric Dirichlet-type random vector $(U_1, \ldots, U_n) \sim \mathcal{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ and a nonnegative random variable Θ, independent of $(U_1, \ldots, U_n)$, such that

$$(X_1, \ldots, X_n) \cong (U_1 \Theta^{1/\alpha_1}, \ldots, U_n \Theta^{1/\alpha_n}).$$

Whenever Definition 3.1 holds we will use the notation

$$(X_1, \dots, X_n) \sim \mathcal{SL}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n; \Theta).$$

3.2. Proposition. *Suppose that $(X_1, \dots, X_n) \sim \mathcal{SL}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n; \Theta)$ and Θ is absolutely continuous with probability density function g. Then the joint density function of $(X_1, \dots, X_n)$ is*

$$\Gamma(p_n) \prod_{i=1}^{n} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \Big(\sum_{i=1}^{n} |x_i|^{\alpha_i}\Big)^{1-p_n} g\Big(\sum_{i=1}^{n} |x_i|^{\alpha_i}\Big), \tag{3.1}$$

where $x_1, \dots, x_n \in \mathbf{R}$.

Proof. Since $\sum_{i=1}^{n} |U_i|^{\alpha_i} = 1$ almost surely, it follows from Definition 3.1 that $\Theta = \sum_{i=1}^{n} |X_i|^{\alpha_i}$ almost surely. Then the result is obtained using the standard method of transformations. ∎

3.3. Examples. (i) Let $\alpha_i \equiv 2$ and $\beta_i \equiv 1$. Then the random vector

$$(X_1, \dots, X_n) \sim \mathcal{SL}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n; \Theta)$$

has a spherically symmetric distribution [3], [12].

(ii) If $\alpha_i \equiv 1$, then $(X_1, \dots, X_n)$ may be viewed as a sign-symmetric Liouville distribution.

(iii) If $\alpha_i \equiv \alpha > 0$ and $\beta_i \equiv 1$, then $(X_1, \dots, X_n)$ has an l_α-isotropic distribution [15]. In the case where $(X_1, \dots, X_n)$ is absolutely continuous, the density function is constant on the l_α-spheres $\{(x_1, \dots, x_n) \in \mathbf{R}^n\colon \sum_{i=1}^{n} |x_i|^{\alpha} = c\}$, c constant.

(iv) Suppose that Θ has a beta distribution with parameters p_n and β/α, where $\alpha, \beta > 0$. Applying Proposition 3.2 and Proposition 2.3 (i) we deduce that the distribution of $(X_1, \dots, X_n)$ is the same as the distribution of $(Y_1, \dots, Y_n)$, where $(Y_1, \dots, Y_n)$ is a subvector of the vector

$$(Y_1, \dots, Y_{n+1}) \sim \mathcal{SD}(\alpha_1, \dots, \alpha_n, \alpha; \beta_1, \dots, \beta_n, \beta).$$

Note that the distribution of $(X_1, \dots, X_n)$ depends on α and β only through the ratio β/α; cf. Remark 2.4.

(v) Similarly to the independence properties of the Liouville distributions [7], the random variables $X_1, \dots, X_n$ are mutually independent if and only if each X_i has density function

$$\frac{\alpha_i b^{\beta_i/\alpha_i}}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \exp\{-b|x_i|^{\alpha_i}\} \quad \text{for } x_i \in \mathbf{R},\ i = 1, \dots, n.$$

In that case, Θ has density function

$$\frac{b^{p_n}}{\Gamma(p_n)} r^{p_n - 1} e^{-br} \quad \text{for } r > 0.$$

The marginal distributions of $(X_1, \ldots, X_n) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ are given in the following result.

3.4. PROPOSITION. *Suppose that* $(X_1, \ldots, X_n) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ *and* $1 \leqslant k < n$. *Then*

$$(X_1, \ldots, X_k) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_k),$$

where

$$\Theta_k = \Big(1 - \sum_{i=k+1}^{n} |U_i|^{\alpha_i}\Big)\Theta.$$

Moreover, Θ_k is absolutely continuous (with respect to Lebesgue measure) and the density function g_k of Θ_k is

$$g_k(r) = \frac{\Gamma(p_n)}{\Gamma(p_k)\Gamma(p_n - p_k)} r^{p_k - 1} \int_0^{\infty} (t-r)_+^{p_n - p_k - 1} t^{1-p_n} \lambda(dt)$$

for $r > 0$, where λ is the distribution function of Θ.

