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ON THE FRACTIONAL ANISOTROPIC WIENER FIELD*

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Abstract. In this paper we study the local properties of the fractional anisotropic Wiener field $\{B^{(\alpha)}(t): t \in R^d\}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $0 < \alpha_i < 2$. It is proved that, with probability 1, the realizations of the field $B^{(\alpha)}$ over any cube $Q \subset R^d$ belong to the anisotropic Hölder class with parameter $\boldsymbol{\alpha}/2$ in the Orlicz norm corresponding to the Young function $\mathcal{M}_2 = \exp(t^2)-1$. Other supporting spaces are treated as well. Moreover, the box dimension of the graph of the realization of $B^{(\alpha)}$ has been calculated; it is proved that, with probability 1, the box dimension of the graph of the realization of $B^{(\alpha)}$ over any cube $Q \subset R^d$ is equal to $d+1-\kappa/2$, where $\kappa = \min(\alpha_1, \ldots, \alpha_d)$.

1. Introduction. By the *fractional anisotropic Wiener field* with the multidimensional parameter $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $0 < \alpha_i < 2$, we mean a Gaussian field $\{B^{(\alpha)}(t): t \in R^d\}$, with continuous realizations, $\mathrm{E}B^{(\alpha)}(t) = 0$, and the covariance kernel

$$\mathrm{E}B^{(\alpha)}(t)\,B^{(\alpha)}(s) = K_{\boldsymbol{\alpha}}(t, s), \qquad \text{where } K_{\boldsymbol{\alpha}} = K_{\alpha_1} \otimes \ldots \otimes K_{\alpha_d},$$

and for $0 < \alpha < 2$, K_α is the covariance kernel of one-dimensional fractional Brownian motion with parameter α, i.e.

$$K_\alpha(t, s) = \tfrac{1}{2}(|t|^\alpha + |s|^\alpha - |t-s|^\alpha).$$

The aim of this paper is to study the local properties of $B^{(\alpha)}$. It is proved that, with probability 1, the restrictions of realizations of $B^{(\alpha)}(\cdot)$ to any cube $Q \subset R^d$ fulfill multiply Hölder conditions with parameter $\boldsymbol{\alpha}/2$ in the Orlicz norm corresponding to the Young function $\mathcal{M}_2 = \exp(t^2)-1$. The detailed calculations are presented for the cube I^d. The same arguments applied to the dilated and shifted field $\{\varrho^{|\alpha|} B^{(\alpha)}(\varrho^{-2} t - c): t \in I^d\}$ $(\varrho > 0, c \in R^d)$ give the result for the arbitrary cube $Q \subset R^d$.

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The one-dimensional problem was recently discussed in the papers [1] (for $\alpha = 1$) and [5] (for all $0 < \alpha < 2$). The analogous problem for the isotropic fractional Lévy's field on R^d was considered in [3], and the case of the fractional Lévy's field on the d-dimensional sphere was studied in [4].

The method used to obtain the results for the field $B^{(\alpha)}(\cdot)$ (Theorem 2.1) reminds the method from the papers mentioned above: at first, we obtain the characterization of the function spaces in terms of the coefficients of the expansion of a function in some basis (here we consider the basis consisting of tensor products of Schauder functions), and then we prove that the coefficients of the expansion of $B^{(\alpha)}$ satisfy these conditions with probability 1.

In the last part of the paper (Section 5, Theorem 5.1) the box dimension of the graph of the realization of $B^{(\alpha)}$ is calculated. The upper estimate follows from the regularity of $B^{(\alpha)}$, but to get the lower estimate we have to study the coefficients of the expansion of $B^{(\alpha)}$ in the so-called multiaffine (or diamond) basis. This method comes from [2], and was used in [3] and [4] to calculate the box dimension of the graph of the realization of the isotropic fractional Lévy's field on R^d and on the sphere.

2. Function spaces and fractional Wiener field. Let us start with some notation: $I = [0, 1]$ and for $d \in N = \{1, 2, \ldots\}$ put $\mathscr{D} = \{1, \ldots, d\}$; given a vector $\boldsymbol{a} = (a_1, \ldots, a_d) \in R^d$ and $A \subset \mathscr{D}$ put $\boldsymbol{a}(A) = (\tilde{a}_1, \ldots, \tilde{a}_d)$, where $\tilde{a}_i = a_i$ if $i \in A$, and $\tilde{a}_i = 0$ if $i \notin A$; in addition, put $|\boldsymbol{a}| = |a_1| + \ldots + |a_d|$. Moreover, for two vectors $\boldsymbol{a} = (a_1, \ldots, a_d) \in R^d$ and $\boldsymbol{b} = (b_1, \ldots, b_d) \in R^d$ we write

$$\boldsymbol{a} \leqslant \boldsymbol{b} \text{ iff } a_i \leqslant b_i \text{ for all } i \in \mathscr{D} \quad \text{and} \quad \boldsymbol{a} < \boldsymbol{b} \text{ iff } a_i < b_i \text{ for all } i \in \mathscr{D};$$

in addition, we use

$$\boldsymbol{a}^{\boldsymbol{b}} = \prod_{i=1}^{d} a_i^{b_i} \quad \text{and} \quad \frac{1}{\boldsymbol{a}} = \left(\frac{1}{a_1}, \ldots, \frac{1}{a_d}\right).$$

We will also need the notation

$$\boldsymbol{0} = (0, \ldots, 0) \in R^d, \quad \boldsymbol{1} = (1, \ldots, 1) \in R^d.$$

By $L_p(I^d)$, $1 \leqslant p < \infty$, we denote the space of functions integrable on I^d with exponent p, and $C(I^d)$ is the space of continuous functions on I^d. By $L_{\mathscr{M}}(I^d)$ we denote the Orlicz space on I^d, corresponding to the Young function $\mathscr{M}$, with the norm

$$\|f\|_{\mathscr{M}} = \sup\Big\{\int_{I^d} f(x)g(x)\,dx\colon \int_{I^d} \mathscr{M}^*\big(g(x)\big)\,dx \leqslant 1\Big\},$$

where $\mathscr{M}^*$ is the complementary Young function to $\mathscr{M}$. For the general theory of Orlicz spaces we refer, e.g., to [11].

