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THE REFLECTED BROWNIAN MOTION ON THE SIERPIŃSKI GASKET

BY

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Abstract. The paper identifies the Dirichlet form of the "path-wise-defined" reflected Brownian motion on the Sierpiński gasket with the Dirichlet form on the Sierpiński gasket introduced by Fukushima and Shima in [3].

1. INTRODUCTION

In [10] we defined a Markov process on the Sierpiński gasket, which we called the *reflected Brownian motion*. Our definition was purely trajectorial and the work did not investigate any properties of its Dirichlet form. Basically, the process was obtained from the free Brownian motion on the infinite gasket by the following procedure: "nothing changes when we are on the first triangle – and when we get to a corner, the process gets reflected instead of going through."

The work by Fukushima and Shima [3], initiated by a paper by Kigami [5], was devoted to the study of the Laplacian on the Sierpiński gasket, defined geometrically and without probabilistic context.

There is a Dirichlet form coming naturally from the Laplacian introduced by Kigami in [5] and it turns out to be (up to a multiplicative constant) the Dirichlet form of the process in study.

The goal of this paper is to identify the two Dirichlet forms: the one corresponding to the reflected Brownian motion, the other defined in [5]. This work therefore justifies the name "reflected Brownian motion" – it is in fact the normally reflected process that comes out.

In Section 2 we introduce the notation concerning the Sierpiński gasket itself. In order to make the paper self-contained, Sections 3 and 4 are devoted to sketchy constructions of the reflected Brownian motion and of the Kigami Dirichlet form. Section 3 is based on the work from [10], Section 4 – from [3] and [5]. The identification theorem is carried out in Section 5. The theorem is obtained by establishing a resolvent identification, which then yields the identity of the Dirichlet forms classically. The main tool we are using is Dynkin's formula, which allows us to straightforward connect the difference operators (the discrete approximation of the Laplacian) with the Brownian motion. The heart of the identification theorem lies in Lemma 2, which follows from certain symmetries of the process.

Finally, let me thank Professor Alain-Sol Sznitman, who conjectured this theorem and patiently read previous versions of this paper.

2. PRELIMINARIES

We start by introducing the notation, which will be a mixture of the one from [1] and the other from [3] and [5].

Let $a_0 = (0, 0)$, $a_1 = (1, 0)$, $a_2 = (\frac{1}{2}, \sqrt{3}/2)$ be the vertices of the equilateral triangle of unit size $\mathscr{I}_0$. Set $W_0 = \{a_0, a_1, a_2\}$.

For $m \in \boldsymbol{Z}_+$ we define inductively

$$W_{m+1} = W_m \cup \{2^m a_1 + W_m\} \cup \{2^m a_2 + W_m\}$$

and we put

$$\mathscr{G}_0 = \bigcup_{m=0}^{\infty} W_m \cup \bigcup_{m=0}^{\infty} \overline{W}_m,$$

where $\overline{W}_m$ denotes the symmetric image of W_m in symmetry with respect to the y-axis.

Now, we let

$$\mathscr{G}_m = 2^{-m} \mathscr{G}_0$$

to be the so-called m-th grid, and

$$\mathscr{G}_\infty = \bigcup_{m \geq 0}^{\infty} \mathscr{G}_m.$$

$\mathscr{G}_\infty$ is called the (infinite) *two-dimensional Sierpiński pregasket*. Its closure in the planar topology is the two-dimensional Sierpiński gasket, which will be denoted by $\mathscr{G}$.

$\mathscr{T}_0$ will denote the collection of all "unit triangles" in $\mathscr{G}$, whose vertices are neighboring points from $\mathscr{G}_0$. For $x \in \mathscr{G}_\infty$ we define the index of x as the number of step at which x appeared, i.e.

$$i(x) = \inf\{m \geq 0\colon\ x \in \mathscr{G}_m\}.$$

Let $\mathscr{F} = \mathscr{G} \cap \mathscr{I}_0$ be the "first triangle" on the gasket and we will write $\mathscr{F}_m$ for $\mathscr{G}_m \cap \mathscr{I}_0$, $\mathscr{F}_\infty$ for $\mathscr{G}_\infty \cap \mathscr{I}_0$, and so on.

For $m \geqslant 1$,

$$\mathscr{F}_m^0 \stackrel{\mathrm{df}}{=} \mathscr{F}_m \setminus \mathscr{F}_0.$$

If $x \in \mathscr{F}_m$, in the sequel we will work with the collection of its neighbors from the m-th grid, $V_{m,x}$:

$$V_{m,x} \stackrel{\mathrm{df}}{=} \{y \in \mathscr{F}_m \colon |x-y| = 1/2^m\}.$$

μ_m will be the normalized counting measure on $\mathscr{F}_m$,

(1) $$\mu_m = \frac{2}{3^{m+1}} \sum_{x \in \mathscr{F}_m} \delta_x.$$

μ_m do converge weakly to a measure μ, which is a multiple of the Hausdorff x^{d_f}-measure on $\mathscr{F}$, d_f being the Hausdorff dimension of the two-dimensional Sierpiński gasket,

$$d_f = \frac{\log 3}{\log 2} = 1.58496\ldots$$

We use the same notation, μ, for the measure on the whole gasket obtained as a natural extension of the measure on $\mathscr{F}$.