Proof. By Definition 3.1 we have

$$(X_1, \ldots, X_k) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k}).$$

Letting $R_k = 1 - \sum_{i=k+1}^{n} |U_i|^{\alpha_i}$ and $U_i' = U_i R_k^{-1/\alpha_i}$, $i = 1, \ldots, k$, then we obtain

$$\Theta R_k = \Theta_k \quad \text{and} \quad (X_1, \ldots, X_k) = \big(U_1'(R_k\Theta)^{1/\alpha_1}, \ldots, U_k'(R_k\Theta)^{1/\alpha_k}\big).$$

Moreover, by applying the techniques used to establish Proposition 2.3 we deduce that R_k has a beta distribution with parameters p_k and $p_n - p_k$.

Next, it follows from the definition of the sign-symmetric Dirichlet-type distributions that $(U_1', \ldots, U_k') \sim \mathscr{S}\mathscr{D}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)$ and is independent of $R_k \Theta = \Theta_k$. Finally, the density function of Θ_k is calculated directly using the representation $\Theta_k = R_k \Theta$ and the known distribution of R_k. ∎

We now study the uniqueness of the representation $\mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ for the sign-symmetric Liouville-type distributions.

3.5. PROPOSITION. *The representation $(X_1, \ldots, X_n) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ of a sign-symmetric Liouville-type random vector $(X_1, \ldots, X_n)$ is unique if*

(i) *$n = 2$ and Θ has a continuous density function, or*

(ii) *$n \geqslant 3$.*

Proof. We will only prove (ii) since the proof of (i) is similar. Thus suppose that

$$(X_1, \ldots, X_n) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$$

and also that

$$(X_1, \ldots, X_n) \sim \mathscr{SD}(\alpha'_1, \ldots, \alpha'_n; \beta'_1, \ldots, \beta'_n; \tilde{\Theta}).$$

Now consider the random vector $(X_1, \ldots, X_{n-1})$. By Proposition 3.4 there exist positive random variables Θ_{n-1} and $\tilde{\Theta}_{n-1}$ such that

$$(X_1, \ldots, X_{n-1}) \sim \mathscr{SL}(\alpha_1, \ldots, \alpha_{n-1}; \beta_1, \ldots, \beta_{n-1}; \Theta_{n-1})$$

and

$$(X_1, \ldots, X_{n-1}) \sim \mathscr{SL}(\alpha'_1, \ldots, \alpha'_{n-1}; \beta'_1, \ldots, \beta'_{n-1}; \tilde{\Theta}_{n-1}).$$

Moreover, Θ_{n-1} and $\tilde{\Theta}_{n-1}$ are absolutely continuous with continuous density functions, say, g_{n-1} and $\tilde{g}_{n-1}$, respectively. Applying Proposition 3.2 with Θ_{n-1} and $\tilde{\Theta}_{n-1}$ we obtain two representations for the density function of $(X_1, \ldots, X_{n-1})$. Setting

$$f_{n-1}(r) = r^{1-p_{n-1}} g_{n-1}(r) \quad \text{and} \quad \tilde{f}(r) = r^{1-p'_{n-1}} \tilde{g}_{n-1}(r) \quad \text{for } r > 0,$$

and then comparing the two representations for the density of $(X_1, \ldots, X_{n-1})$ we obtain the identity

$$f_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha_i}\Big) \prod_{i=1}^{n-1} |x_i|^{\beta_i - 1} \equiv c_1 \tilde{f}_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha'_i}\Big) \prod_{i=1}^{n-1} |x_i|^{\beta'_i - 1}, \tag{3.2}$$

where c_1 is a constant.

Now suppose that there exists $j < n$ such that $\beta_j \neq \beta'_j$, and without loss of generality assume that $\beta_j > \beta'_j$; by renumbering the variables in (3.2) we can even assume that $j = 1$. Then (3.2) reduces to

$$f_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha_i}\Big) |x_1|^{\beta_1 - \beta'_1} \prod_{i=2}^{n-1} |x_i|^{\beta_i - 1} \equiv c_1 \tilde{f}_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha'_i}\Big) \prod_{i=2}^{n-1} |x_i|^{\beta'_i - 1}. \tag{3.3}$$

As $|x_1| \to 0$, the left-hand side of (3.3) tends to zero for every fixed $x_2, \ldots, x_{n-1}$. This implies that the right-hand side of (3.3) is identically zero, which eventually leads to the conclusion $\tilde{g}_{n-1} \equiv 0$. This is clearly a contradiction. Therefore $\beta_1 \leqslant \beta'_1$. If $\beta_1 < \beta'_1$, then a similar argument leads to the conclusion $g_{n-1} \equiv 0$, again a contradiction. Therefore $\beta_1 = \beta'_1$. By considering the random vector $(X_2, \ldots, X_n)$ instead of $(X_1, \ldots, X_{n-1})$ and following the above procedure, we eventually obtain $\beta_j = \beta'_j$ for all $j = 1, \ldots, n$.