We are interested in some special family of Young functions. Namely, let for $\gamma > 0$

$$\mathcal{M}_\gamma(u) = \begin{cases} \exp\{|u|^\gamma\}-1 & \text{for } 1 \leqslant \gamma < \infty, \\ E_\gamma(u)-E_\gamma(0) & \text{for } 0 < \gamma < 1, \end{cases}$$

where $E_\gamma(-u) = E_\gamma(u)$ is the extension of the convex part of $\exp\{u^\gamma\}$ on (u_γ, ∞) by its tangent line at $u_\gamma > 0$, and u_γ is the point at which the function $\exp\{u^\gamma\}$ changes the concavity to the convexity. For these Young functions there is an equivalent norm on $L_{\mathcal{M}_\gamma}(I^d)$:

(1) $$\|f\|^*_{\mathcal{M}_\gamma} = \sup_{p \geqslant 1} \frac{\|f\|_p}{p^{1/\gamma}}.$$

For the equivalence of the norms $\|\cdot\|_{\mathcal{M}_\gamma}$ and $\|\cdot\|^*_{\mathcal{M}_\gamma}$ see [8] or [1].

For $f\colon I^d \to R$, $i \in \mathcal{D}$ and $h \in R$, the progressive difference in direction e_i (where $e_i = (\delta_{1,i}, \ldots, \delta_{d,i}) \in R^d$ denotes the i-th coordinate vector in R^d) is defined by the standard formula

$$\Delta_{h,i} f(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}+he_i)-f(\boldsymbol{x}) & \text{if } \boldsymbol{x},\ \boldsymbol{x}+he_i \in I^d, \\ 0 & \text{if } \boldsymbol{x} \in I^d, \text{ but } \boldsymbol{x}+he_i \notin I^d. \end{cases}$$

For $\boldsymbol{h} = (h_1, \ldots, h_d) \in R^d$ and $A = \{i_1, \ldots, i_k\} \subset \mathcal{D}$ we set

$$\Delta_{\boldsymbol{h},A} f = \Delta_{h_{i_1},i_1} \circ \ldots \circ \Delta_{h_{i_k},i_k} f.$$

For $f \in L_p(I^d)$, $1 \leqslant p < \infty$, or $f \in C(I^d)$ if $p = \infty$, the *moduli of smoothness in the* L_p*- and* $L_{\mathcal{M}_\gamma}$*-norms in the directions* A are defined as follows:

$$\omega_{p,A}(f, \boldsymbol{t}) = \sup_{0<\boldsymbol{h}\leqslant \boldsymbol{t}} \|\Delta_{\boldsymbol{h},A} f\|_p \quad \text{for } \boldsymbol{t} \in R^d,\ \boldsymbol{0} < \boldsymbol{t} \leqslant \boldsymbol{1},$$

$$\omega_{\mathcal{M}_\gamma,A}(f, \boldsymbol{t}) = \sup_{0<\boldsymbol{h}\leqslant \boldsymbol{t}} \|\Delta_{\boldsymbol{h},A} f\|_{\mathcal{M}_\gamma} \quad \text{for } \boldsymbol{t} \in R^d,\ \boldsymbol{0} < \boldsymbol{t} \leqslant \boldsymbol{1}.$$

It follows from the equivalence of the norms $\|\cdot\|_{\mathcal{M}_\gamma}$ and $\|\cdot\|^*_{\mathcal{M}_\gamma}$ that

$$\omega_{\mathcal{M}_\gamma,A}(f, \boldsymbol{t}) \sim \sup_{p \geqslant 1} \frac{\omega_{p,A}(f, \boldsymbol{t})}{p^{1/\gamma}}.$$

Now let $\boldsymbol{0} < \boldsymbol{\beta} < \boldsymbol{1}$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)$, and $\lambda \in R$. Define

(2) $$\omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}) = \boldsymbol{t}^{\boldsymbol{\beta}}\left(1+\ln\frac{1}{\boldsymbol{t}}\right)^\lambda = \prod_{i=1}^{d} t_i^{\beta_i}\left(1+\sum_{i=1}^{d} \ln\frac{1}{t_i}\right)^\lambda.$$

We are going to consider some anisotropic generalized Hölder classes in the L_p- and $L_{\mathcal{M}_\gamma}$-norms, described in terms of $\omega_{p,A}(f, \boldsymbol{t})$, $\omega_{\mathcal{M}_\gamma,A}(f, \boldsymbol{t})$ and $\omega_{\boldsymbol{\beta},\lambda}(\cdot)$. More precisely, let for a function $\psi\colon [0, 1]^d \to R$, $A \subset \mathcal{D}$, and $\boldsymbol{t} \in [0, 1]^d$,

$$\psi(\boldsymbol{t}; A) = \psi\big(\boldsymbol{t}(A)+\boldsymbol{1}(\mathcal{D}\setminus A)\big).$$