We also introduce a new metric on the gasket, which better suits our purposes: for $x, y \in \mathscr{G}_\infty$ *define* $d(x, y)$ *to be the infimum over the Euclidean length of all paths, joining* x *and* y *on the gasket.*

It is then extended by a limit procedure to a metric on $\mathscr{G}$ and

$$|x-y| \leqslant d(x, y) \leqslant 2|x-y|.$$

The Brownian motion on the free infinite Sierpiński gasket was first defined by Kusuoka in [7] and Goldstein [4]. However, we stick to construction by Barlow and Perkins [1], as their work gives very precise estimates on the transition function, distribution of hitting times, and so on.

There are two additional numbers related to this process (and to the gasket itself): the dimension of the walk,

$$d_w = \frac{\log 5}{\log 2} = 2.32193\ldots,$$

and the spectral dimension (asymptotic frequency of eigenvalues of the Laplacian),

$$d_s = \frac{2\log 3}{\log 5} = 1.36521\ldots$$

This process is a strongly Markov, Feller process having $\mathscr{G}$ as its state-space. It will be denoted by $\big((P_x)_{x \in \mathscr{G}}, (Z_t)_{t \geqslant 0}\big)$, its transition density by

$p(t, x, y)$, the semigroup (acting on $L^2(\mathscr{G}, d\mu)$) by $(P_t)_{t \geq 0}$, and the resolvent by $(R_\alpha)_{\alpha > 0}$. The properties of this process that we shall appeal to will be listed as needed.

3. CONSTRUCTION OF THE REFLECTED BROWNIAN MOTION

In this section, we will sketch the construction of the reflected Brownian motion on $\mathscr{F}$. For the precise definitions, proofs of the properties listed below, and related questions we refer the reader to [10].

First, one needs to introduce some labeling on $\mathscr{G}_0$.

3.1. Preparatory labeling of the gasket. We will introduce a labeling of the grid $\mathscr{G}_0$ of size 1. Our labeling will distinguish between the vertices of the 0-triangles, although the process is locally symmetric with respect to rotation by the angle 120°. This procedure will allow us to construct the "reflected Brownian motion" on the Sierpiński gasket.

First observe that $\mathscr{G}_0 \subset \mathbf{Z}e_1 + \mathbf{Z}e_2$ as for every point $x \in \mathscr{G}_0$, $x = ne_1 + me_2$, $n, m \in \mathbf{Z}$ ($e_1 = (1, 0)$, $e_2 = (\frac{1}{2}, \sqrt{3}/2)$), and this representation is clearly well defined.

We put the labels as follows (see Fig. 1). Consider the commutative 3-group A_3 which consists of even permutations of 3 elements $\{a, b, c\}$. Then $A_3 = \{\mathrm{id}, (a, b, c), (a, c, b)\}$, and we put $p_1 = (a, b, c)$ and $p_2 = (a, c, b)$.

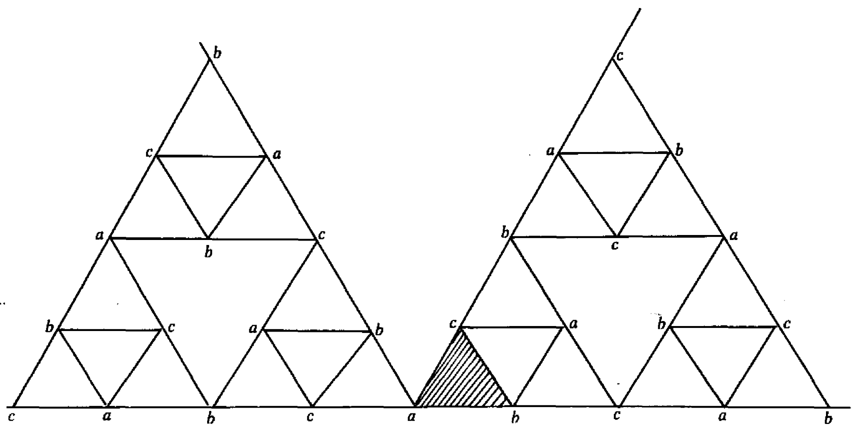

Fig. 1

Clearly, $p_1^3 = \mathrm{id}$ and $p_2^3 = \mathrm{id}$. The mapping

$$\mathscr{G}_0 \ni x = ne_1 + me_2 \mapsto p_1^n \circ p_2^m \in A_3$$

is well defined. We associate with each point $x = ne_1 + me_2$ the value of $(p_1^n \circ p_2^m)(a)$. Consequently, every triangle of size 1 from $\mathscr{T}_0$, with vertices from

$\mathcal{G}_0$, has its vertices labeled a, b, c, in the way corresponding to the location of this triangle in the gasket.