Applying this conclusion to (3.3), we are now faced with the identity

$$f_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha_i}\Big) \equiv c_2 \tilde{f}_{n-1}\Big(\sum_{i=1}^{n-1} |x_i|^{\alpha'_i}\Big),$$

where c_2 is a constant. By equating the level curves of both sides of this identity we find that for any $r > 0$

$$\Big\{(x_1, \ldots, x_{n-1}) \in \boldsymbol{R}^{n-1}\colon \sum_{i=1}^{n-1} |x_i|^{\alpha_i} = r\Big\} = \Big\{(x_1, \ldots, x_{n-1}) \in \boldsymbol{R}^{n-1}\colon \sum_{i=1}^{n-1} |x_i|^{\alpha'_i} = c_3 r\Big\},$$

where the constant c_3 is possibly dependent on r. It is well known that this last identity holds if and only if $\alpha_i = \alpha'_i$ for all $i = 1, \ldots, n-1$. As before, we also deduce that $\alpha_n = \alpha'_n$ by working with the random vector $(X_2, \ldots, X_n)$.

In conclusion, the parameters α_i and β_i are uniquely determined by the distribution of $(X_1, \ldots, X_n)$. In turn, the distribution of Θ is also uniquely determined by the constraint $\Theta = \sum_{i=1}^{n} |X_i|^{\alpha_i}$ almost surely. ∎

We now turn to the behavior of the conditional distributions of sign-symmetric Liouville-type random vectors.

3.6. PROPOSITION. *Suppose that* $(X_1, \ldots, X_n) \sim \mathcal{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ *where* Θ *is absolutely continuous with density function* g, *and* $1 \leqslant k < n$. *Then the conditional distribution of* $(X_1, \ldots, X_k)$ *given* $\{X_{k+1}, \ldots, X_n\}$ *is* $\mathcal{SL}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_c)$, *where* Θ_c *is the conditional random variable* $\sum_{i=1}^{k} |X_i|^{\alpha_i}$ *given* $\sum_{k+1}^{n} |X_i|^{\alpha_i}$.

Proof. Since Θ is absolutely continuous, the joint density function of $(X_1, \ldots, X_n)$ is given by Proposition 3.2. Also the marginal density function of $(X_{k+1}, \ldots, X_n)$ is obtained from Proposition 3.4. Dividing the joint density by the marginal density, we find that the conditional density of $(X_1, \ldots, X_k)$ given $\{X_{k+1} = x_{k+1}, \ldots, X_n = x_n\}$ may be written as

$$\Gamma(p_k) \prod_{1}^{k} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \Big(\sum_{1}^{k} |x_i|^{\alpha_i}\Big)^{1-p_k}$$
$$\times \Big(\sum_{1}^{k} |x_i|^{\alpha_i}\Big)^{p_k - 1} \Big(\sum_{k+1}^{n} |x_i|^{\alpha_i}\Big)^{p_n - p_k - 1} \Big(\sum_{1}^{n} |x_i|^{\alpha_i}\Big)^{1-p_n} \frac{g\big(\sum_{1}^{n} |x_i|^{\alpha_i}\big)}{g_{n-k}\big(\sum_{k+1}^{n} |x_i|^{\alpha_i}\big)},$$

where g_k is defined in Proposition 3.4.

Noting the general expression for the density given in Proposition 3.2, we see that the above expression for the conditional density is the same as the joint density of $\mathcal{SL}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_c)$, where Θ_c has the density

$$x^{p_k - 1} r^{p_n - p_k - 1} (x+r)^{1-p_n} \frac{g(x+r)}{g_{n-k}(r)}$$

and $r = \sum_{i=k+1}^{n} |x_i|^{\alpha_i}$. It is straightforward to verify that this last expression is the density of the random variable $\sum_{i=1}^{k} |X_i|^{\alpha_i}$ given $\sum_{k+1}^{n} |X_i|^{\alpha_i} = r$. ∎

In ending this section, we note that the calculation of the moments of a sign-symmetric Liouville-type random vector $(X_1, \ldots, X_n)$ proceeds directly from Definition 3.1 and the formulas for the moments of the sign-symmetric Dirichlet-type random variables. In particular, if $\mathrm{E}(\Theta^{2k+1}) < \infty$, then, for any $k \in N$, $\mathrm{E}(X_i^{2k+1}) = 0$, $i = 1, \ldots, n$.

4. Infinite sequences with sign-symmetric Liouville-type distributions. In this section we treat the properties of infinite sequences of random variables for which every finite subsequence follows a sign-symmetric Liouville-type distributions.