The anisotropic Hölder classes in the L_p- and $L_{\mathcal{M}_\gamma}$-norms are now defined as follows:

$$\mathrm{Lip}_p(\boldsymbol{\beta}, \lambda) = \{f \in L_p(I^d)\colon \ \forall (\emptyset \neq A \subset \mathcal{D}) \ \omega_{p,A}(\boldsymbol{t}, f) = O(\omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}; A))\},$$
$$\mathrm{lip}_p(\boldsymbol{\beta}, \lambda) = \{f \in \mathrm{Lip}_p(\boldsymbol{\beta}, \lambda)\colon \ \forall (\emptyset \neq A \subset \mathcal{D}) \ \omega_{p,A}(\boldsymbol{t}, f) = o(\omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}; A))\},$$
$$\mathrm{Lip}_{\mathcal{M}_\gamma}(\boldsymbol{\beta}, \lambda) = \{f \in L_{\mathcal{M}_\gamma}(I^d)\colon \ \forall (\emptyset \neq A \subset \mathcal{D}) \ \omega_{\mathcal{M}_\gamma,A}(\boldsymbol{t}, f) = O(\omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}; A))\},$$
$$\mathrm{lip}_{\mathcal{M}_\gamma}(\boldsymbol{\beta}, \lambda) = \{f \in \mathrm{Lip}_{\mathcal{M}_\gamma}(\boldsymbol{\beta}, \lambda)\colon \ \forall (\emptyset \neq A \subset \mathcal{D}) \ \omega_{\mathcal{M}_\gamma,A}(\boldsymbol{t}, f) = o(\omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}; A))\},$$

where $O(\boldsymbol{t}(A))$ and $o(\boldsymbol{t}(A))$ refer to $\min(t_i\colon i \in A) \to 0$,

$$\mathrm{lip}^*_\gamma(\boldsymbol{\beta}, \lambda) = \{f \in \mathrm{Lip}_{\mathcal{M}_\gamma}(\boldsymbol{\beta}, \lambda)\colon \ \|f\|_p = o(p^{1/\gamma}) \text{ as } p \to \infty,$$
$$\forall (\emptyset \neq A \subset \mathcal{D}) \ \omega_{p,A}(\boldsymbol{t}, f) = o(p^{1/\gamma}\, \omega_{\boldsymbol{\beta},\lambda}(\boldsymbol{t}; A)) \text{ as } \min(t_i\colon i \in A, 1/p) \to 0\}.$$

The following theorem presents the results on the supporting function spaces for the fractional anisotropic Wiener field $B^{(\boldsymbol{\alpha})}$.

THEOREM 2.1. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $0 < \alpha_i < 2$ *for* $i = 1, \ldots, d$, *and* $1 \leqslant p < \infty$, $1/p < \alpha_i/2$ *for* $i = 1, \ldots, d$; *then*

$$\Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \in \mathrm{Lip}_p(\boldsymbol{\alpha}/2, 0)\} = 1, \qquad \Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \notin \mathrm{lip}_p(\boldsymbol{\alpha}/2, 0)\} = 1,$$
$$\Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \in \mathrm{Lip}_{\mathcal{M}_2}(\boldsymbol{\alpha}/2, 0)\} = 1, \qquad \Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \notin \mathrm{lip}_{\mathcal{M}_2}(\boldsymbol{\alpha}/2, 0)\} = 1,$$
$$\Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \in \mathrm{Lip}_\infty(\boldsymbol{\alpha}/2, 1/2)\} = 1, \qquad \Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \notin \mathrm{lip}_\infty(\boldsymbol{\alpha}/2, 1/2)\} = 1,$$
$$\Pr\{B^{(\boldsymbol{\alpha})}|_{I^d} \in \mathrm{lip}^*_2(\boldsymbol{\alpha}/2, 1/2)\} = 1.$$

The idea of proof of Theorem 2.1 is the following: there are some characterizations of the anisotropic Hölder classes in the L_p- and $L_{\mathcal{M}_\gamma}$-norms by the coefficients of the expansion of a function in the basis consisting of tensor products of Schauder functions (these results are presented in Section 3). Then we prove that the coefficients of $B^{(\boldsymbol{\alpha})}$ in this basis fulfill, with probability 1, the conditions required in these characterizations (Section 4). Putting these results together gives the proof of Theorem 2.1.

COROLLARY 2.2. *Applying the method of proof of Theorem* 2.1 *to the shifted and dilated field* $\{\varrho^{|\boldsymbol{\alpha}|} B^{(\boldsymbol{\alpha})}(\varrho^{-2}\boldsymbol{t} - \boldsymbol{c})\colon \boldsymbol{t} \in I^d\}$ $(\varrho > 0, \boldsymbol{c} \in R^d)$, *we can prove that*

$$\Pr\{\forall (Q \subset R^d) \ B^{(\boldsymbol{\alpha})}|_Q \in \mathrm{Lip}_p(\boldsymbol{\alpha}/2, 0)(Q), \ B^{(\boldsymbol{\alpha})}|_Q \notin \mathrm{lip}_p(\boldsymbol{\alpha}/2, 0)(Q)\} = 1,$$
$$\Pr\{\forall (Q \subset R^d) \ B^{(\boldsymbol{\alpha})}|_Q \in \mathrm{Lip}_{\mathcal{M}_2}(\boldsymbol{\alpha}/2, 0)(Q), \ B^{(\boldsymbol{\alpha})}|_Q \notin \mathrm{lip}_{\mathcal{M}_2}(\boldsymbol{\alpha}/2, 0)(Q)\} = 1,$$
$$\Pr\{\forall (Q \subset R^d) \ B^{(\boldsymbol{\alpha})}|_Q \in \mathrm{Lip}_\infty(\boldsymbol{\alpha}/2, 1/2)(Q), \ B^{(\boldsymbol{\alpha})}|_Q \notin \mathrm{lip}_\infty(\boldsymbol{\alpha}/2, 1/2)(Q)\} = 1,$$
$$\Pr\{\forall (Q \subset R^d) \ B^{(\boldsymbol{\alpha})}|_Q \in \mathrm{lip}^*_2(\boldsymbol{\alpha}/2, 1/2)(Q)\} = 1,$$

where $Q \subset R^d$ *is a cube in* R^d, *the function spaces appearing above are the function spaces over the cube* Q, *and they are defined in the same way as the function spaces over* I^d.

3. The characterization of function spaces. Now we present the characterization of the anisotropic Hölder classes in the L_p- and $L_{\mathcal{M}_\gamma}$-norms in terms of the coefficients of the expansion of a function in the basis consisting of tensor products of Schauder functions.