4.1. Definition. An infinite sequence of random variables $\{U_i\colon i\in N\}$ is said to have a *sign-symmetric Dirichlet-type distribution* if for every $n\in N$ there exists a symmetric random variable W_n such that the random vector $(U_1, \ldots, U_n, W_n)$ has a sign-symmetric Dirichlet-type distribution.

4.2. Proposition. *Suppose that the sequence $\{U_i\colon i\in N\}$ has a sign-symmetric Dirichlet-type distribution. Then there exist sequences of positive numbers α_i, β_i, γ_i and δ_i, $i\in N$, such that for every $n\in N$ the random vector $(U_1, \ldots, U_n, W_n) \sim \mathcal{SD}(\alpha_1, \ldots, \alpha_n, \gamma_n; \beta_1, \ldots, \beta_n, \delta_n)$. Moreover, the sequences α_i, β_i and δ_i/γ_i are uniquely determined by the distribution of $\{U_i\}$.*

Proof. Choose an $n\in N$. By an application of Proposition 2.3 (i) we find that the density function of $(U_1, \ldots, U_n)$ exists and is proportional to

$$\Big(1-\sum_{i=1}^{n}|u_i|^{\alpha_i}\Big)_+^{(\delta_n/\gamma_n)-1}\prod_{i=1}^{n}|u_i|^{\beta_i-1}.$$

Since n is arbitrary, this also establishes the existence of the sequences α_i, β_i, γ_i and δ_i. Moreover, the uniqueness of these sequences follows from Proposition 3.5. ∎

4.3. Definition. An infinite sequence of random variables $\{X_i\colon i\in N\}$ is said to have a *sign-symmetric Liouville-type distribution* if for every $n\in N$ the random vector $(X_1, \ldots, X_n)$ has a sign-symmetric Liouville-type distribution.

4.4. Proposition. *Suppose that $\{X_i\colon i\in N\}$ has a sign-symmetric Liouville-type distribution. Then there exist sequences of positive numbers α_i and β_i, $i\in N$, and an increasing sequence of random variables $\Theta_1 \leqslant \Theta_2 \leqslant \ldots$ such that, for every $n\in N$,*

$$(X_1, \ldots, X_n) \sim \mathcal{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta_n).$$

Proof. The existence of the sequences α_i and β_i, $i\in N$, follows as in Proposition 4.2.

Next, again choose $n\in N$. By the definition of the sign-symmetric Liouville-type vector, there exists a sign-symmetric Dirichlet-type vector $(U_1, \ldots, U_n) \sim \mathcal{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ and a nonnegative random variable Θ_n such that

$$(X_1, \ldots, X_n) = (U_1\,\Theta_n^{1/\alpha_1}, \ldots, U_n\,\Theta_n^{1/\alpha_n}).$$

Moreover, this representation also holds for $(X_1, \ldots, X_{n+1})$, so that there exists a sign-symmetric Dirichlet-type random vector $(U_1', \ldots, U_{n+1}')$, with distribution $\mathcal{SD}(\alpha_1', \ldots, \alpha_{n+1}'; \beta_1', \ldots, \beta_{n+1}')$, and a nonnegative random variable Θ_{n+1} such that

$$(X_1, \ldots, X_{n+1}) = (U_1'\,\Theta_{n+1}^{1/\alpha_1'}, \ldots, U_{n+1}'\,\Theta_{n+1}^{1/\alpha_n'}). \tag{4.1}$$

Now we calculate the marginal distribution of $(X_1, \ldots, X_n)$ starting from (4.1). Then we obtain

$$(X_1, \ldots, X_n) = \left(V_1 (R\Theta_{n+1})^{1/\alpha_1'}, \ldots, V_n (R\Theta_{n+1})^{1/\alpha_{n+1}'}\right),$$

where $R = (1-|U_{n+1}'|^{\alpha_{n+1}'})$ and $V_i = R^{-1/\alpha_i'} U_i'$, $i = 1, \ldots, n$. Then it is evident that

$$(V_1, \ldots, V_n) \sim \mathscr{S}\mathscr{D}(\alpha_1', \ldots, \alpha_n'; \beta_1', \ldots, \beta_n').$$

Next, it follows from Proposition 3.5 that $\alpha_i = \alpha_i'$ and $\beta_i = \beta_i'$, $i = 1, \ldots, n$, and also that $\Theta_n = (1-|U_{n+1}'|^{\alpha_{n+1}'})\Theta_{n+1} \leqslant \Theta_{n+1}$. ∎

We will use the notation $\{X_i\colon i \in N\} \sim \mathscr{S}\mathscr{L}(\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\})$ whenever a sequence of random variables $X_1, X_2, \ldots$ has a sign-symmetric Liouville--type distribution as described in Definition 4.3. If the associated random variables Θ_i each has a beta distribution with parameters p_i and δ_i/γ_i, $i = 1, 2, \ldots$, then it follows as in the proof of Proposition 4.4 that the distribution of the sequence $\{X_i\}$ reduces to a sign-symmetric Dirichlet-type distribution, which will be written as $\{X_i\colon i \in N\} \sim \mathscr{S}\mathscr{D}(\{\alpha_i\}; \{\beta_i\})$.

4.5. THEOREM. *Suppose that $\{X_i\colon i \in N\}$ is a sequence of random variables such that*

$$\{X_i\colon i \in N\} \sim \mathscr{S}\mathscr{L}(\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\}),$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ satisfy $\sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty$. Then there exists a probability measure λ on $\boldsymbol{R}_+$ such that, for all $n \in N$, the joint density function of the random vector $(X_1, \ldots, X_n)$ is

$$\prod_{i=1}^{n} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \int_0^{\infty} s^{p_n} \exp\{-s \sum_{i=1}^{n} |x_i|^{\alpha_i}\} \lambda(ds).$$

Equivalently, there exists a nonnegative random variable Θ and a sequence of mutually independent random variables $\{Y_i\colon i \in N\}$, all independent of Θ, where each Y_i has density function (1.1), *and such that*

$$(X_1, \ldots, X_n) = (Y_1 \Theta^{1/\alpha_1}, \ldots, Y_n \Theta^{1/\alpha_n}) \quad \textit{for all } n \in N.$$

Proof. The method of proof will be similar to the one used in Theorem 2 of [15]. Thus, choose $k \in N$. Then for any $n > k$ we may regard the distribution of $(X_1, \ldots, X_k)$ as a k-dimensional marginal distribution of $(X_1, \ldots, X_n)$; this defines the random variable Θ_n for any $n > k$. By Proposition 3.4,

$$\Theta_n = (1-|U_{n+1}|^{\alpha_{n+1}})\Theta_{n+1};$$

hence Θ_n is absolutely continuous and with a continuous density function g_n. Furthermore, the density function g_k of Θ_k is given in Proposition 3.4.

Letting $f_i(r) \equiv r^{1-p_i} g_i(r)$, $i \in N$, we can write the formula for f_k as

$$(4.2) \qquad f_k(r) = \frac{\Gamma(p_n)}{\Gamma(p_k)\Gamma(p_n - p_k)} \int_0^\infty (t-r)_+^{p_n - p_k - 1} f_n(t)\, dt.$$

By [20] it follows that the function f_k is $(p_n - p_k)$-monotone. Letting $n \to \infty$ and noting that $p_n - p_k = \sum_{i=k+1}^n \beta_i/\alpha_i \to \infty$, it also follows that f_k is completely monotone. That is, there exists a positive measure λ_k on $\boldsymbol{R}_+$ such that f_k is the Laplace transform of λ_k,

$$(4.3) \qquad f_k(r) = \int_0^\infty e^{-sr} \lambda_k(ds)$$

for $r > 0$. Now choose $n = k+1$ in (4.2), substitute (4.3) into the right-hand side of the resulting formula, and interchange the order of integration. Then we obtain

$$\begin{aligned}(4.4) \qquad f_k(r) &= \frac{\Gamma(p_{k+1})}{\Gamma(p_k)\Gamma(\beta_{k+1}/\alpha_{k+1})} \int_0^\infty t^{(\beta_{k+1}/\alpha_{k+1})-1} f_{k+1}(r+t)\, dt \\ &= \frac{\Gamma(p_{k+1})}{\Gamma(p_k)\Gamma(\beta_{k+1}/\alpha_{k+1})} \int_0^\infty t^{(\beta_{k+1}/\alpha_{k+1})-1} \int_0^\infty e^{-s(r+t)} \lambda_{k+1}(ds)\, dt \\ &= \frac{\Gamma(p_{k+1})}{\Gamma(p_k)} \int_0^\infty s^{-(\beta_{k+1}/\alpha_{k+1})} e^{-sr} \lambda_{k+1}(ds).\end{aligned}$$

Comparing (4.4) with (4.3) and applying the uniqueness of the Laplace transform, we get

$$\lambda_{k+1}(ds) = \frac{\Gamma(p_k)}{\Gamma(p_{k+1})} s^{\beta_{k+1}/\alpha_{k+1}} \lambda_k(ds).$$