Let $\{\phi_k, k \geqslant 0\}$ be the family of Schauder functions on I, normed in L_∞, i.e. $\phi_0(t) = 1$, $\phi_1(t) = t$, and for $k \geqslant 2$, $k = 2^j + n$ with $j \geqslant 0$ and $1 \leqslant n \leqslant 2^j$

$$\phi_k(t) = \max(0, 1 - |2^{j+1} t - 2n + 1|).$$

In several dimensions, we consider the family $\{\phi_{\boldsymbol{k}}, \boldsymbol{k} \geqslant \boldsymbol{0}\}$ of tensor products of Schauder functions, i.e. $\phi_{\boldsymbol{k}} = \phi_{k_1} \otimes \ldots \otimes \phi_{k_d}$ for $\boldsymbol{k} = (k_1, \ldots, k_d)$.

To describe the anisotropic Hölder classes in terms of the coefficients of a function in the basis $\{\phi_{\boldsymbol{k}}, \boldsymbol{k} \geqslant \boldsymbol{0}\}$, the following decomposition of the set of indices is needed. Let for $j \in M = \{-2, -1, 0, 1, \ldots\}$

$$\tilde{N}_j = \begin{cases} \{j+2\} & \text{for } j = -2 \text{ or } j = -1, \\ \{2^j + n\colon\ n = 1, \ldots, 2^j\} & \text{for } j \geqslant 0, \end{cases}$$

and for a vector $\boldsymbol{j} = (j_1, \ldots, j_d)$ we put

$$\tilde{N}_{\boldsymbol{j}} = \tilde{N}_{j_1} \times \ldots \times \tilde{N}_{j_d}.$$

The formulae for the coefficients of a continuous function $f \in C(I^d)$ in the basis $\{\phi_{\boldsymbol{k}}, \boldsymbol{k} \geqslant \boldsymbol{0}\}$ will be needed. Let for $f \in C(I^d)$, $i \in \mathcal{D}$, $\boldsymbol{x} \in I^d$ and $k \geqslant 0$

$$c_{i,k}(f)(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x} - x_i \boldsymbol{e}_i) & \text{for } k = 0, \\ f(\boldsymbol{x} + (1 - x_i)\boldsymbol{e}_i) - f(\boldsymbol{x} - x_i \boldsymbol{e}_i) & \text{for } k = 1, \end{cases}$$

and for $k \in \tilde{N}_j$ with $j \geqslant 0$, $k = 2^j + n$

$$c_{i,k}(f)(\boldsymbol{x}) = f\left(\boldsymbol{x} + \left(\frac{2n-1}{2^{j+1}} - x_i\right)\boldsymbol{e}_i\right) - \frac{f\left(\boldsymbol{x} + ((n-1)/2^j - x_i)\boldsymbol{e}_i\right) + f\left(\boldsymbol{x} + (n/2^j - x_i)\boldsymbol{e}_i\right)}{2}.$$

For $\boldsymbol{k} = (k_1, \ldots, k_d)$ we put

$$c_{\boldsymbol{k}}(f) = c_{1,k_1} \circ \ldots \circ c_{d,k_d}(f).$$

Then for any $f \in C(I^d)$ we have

$$f = \sum_{\boldsymbol{j} \in M^d} \sum_{\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}} c_{\boldsymbol{k}}(f)\, \phi_{\boldsymbol{k}}. \tag{3}$$

Remark. Each time when we write the sum of the d-dimensional set of indices M^d we mean that this set is ordered in such a way that, for $\boldsymbol{j}, \boldsymbol{j}' \in M^d$, $\boldsymbol{j} = (j_1, \ldots, j_d)$, $\boldsymbol{j}' = (j'_1, \ldots, j'_d)$, if $\max(j_1, \ldots, j_d) < \max(j'_1, \ldots, j'_d)$, then $\boldsymbol{j}$ precedes $\boldsymbol{j}'$.

For f given by (3) we put

$$\tau_{j,p}(f) = 2^{-|j|/p}\Big(\sum_{k\in\tilde N_j} |c_k(f)|^p\Big)^{1/p}, \qquad \tau_{j,\mathcal{M}_\gamma}(f) = \sup_{p\geq 1} \frac{\tau_{j,p}(f)}{p^{1/\gamma}}.$$

Now we can formulate the characterization of the Hölder classes in terms of the coefficients $\{c_{\boldsymbol k}(f)\colon \boldsymbol{k} \geq \boldsymbol{0}\}$.

LEMMA 3.1. *Let $\boldsymbol 0 < \boldsymbol\beta < \boldsymbol 1$, $\lambda \in R$, and let the function $\omega_{\beta,\lambda}$ be defined as in (2). Moreover, let*

$$\boldsymbol t_j = (2^{-\max(j_1,0)}, \ldots, 2^{-\max(j_d,0)}).$$

Let $1 \leq p \leq \infty$ be such that $1/p < \beta_i$ for all $i = 1, \ldots, d$. Then

$$f \in \mathrm{Lip}_p(\boldsymbol\beta, \lambda) \quad \textit{iff} \quad \tau_{j,p}(f) = O\big(\omega_{\beta,\lambda}(\boldsymbol t_j)\big) \textit{ as } |j| \to \infty,$$

$$f \in \mathrm{lip}_p(\boldsymbol\beta, \lambda) \quad \textit{iff} \quad \tau_{j,p}(f) = o\big(\omega_{\beta,\lambda}(\boldsymbol t_j)\big) \textit{ as } |j| \to \infty.$$

Moreover, for any $0 < \gamma < \infty$

$$f \in \mathrm{Lip}_{\mathcal{M}_\gamma}(\boldsymbol\beta, \lambda) \quad \textit{iff} \quad \tau_{j,\mathcal{M}_\gamma}(f) = O\big(\omega_{\beta,\lambda}(\boldsymbol t_j)\big) \textit{ as } |j| \to \infty,$$

$$f \in \mathrm{lip}_{\mathcal{M}_\gamma}(\boldsymbol\beta, \lambda) \quad \textit{iff} \quad \tau_{j,\mathcal{M}_\gamma}(f) = o\big(\omega_{\beta,\lambda}(\boldsymbol t_j)\big) \textit{ as } |j| \to \infty,$$

$$f \in \mathrm{lip}^*_\gamma(\boldsymbol\beta, \lambda) \quad \textit{iff} \quad \tau_{j,p}(f) = o\big(\omega_{\beta,\lambda}(\boldsymbol t_j)\big) \textit{ as } \max(p, |j|) \to \infty.$$

Proof. For the L_p-norm and $\lambda = 0$ this lemma was proved in [9], and the proof for other cases follows the same idea. For the sake of completeness we present here the sketch of the proof.