Using this recurrence relation we obtain

$$\lambda_k(ds) = \left(\prod_{i=2}^k \frac{\Gamma(p_{i-1})}{\Gamma(p_i)} s^{\beta_i/\alpha_i} \right) \lambda_1(ds) = \frac{\Gamma(p_1)}{\Gamma(p_k)} s^{p_k - p_1} \lambda_1(ds).$$

Since $g_1(r) = r^{p_1 - 1} f_1(r)$, $r > 0$, is the density function of $|X_1|^{\alpha_1}$, we have

$$1 = \int_0^\infty r^{p_1-1} f_1(r)\, dr = \int_0^\infty r^{p_1-1} \int_0^\infty e^{-sr} \lambda_1(ds)\, dr = \Gamma(p_1) \int_0^\infty s^{-p_1} \lambda_1(ds),$$

proving that the measure λ defined by $\lambda(ds) = \Gamma(p_1) s^{-p_1} \lambda_1(ds)$, $s > 0$, is a probability measure.

Now we express λ_k in terms of λ, and then, by (4.3), we obtain

$$f_k(r) = \frac{1}{\Gamma(p_k)} \int_0^\infty s^{p_k} e^{-sr} \lambda(ds).$$

This completes the proof of the first part of the theorem.

To see that the second statement in the theorem implies the first one, it is enough to notice that the desired measure λ is the distribution of Θ. The converse follows by defining a random variable Θ independent of $\{Y_i\colon i \in N\}$ and with distribution λ. ∎

4.6. COROLLARY. *If the α_i and β_i satisfy $\sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty$, then there exists no infinite sequence $\{U_i\colon i \in N\} \sim \mathscr{S}\mathscr{D}(\{\alpha_i\}; \{\beta_i\})$, a sign-symmetric Dirichlet-type sequence. In particular, there exists no exchangeable infinite sequence of positive random variables such that every finite subsequence has a Dirichlet distribution.*

Proof. Assume that such a sequence exists. Then from Proposition 4.2 we obtain a formula for the marginal density of U_1. Equating this formula for the marginal density of U_1 with the one obtained from Theorem 4.5, we have

$$\frac{\alpha_1}{2} \frac{\Gamma(\beta_1/\alpha_1 + \delta_1/\gamma_1)}{\Gamma(\beta_1/\alpha_1)\Gamma(\delta_1/\gamma_1)} |x|^{\beta_1 - 1} (1 - |x|^{\alpha_1})_+^{(\delta_1/\gamma_1) - 1}$$
$$= \frac{\alpha_1}{2\Gamma(\beta_1/\alpha_1)} |x|^{\beta_1 - 1} \int_0^{\infty} s^{\beta_1/\alpha_1} \exp\{-s|x|^{\alpha_1}\}\,\lambda(ds)$$

for some probability measure λ on $\boldsymbol{R}_+$. But the left-hand side of this last formula is a density function with support in the interval $[-1, 1]$, while the right-hand side has unbounded support on $\boldsymbol{R}_+$. Therefore we have a contradiction.

Finally, the second statement is proved by choosing $\alpha_i \equiv \alpha$ and $\beta_i \equiv \beta$ for $\alpha, \beta > 0$. ∎

Now we turn to the case where the infinite sequences α_i and β_i satisfy $\sum_{i=1}^{\infty} \beta_i/\alpha_i = p < \infty$. That is, $p_n \to p < \infty$ as $n \to \infty$.

4.7. Remark. Suppose that $p_n \to p < \infty$ as $n \to \infty$. Define a consistent family of measures $\{\mu_n\colon n \in N\}$ such that μ_n on $\boldsymbol{R}^{\{1,\ldots,n\}}$ has density function

$$\frac{\Gamma(p)}{\Gamma(p - p_n)} \prod_1^n \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \Big(1 - \sum_{i=1}^{n} |x_i|^{\alpha_i}\Big)_+^{p - p_n - 1}. \tag{4.5}$$

By Kolmogorov's extension theorem ([6], p. 121) there exists a sequence of random variables $U_1, U_2, \ldots$ such that for every $n \in N$ the measure μ_n is the distribution of the random vector $(U_1, \ldots, U_n)$. Then there exists an infinite sequence of random variables with a sign-symmetric Dirichlet-type distribution.

Consider now the sequence of random variables $R_n = \sum_{i=1}^{n} |U_i|^{\alpha_i}$. Then it follows from (4.5) that $\mathrm{E}(R_n) = p_n/p \to 1$ and

$$\mathrm{Var}(R_n) = \frac{p_n(p - p_n)}{p^2(p+1)} \to 0 \quad \text{as } n \to \infty.$$

Therefore $R_n \to 1$ in probability and also in distribution. Since the sequence R_n is increasing, we also have $R_n \to 1$ almost surely.