Let $\{f_k, k \geq 0\}$ denote the Franklin system on I, i.e. $\{f_k, k \geq 0\}$ is the system obtained by the Gram–Schmidt orthonormalization (in $L_2(I)$) of Schauder functions $\{\phi_k, k \geq 0\}$, and let $\{f_{\boldsymbol k}\colon \boldsymbol k \geq \boldsymbol 0\}$ be the family of tensor products of Franklin functions. Let for $f \in C(I^d)$

$$\eta_{j,p}(f) = 2^{|j|(1/2-1/p)}\Big(\sum_{k\in\tilde N_j} |(f, f_k)|^p\Big)^{1/p}.$$

It was proved in [9] that there exists a constant $C > 0$, independent of f and p, such that

$$\eta_{j,p}(f) \leq C\omega_{p,A_j}(f, \boldsymbol t_j),$$

where $A_j = \{i\colon j_i \geq 0\}$, and for any $\emptyset \neq A \subset \mathcal{D}$

$$\omega_{p,A}(f, \boldsymbol t_\mu) \leq C\boldsymbol t_\mu^{I(A)} \sum_{j\in M^d} \eta_{j,p}(f) \prod_{i\in A} 2^{\min(\mu_i, j_i)},$$

$$\eta_{j,p}(f) \leq C \sum_{\xi\geq j} \tau_{\xi,p}(f), \qquad \tau_{j,p}(f) \leq C \sum_{\xi\geq j} 2^{(|\xi|-|j|)/p}\eta_{\xi,p}(f).$$

The required characterizations for the L_p-norm follow now from these inequalities. The characterizations for the $L_{\mathcal{M}_\gamma}$-norm follow from the above inequalities and the equivalence (1). ■

4. The asymptotics of the basic coefficients of $B^{(\alpha)}$. Let $\{B^{(\alpha)}(t)\colon t \in I^d\}$ be the fractional anisotropic Wiener field with parameter $\alpha = (\alpha_1, \ldots, \alpha_d)$, and let $\{b_k^{(\alpha)}, k \geqslant 0\}$ be a sequence of the coefficients of $B^{(\alpha)}$ in the tensor product Schauder basis, i.e.

$$B^{(\alpha)} = \sum_{j \in M^d} \sum_{k \in \tilde{N}_j} b_k^{(\alpha)} \phi_k.$$

LEMMA 4.1. *The sequence $\{b_k^{(\alpha)}, k \geqslant 0\}$ is a Gaussian sequence, with $\mathrm{E} b_k^{(\alpha)} = 0$ and the variance given by the formula*

$$\mathrm{E}\, |b_k^{(\alpha)}|^2 = \prod_{i=1}^{d} a_{k_i}^{(\alpha_i)}, \tag{4}$$

where for $0 < \alpha < 2$

$$a_k^{(\alpha)} = \begin{cases} 0 & \text{for } k = 0, \\ 1 & \text{for } k = 1, \\ (2^{-\alpha} - 2^{-2}) 2^{-j\alpha} & \text{for } k \in \tilde{N}_j, j \geqslant 0. \end{cases}$$

Moreover, there exists $C > 0$ such that for all j and $k, l \in \tilde{N}_j$

$$|\mathrm{E} b_k^{(\alpha)} b_l^{(\alpha)}| \leqslant C \frac{1}{2^{j \cdot \alpha}} R_\alpha(k - l), \tag{5}$$

where $j \cdot \alpha = j_1 \alpha_1 + \ldots + j_d \alpha_d$, and

$$R_\alpha(t) = \prod_{i=1}^{d} \frac{1}{1 + |t_i|^{4 - \alpha_i}}.$$

Proof. These estimates follow from the formulae for the coefficients of a function in the tensor product Schauder basis, the formula for the covariance of the field $B^{(\alpha)}$ and the estimates for the progressive difference of order 4 with the step 1 of the function $|\cdot + n|^\alpha$, $0 < \alpha < 2$ (cf. lemme IV.2 of [5]). ∎

Now the asymptotic behaviour of the sequence $\{b_k^{(\alpha)}, k \geqslant 0\}$ will be studied. Let us note that if $k = (k_1, \ldots, k_d)$ with $k_i = 0$ for some $i \in \mathscr{D}$, then $\Pr\{b_k^{(\alpha)} = 0\} = 1$. For $k > 0$ let us introduce

$$g_k = \frac{b_k^{(\alpha)}}{\sqrt{\mathrm{E}\, |b_k^{(\alpha)}|^2}}.$$

Moreover, let us put $n_j = \# \tilde{N}_j$, $\mu_p = \mathrm{E}\, |g|^p$, where $g \in N(0, 1)$,

$$G(j, p) = \frac{1}{n_j} \sum_{k \in \tilde{N}_j} |g_k|^p,$$

and let $\tilde{M} = \{-1, 0, 1, \ldots\}$.

LEMMA 4.2. *For each* p, $1 \leqslant p < \infty$,

$$\Pr\{G(j, p) \to \mu_p \text{ as } |j| \to \infty, j \in \tilde{M}^d\} = 1.$$

Proof. Let $\varepsilon > 0$ and $j \in \tilde{M}^d$ be given. Then

$$\Pr\{|G(j, p) - \mu_p| > \varepsilon\} \leqslant \frac{1}{(n_j \varepsilon)^2} \mathrm{E}\Big(\sum_{k \in \tilde{N}_j} (|g_k|^p - \mu_p)\Big)^2$$
$$= \frac{1}{(n_j \varepsilon)^2} \sum_{k, k' \in \tilde{N}_j} \mathrm{E}(|g_k|^p - \mu_p)(|g_{k'}|^p - \mu_p).$$

As the random vector $(g_k, g_{k'})$ has a normal distribution, we get from lemme II.2 of [5] (or Theorem 4.6 of [1]; actually, it is equivalent to Gebelein's inequality, cf. [6], p. 66) and from the estimates (4) and (5) the inequality

$$|\mathrm{E}(|g_k|^p - \mu_p)(|g_{k'}|^p - \mu_p)| \leqslant C(\mu_{2p} - \mu_p^2) R_\alpha(k - l). \tag{6}$$

Let us observe that

$$\sum_{k, k' \in \tilde{N}_j} R_\alpha(k - l) \sim n_j, \tag{7}$$

which implies

$$\Pr\{|G(j, p) - \mu_p| > \varepsilon\} \leqslant C \frac{\mu_{2p} - \mu_p^2}{\varepsilon^2} \frac{1}{n_j}.$$

As $n_j \sim 2^{|j|}$, Lemma 4.2 follows from the last inequality and Borel–Cantelli lemma. ■

COROLLARY 4.3. *For each* $1 \leqslant p < \infty$ *we have*

$$\Pr\{\sup_{j \in \tilde{M}^d} (G(j, p))^{1/p} \geqslant \mu_p^{1/p}\} = 1$$

and

$$\Pr\{\sup_{1 \leqslant p < \infty} \sup_{j \in \tilde{M}^d} \frac{1}{\sqrt{p}} (G(j, p))^{1/p} \geqslant \mu_{\exp}\} = 1,$$

where

$$\mu_{\exp} = \sup_{1 \leqslant p < \infty} \frac{1}{\sqrt{p}} \mu_p^{1/p}.$$

LEMMA 4.4. *We have*

$$\Pr\{\sup_{1 \leqslant p < \infty} \sup_{j \in \tilde{M}^d} \frac{1}{\sqrt{p}} (G(j, p))^{1/p} < \infty\} = 1.$$

Proof. Let us observe that

$$\sup_{1 \leqslant p < \infty} \sup_{j \in \tilde{M}^d} \frac{1}{\sqrt{p}} (G(j, p))^{1/p} < \infty \quad \text{iff} \quad \sup_{p \in N} \sup_{j \in \tilde{M}^d} \frac{1}{\sqrt{p}} (G(j, 2p))^{1/(2p)} < \infty.$$

Let $p \in N$, $j \in \tilde{M}^d$, $\zeta \in R$, $\zeta \geqslant \zeta_0$ (where $\zeta_0 > 0$ is chosen so that for all $p \in N$ and $\zeta \geqslant \zeta_0$ we have $(\sqrt{p}\zeta)^{2p} - \mu_{2p} \geqslant \frac{1}{2}(\sqrt{p}\zeta)^{2p}$) be given. Then

$$\Pr\left\{\frac{1}{\sqrt{p}}(G(j, 2p))^{1/(2p)} > \zeta\right\} \leqslant \Pr\{|G(j, 2p) - \mu_{2p}| \geqslant (\sqrt{p}\zeta)^{2p} - \mu_{2p}\}$$

$$\leqslant \Pr\{|G(j, 2p) - \mu_{2p}| \geqslant \tfrac{1}{2}(\sqrt{p}\zeta)^{2p}\}.$$

Using the Tchebyshev inequality, (6) and (7) we obtain

$$\Pr\left\{\frac{1}{\sqrt{p}}(G(j, 2p))^{1/(2p)} > \zeta\right\}$$

$$\leqslant \frac{4}{(\sqrt{p}\zeta)^{4p}} \frac{1}{n_j^2} \sum_{k,k' \in \tilde{N}_j} \left|\mathrm{E}\,(|g_k|^{2p} - \mu_{2p})(|g_{k'}|^{2p} - \mu_{2p})\right| \leqslant C \frac{\mu_{4p} - \mu_{2p}^2}{(\sqrt{p}\zeta)^{4p}} \frac{1}{n_j}.$$

As $\mu_{2p} = (2p)!/(p!\,2^p)$ and $p! \sim (2\pi n)^{1/2}(n/e)^n$, we get

$$\Pr\left\{\frac{1}{\sqrt{p}}(G(j, 2p))^{1/(2p)} > \zeta\right\} \leqslant C \frac{1}{n_j}\left(\frac{4}{e\zeta^2}\right)^{2p}.$$

Let $\zeta > \sqrt{4/e}$; then

$$\sum_{p \in N} \sum_{j \in \tilde{M}^d} \frac{1}{n_j}\left(\frac{4}{e\zeta^2}\right)^{2p} < \infty.$$

Now the last inequality and the Borel–Cantelli lemma complete the proof of Lemma 4.4. ■

LEMMA 4.5. *There exists* $C > 0$ *such that*

$$\Pr\left\{\varliminf_{|j| \to \infty, j \in \tilde{M}^d} \frac{\sup_{k \in \tilde{N}_j} |g_k|}{\sqrt{1 + \ln n_j}} \geqslant C\right\} = 1.$$

Moreover,

$$\Pr\left\{\varlimsup_{|j| \to \infty, j \in \tilde{M}^d} \frac{\sup_{k \in \tilde{N}_j} |g_k|}{\sqrt{1 + \ln n_j}} \leqslant \sqrt{2}\right\} = 1.$$

Proof. First, let $c > \sqrt{2}$; then

$$\Pr\left\{\frac{1}{\sqrt{1 + \ln n_j}} \sup_{k \in \tilde{N}_j} |g_k| > c\right\} \leqslant \sum_{k \in \tilde{N}_j} \Pr\{|g_k| > c\sqrt{1 + \ln n_j}\}$$

$$\leqslant n_j \exp\left(-\frac{c^2(1 + \ln n_j)}{2}\right).$$

This gives

$$\sum_{j\in\tilde{M}^d} \Pr\left\{\frac{1}{\sqrt{1+\ln n_j}}\sup_{k\in\tilde{N}_j}|g_k| > c\right\} < \infty,$$

and the second statement follows from the Borel–Cantelli lemma.