Define the infinite sequence of random variables $\{X_i\colon i \in N\} = \{U_i \Theta^{1/\alpha_i}\colon i \in N\}$, where $\{U_i\colon i \in N\}$ is a sequence of sign-symmetric Dirichlet-type random variables, Θ is a positive random variable with distribution λ, and Θ is independent of $\{U_i\colon i \in N\}$. Then for each $n \in N$ the joint density function of the random vector $(X_1, \ldots, X_n)$ is

$$\frac{\Gamma(p)}{\Gamma(p-p_n)} \prod_1^n \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \int_0^\infty \Big(s - \sum_{i=1}^n |x_i|^{\alpha_i}\Big)_+^{p-p_n-1} s^{1-p} \lambda(ds).$$

This proves that the sequence $\{X_i\colon i \in N\} \sim \mathscr{S}\mathscr{L}(\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\})$. Moreover, as $n \to \infty$, $\Theta_n = \sum_{i=1}^n |X_i|^{\alpha_i} = R_n \Theta \to \Theta$ almost surely.

Note also that $(U_1, \ldots, U_n) \sim \mathscr{S}\mathscr{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; R_n)$, with the random variable $R_n = \sum_{i=1}^n |U_i|^{\alpha_i} = 1 - \sum_{n+1}^\infty |U_i|^{\alpha_i}$ almost surely.

The following result provides a converse to the construction of the sign-symmetric Liouville-type distributions given in the previous example.

4.8. THEOREM. *Suppose that $\{X_i\colon i \in N\}$ is a sequence of random variables such that*

$$\{X_i\colon i \in N\} \sim \mathscr{S}\mathscr{L}(\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\}),$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ satisfy $p_n \to p < \infty$ as $n \to \infty$. Then there exist a sign-symmetric Dirichlet-type sequence $\{U_i\colon i \in N\} \sim \mathscr{S}\mathscr{D}(\{\alpha_i\}; \{\beta_i\})$ and a non-negative random variable Θ, independent of $\{U_i\colon i \in N\}$, such that $\{X_i\colon i \in N\} = \{U_i \Theta^{1/\alpha_i}\colon i \in N\}$.

Proof. From Definition 4.3 we infer that for every $n \in N$ there exists a sign-symmetric Dirichlet-type random vector

$$(W_{1,n}, \ldots, W_{n,n}) \sim \mathscr{S}\mathscr{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$$

and a positive random variable Θ_n independent of $(W_{1,n}, \ldots, W_{n,n})$ such that

$$(X_1, \ldots, X_n) = (W_{1,n}\Theta_n^{1/\alpha_1}, \ldots, W_{n,n}\Theta_n^{1/\alpha_n}).$$

Consider now the sign-symmetric Dirichlet-type sequence of random variables constructed as in Remark 4.7. For this sequence, we know that for each $n \in N$

$$\left(\frac{U_1}{(\sum_{j=1}^n |U_j|^{\alpha_j})^{1/\alpha_1}}, \ldots, \frac{U_n}{(\sum_{j=1}^n |U_j|^{\alpha_j})^{1/\alpha_n}}\right) \sim \mathscr{S}\mathscr{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).$$

Now, for $i = 1, \ldots, n$, we choose

$$W_{i,n} = \frac{U_i}{(\sum_{j=1}^n |U_j|^{\alpha_j})^{1/\alpha_i}}.$$

Therefore, for each $n \in N$

$$(4.6)\qquad (X_1, \ldots, X_n) = \left(U_1\left(\frac{\Theta_n}{\sum_{j=1}^n |U_j|^{\alpha_j}}\right)^{1/\alpha_1}, \ldots, U_n\left(\frac{\Theta_n}{\sum_{j=1}^n |U_j|^{\alpha_j}}\right)^{1/\alpha_n}\right)$$

almost surely. Writing out (4.6) for two positive integers n and k, $k < n$, we have

$$(4.7)\qquad \Theta_n = \left(\frac{\sum_{j=1}^n |U_j|^{\alpha_j}}{\sum_{j=1}^k |U_j|^{\alpha_j}}\right)\Theta_k$$

almost surely. Next, we let $n \to \infty$ with k fixed, and apply the result (in Remark 4.7) that $\sum_{i=1}^n |U_i|^{\alpha_i} \to 1$ almost surely. Then the right-hand side of (4.6) converges almost surely. Therefore there exists a random variable Θ such that Θ_n converges to Θ almost surely. Then (4.7) implies

$$\Theta_k = \Theta \sum_{j=1}^{k} |U_j|^{\alpha_j}$$

almost surely. Consequently, it follows from (4.6) (with n replaced by k) that

$$(X_1, \ldots, X_k) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k})$$

almost surely.

It remains to prove that Θ is independent of $\{U_i\colon i \in N\}$. To see this, consider $(X_1, \ldots, X_k)$ as a k-dimensional marginal subvector of the random vector

$$(X_1, \ldots, X_n) \sim \mathscr{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta_n), \qquad k < n.$$

By Proposition 3.4, the density function of Θ_k is

$$g_k(r) = \frac{\Gamma(p_n)}{\Gamma(p_k)\Gamma(p_n - p_k)} r^{p_k - 1} \int_0^\infty (s-r)_+^{p_n - p_k - 1} s^{1-p_n} \lambda_n(ds) \qquad \text{for } r > 0,$$

where λ_n is the distribution function of Θ_n. Letting $n \to \infty$, since $\Theta_n \to \Theta$, we get $\lambda_n \to \lambda$, where λ is the distribution function of Θ. Since $p_n \to p$, letting $n \to \infty$ in the above formula for g_k, we obtain

$$g_k(r) = \frac{\Gamma(p)}{\Gamma(p_k)\Gamma(p - p_k)} r^{p_k - 1} \int_0^\infty (s-r)_+^{p - p_k - 1} s^{1-p} \lambda(ds)$$

for $r > 0$. Then the joint density function of $(X_1, \ldots, X_k)$ is

$$\frac{\Gamma(p)}{\Gamma(p-p_k)} \prod_1^k \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \int_0^\infty \Big(s - \sum_{i=1}^k |x_i|^{\alpha_i}\Big)_+^{p - p_k - 1} s^{1-p} \lambda(ds)$$

for $x_1, \ldots, x_n \in \boldsymbol{R}$; and by Remark 4.7 this is also the marginal density of $(U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k})$, where Θ is independent of $\{U_i\colon i \in N\}$. Since k was chosen arbitrarily, the proof is complete. ■

REFERENCES

[1] S. M. Berman, *Second order random fields over l_p with homogeneous and isotropic increments*, Z. Wahrsch. verw. Gebiete 12 (1969), pp. 107–126.

[2] — *Stationarity, isotropy and sphericity in l_p*, ibidem 54 (1980), pp. 21–23.

[3] M. A. Chmielewski, *Elliptically symmetric distributions: A review and bibliography*, Internat. Statist. Rev. 49 (1981), pp. 67–74.

[4] K.-T. Fang and B.-Q. Fang, *Generalized symmetrized Dirichlet distributions*, Acta Math. Appl. Sinica 4 (1988), pp. 316–322.

[5] K.-T. Fang, S. Kotz and K. W. Ng, *Symmetric Multivariate and Related Distributions*, Chapman & Hall, London 1990.

[6] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York 1966.

[7] R. D. Gupta and D. St. P. Richards, *Multivariate Liouville distributions*, J. Multivariate Anal. 23 (1987), pp. 233–256.

[8] — *The Dirichlet distributions and polynomial regression*, ibidem 32 (1990), pp. 95–102.

[9] — *Multivariate Liouville distributions, II*, Probab. Math. Statist. 12 (1991), pp. 291–309.

[10] — *Multivariate Liouville distributions, III*, J. Multivariate Anal. 43 (1992), pp. 29–57.

[11] N. L. Johnson and S. Kotz, *Distributions in Statistics: Continuous Multivariate Distributions*, Wiley, New York 1972.

[12] D. Kelker, *Distribution theory of spherical distributions and a location-scale parameter generalization*, Sankhyā, Ser. A, 32 (1970), pp. 419–430.

[13] J. K. Misiewicz, *Elliptically contoured measures on $\boldsymbol{R}^\infty$*, Bull. Acad. Polon. Sci. Math. 30 (1982), pp. 283–290.

[14] — *Characterization of the elliptically contoured measures on infinite-dimensional Banach spaces*, Probab. Math. Statist. 4 (1984), pp. 47–56.

[15] — and R. Cooke, *l_p-invariant probability measures*, Report 88–91, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1988.

[16] J. Osiewalski and M. F. J. Steel, *Robust Bayesian inference in l_q-spherical models*, Biometrika 80 (1993), pp. 456–460.

[17] I. J. Schoenberg, *Metric spaces and completely monotonic functions*, Ann. of Math. 38 (1938), pp. 811–841.

[18] B. D. Sivazlian, *A class of multivariate distributions*, Austral. J. Statist. 23 (1981), pp. 251–254.

[19] — *On a multivariate extension of the gamma and beta distributions*, SIAM J. Appl. Math. 41 (1981), pp. 205–209.

[20] R. E. Williamson, *Multiply monotone functions and their Laplace transforms*, Duke Math. J. 23 (1956), pp. 189–207.

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