To prove the first part of the lemma, let us choose δ, $0<\delta<4-\alpha_i$ for $i=1,\ldots,d$; moreover, for $0<s<1$ let us put

$$\tilde{N}_{j_i}(s) = \begin{cases} \{j_i+1\} & \text{for } j_i<0,\\ \{2^{j_i}+t_i[2^{j_i s}]\colon\ t_i=1,\ldots,[2^{j_i(1-s)}]\} & \text{for } j_i\geqslant 0,\end{cases}$$

and $\tilde{N}_j(s) = \tilde{N}_{j_1}(s)\times\ldots\times\tilde{N}_{j_d}(s)$. Then for all $\boldsymbol{k}, \boldsymbol{k}'\in\tilde{N}_j(s)$ we have

$$|\mathrm{E}g_{\boldsymbol{k}}g_{\boldsymbol{k}'}| \leqslant \varrho_j, \qquad \varrho_j = C_1(1/n_j)^{s\delta}.$$

Putting

$$n_j(s) = \#\tilde{N}_j(s), \qquad z(j) = \frac{n_j(s)}{1+\varrho_j\big(n_j(s)-1\big)}$$

and using Slepian's lemma (cf. [10], p. 74) and lemme II.9 of [5] we get

$$\begin{aligned}\Pr\{\sup_{k\in\tilde{N}_j}|g_k| \leqslant C\sqrt{1+\ln n_j}\} &\leqslant \Pr\{\sup_{k\in\tilde{N}_j(s)} g_k \leqslant C\sqrt{1+\ln n_j}\}\\ &\leqslant \Pr\{g\leqslant C\sqrt{1+\ln n_j}\}^{z(j)} \qquad (\text{where } g\in N(0,1))\\ &\leqslant \left(1-\frac{C\sqrt{1+\ln n_j}}{\sqrt{2\pi}}\exp\big(-2C^2(1+\ln n_j)\big)\right)^{z(j)}.\end{aligned}$$

Let us choose $0<s<1$ and $C>0$ such that

$$\sum_{j\in\tilde{M}^d}\left(1-\frac{C\sqrt{1+\ln n_j}}{\sqrt{2\pi}}\exp\big(-2C^2(1+\ln n_j)\big)\right)^{z(j)} < \infty.$$

Then the first part of the lemma is a consequence of the Borel–Cantelli lemma. ∎

COROLLARY 4.6. *There exists a constant $C>0$ such that*

$$\Pr\left\{C < \sup_{j\in\tilde{M}^d}\sup_{k\in\tilde{N}_j}\frac{|g_k|}{\sqrt{1+\ln n_j}} < \infty\right\} = 1.$$

Let us put

$$H(j,p) = \frac{1}{\sqrt{p(1+\ln n_j)}}\big(G(j,p)\big)^{1/p}.$$

LEMMA 4.7. *We have*

$$\Pr\{\lim_{j\in\tilde{M}^d, \max(p,|j|)\to\infty} H(j,p)=0\}=1.$$

Proof. From Lemma 4.4 we obtain

$$\sup_{j\in\tilde{M}^d}\sup_{1\leqslant p<\infty}\frac{1}{\sqrt{p}}\big(G(j,p)\big)^{1/p}<\infty$$

with probability 1, so

$$\Pr\{\lim_{|j|\to\infty, j\in\tilde{M}^d}\sup_{1\leqslant p<\infty} H(j,p)=0\}=1. \tag{8}$$

It follows from Corollary 4.6 that, with probability 1,

$$\sup_{j\in\tilde{M}^d}\frac{\sup_{k\in\tilde{N}_j}|g_k|}{\sqrt{1+\ln n_j}}<\infty,$$

which implies

$$\Pr\{\lim_{p\to\infty}\sup_{j\in\tilde{M}^d} H(j,p)=0\}=1. \tag{9}$$

The equalities (8) and (9) imply the lemma. ∎

Proof of Theorem 2.1. Theorem 2.1 is now a consequence of the estimate for the variance of $b_k^{(\alpha)}$ from Lemma 4.1, the characterization of anisotropic Hölder classes from Lemma 3.1, and the estimates from Lemmas 4.2, 4.4, 4.5, 4.7 and Corollaries 4.3 and 4.6. ∎

5. The box dimension of the graph of $B^{(\alpha)}(\cdot)$. The box dimension of a bounded subset $F\subset R^{d+1}$ is defined as follows. Let, for $\delta>0$, $\mathcal{N}_\delta(F)$ denote the minimal number of sets of diameter not exceeding δ needed to cover F. Then the *box dimension of* F, denoted by $\dim_b F$ is defined as

$$\dim_b F=\lim_{\delta\to 0}\frac{\log\mathcal{N}_\delta(F)}{\log\delta^{-1}}$$

if this limit exists; otherwise, one can consider the upper and lower box dimensions of F, defined as the upper and lower limits of $(\log\mathcal{N}_\delta(F))/\log\delta^{-1}$ as $\delta\to 0$, and denoted by $\overline{\dim}_b F$ and $\underline{\dim}_b F$, respectively. (For more details cf. [7].)

For the function $f\colon U\mapsto R$, $U\subset R^d$, we denote by $\Gamma(f)$ its graph, i.e.

$$\Gamma(f)=\{(x, f(x))\colon x\in U\}.$$

The following theorem gives the result on the box dimension of the graph of the realization of $B^{(\alpha)}$.

THEOREM 5.1. *Let* $\alpha=(\alpha_1,\ldots,\alpha_d)$, $0<\alpha_i<2$, $\kappa=\min(\alpha_1,\ldots,\alpha_d)$. *Then*

$$\Pr\{\dim_b\Gamma(B^{(\alpha)}|_{I^d})=d+1-\kappa/2\}=1.$$

Proof. Let us note that $E|B^{(\alpha)}(t)-B^{(\alpha)}(s)|^2 \leqslant C\|t-s\|^{\kappa}$ (cf. the formula for K_α), and it follows from the Kolmogorov criterion that

$$\Pr\left\{\forall (0<\varepsilon<\kappa/2) \sup_{t,s\in I^d, t\neq s} \frac{|B^{(\alpha)}(t)-B^{(\alpha)}(s)|}{\|t-s\|^{\varepsilon}} < \infty\right\} = 1$$

(cf. also Theorem 2.1). As for $f\colon I^d \to R$ such that $|f(t)-f(s)| = O(\|t-s\|^{\varepsilon})$ we have

$$\overline{\dim}_b \Gamma(f) \leqslant d+1-\varepsilon,$$

we infer that, with probability 1,

$$\overline{\dim}_b \Gamma(B^{(\alpha)}|_{I^d}) \leqslant d+1-\kappa/2. \tag{10}$$

To obtain the lower estimate, we use the method of calculating the box dimension of the graph of a function from [2], with the use of the coefficients of a function in the so-called diamond basis (for more properties of this basis cf., e.g., [12]).

Let us recall the definition of the diamond basis; let

$$\Psi(x) = \max(0, 1-|x_1|)\cdot\ldots\cdot\max(0, 1-|x_d|).$$

In addition, let us put

$$W_0 = \{k \geqslant 0\colon\ k = (k_1, \ldots, k_d),\ \max(k_1, \ldots, k_d) \leqslant 1\},$$

and for $j > 0$

$$W_j = \{k \geqslant 0\colon\ k = (k_1, \ldots, k_d),\ 2^{j-1} < \max(k_1, \ldots, k_d) \leqslant 2^j\},$$

$p(k) = k$ for $k \in W_0$, and for $k \in W_j$, $j > 0$,

$$p(k) = \frac{1}{2^j}(p_j(k_1), \ldots, p_j(k_d)),$$

where

$$p_j(k) = \begin{cases} 2k & \text{for } 0 \leqslant k \leqslant 2^{j-1}, \\ 2(k-2^{j-1})-1 & \text{for } 2^{j-1}+1 \leqslant k \leqslant 2^j. \end{cases}$$

The *diamond basis* is the family of functions $\{\psi_k,\ k \geqslant 0\}$, defined on I^d by the formula

$$\psi_k(t) = \Psi\big(2^j(t-p(k)\big) \quad \text{for } k \in W_j,\ j \geqslant 0.$$

For each $f \in C(I^d)$ there exists a unique sequence $\{u_k,\ k \geqslant 0\}$ such that

$$f = \sum_{j=0}^{\infty} \sum_{k\in W_j} u_k \psi_k; \tag{11}$$

actually, the coefficients $\{u_k,\ k \geqslant 0\}$ are some linear combinations of the values of f.

It was shown in [2] that if for $f \in C(I^d)$ given by (11) and for $0 < \varepsilon < 1$ we have

$$\underline{\lim}_{j\to\infty} \frac{2^{j\varepsilon}}{2^{jd}} \sum_{k\in W_j} |u_k| > 0,$$

then

$$\underline{\dim}_b \Gamma(f) \geqslant d+1-\varepsilon.$$

Therefore, we need to show that

$$\Pr\left\{\underline{\lim}_{j\to\infty} \frac{2^{j\kappa/2}}{2^{jd}} \sum_{k\in W_j} |u_k^{(\alpha)}| > 0\right\} = 1,$$

where $\{u_k^{(\alpha)},\ k \geqslant 0\}$ is a sequence of the coefficients of $B^{(\alpha)}$ in the diamond basis. Actually, let $i\in\mathscr{D}$ be such that $\kappa = \alpha_i$, and

$$W_j^* = \{k\in W_j\colon\ k_i > 2^{j-1},\ k_l \leqslant 2^{j-1} \text{ for } l \neq i,\ 2^{-j} p_j(k_l) \geqslant \tfrac{1}{2} \text{ for } l\in\mathscr{D}\}.$$

For $k\in W_j^*$, $j > 0$, we have

$$u_k^{(\alpha)} = B^{(\alpha)}(p(k)) - \frac{B^{(\alpha)}(p(k)+2^{-j}e_i)+B^{(\alpha)}(p(k)-2^{-j}e_i)}{2}.$$

Using this formula we verify that $\{u_k^{(\alpha)},\ k\in W_j^*\}$ is a Gaussian family, with $\mathrm{E}u_k^{(\alpha)} = 0$, and, uniformly in j and k, $l\in W_j^*$

$$\mathrm{E}\,|u_k^{(\alpha)}|^2 \sim 2^{-j\kappa}, \qquad |\mathrm{E}u_k^{(\alpha)} u_l^{(\alpha)}| \leqslant C\frac{2^{-j\kappa}}{1+|k_i-l_i|^{4-\kappa}}.$$

Proceeding as in the proof of Lemma 4.2, we get

$$\Pr\left\{\underline{\lim}_{j\to\infty} \frac{2^{j\kappa/2}}{2^{jd}} \sum_{k\in W_j^*} |u_k^{(\alpha)}| > 0\right\} = 1,$$

which implies that, with probability 1,

$$\underline{\dim}_b \Gamma(B^{(\alpha)}|_{I^d}) \geqslant d+1-\kappa/2, \tag{12}$$

and Theorem 5.1 follows from (10) and (12). ∎

COROLLARY 5.2. *Applying the method of proof of Theorem 5.1 to the shifted and dilated field* $\{\varrho^{|\alpha|} B^{(\alpha)}(\varrho^{-2} t - c)\colon\ t\in I^d\}$ $(\varrho > 0,\ c\in R^d)$, *we can prove that*

$$\Pr\{\forall (Q\subset R^d)\ \dim_b \Gamma(B^{(\alpha)}|_Q) = d+1-\kappa/2\} = 1,$$

where $Q\subset R^d$ *denotes an arbitrary cube in* R^d.

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