For an arbitrary $x \in \mathcal{G} \setminus \mathcal{G}_0$, x belongs to exactly one triangle $\Delta_0(x) \in \mathcal{T}_0$ (see Fig. 2), and x can be written as

$$x = x_a \cdot a(x) + x_b \cdot b(x) + x_c \cdot c(x),$$

where $a(x)$, $b(x)$, $c(x)$ are the corresponding vertices of $\Delta_0(x)$ (with introduced labeling), x_a, x_b, $x_c \in (0, 1)$.

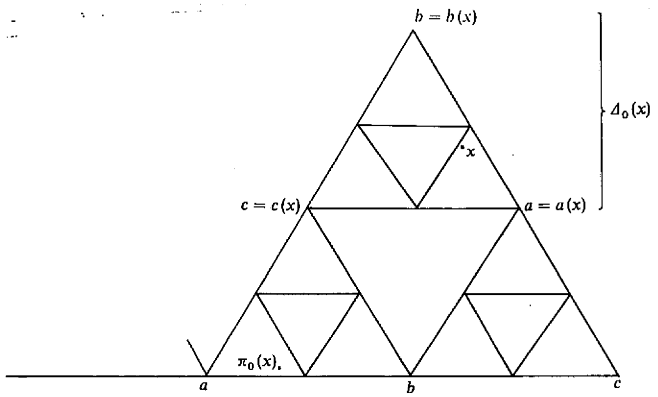

Fig. 2

We define a projection map π_0 from the Sierpiński gasket onto its intersection with the first triangle $\mathcal{F}_0$ by setting

(2) $$\pi_0(x) = x_a \cdot a(0) + x_b \cdot b(0) + x_c \cdot c(0),$$

where $a(0) = (0, 0)$, $b(0) = (\frac{1}{2}, \sqrt{3}/2)$, $c(0) = (0, 1)$.

If $x \in \mathcal{G}_0$, then x itself has a label and we can map it to a corresponding vertex of the "first" (shaded on Fig. 1) triangle.

3.2. Construction of the reflected Brownian motion. We do use the mapping π defined in the previous subsection. Observe that this mapping behaves as a usual projection with one exception: it distinguishes between the vertices of the projected triangle. For this reason, π will be called the *folding projection.*

The *reflected Brownian motion* on $\mathcal{F}$ will be defined as a family of measures on $\big(C(\boldsymbol{R}_+, \mathcal{F}), \mathcal{B}(C(\boldsymbol{R}_+, \mathcal{F}))\big)$, given by

$$\{Q_x\}_{x \in \mathcal{F}} = \{\pi(P_x)\}_{x \in \mathcal{F}},$$

and the process itself by $X_t = \pi(Z_t)$. Its transition density q is given by

(3) $$q(t, x, y) = \begin{cases} \sum_{y' \in \pi^{-1}(y)} p(t, x, y') & \text{for } x \in \mathcal{F},\ y \notin \mathcal{F}_0, \\ 2\sum_{y' \in \pi^{-1}(y)} p(t, x, y') & \text{for } x \in \mathcal{F},\ y \in \mathcal{F}_0. \end{cases}$$

What makes this correct is the fact that (see Theorem 4 from [10]) for all $x, y \in \mathcal{G}$ such that $\pi(x) = \pi(y)$ the equality $\pi(P_x) = \pi(P_y)$ holds true. Moreover, for such x and y and all $z \in \mathcal{G}$,

$$\sum_{z' \in \pi^{-1}(z)} p(t, x, z') = \sum_{z' \in \pi^{-1}(z)} p(t, y, z'). \tag{4}$$

The transition density q, as defined by (3), is symmetric with respect to x and y and continuous in all its variables.

The semigroup of selfadjoint operators on $L^2(\mathcal{F}, d\mu)$ connected with the reflected Brownian motion on $\mathcal{F}$ will be denoted by $(P_t^R)_{t \geq 0}$, its resolvent by $(R_\alpha^R)_{\alpha > 0}$, and the Dirichlet form of this process by

$$\mathcal{E}^R(f, f) = \lim_{t \to 0} \frac{1}{2t} \int_{\mathcal{F}} \int_{\mathcal{F}} (f(x) - f(y))^2 q(t, x, y) \, d\mu(x) \, d\mu(y).$$

Its domain $D(\mathcal{E}^R)$ consists of those functions from $L^2(\mathcal{F}, d\mu)$ for which the limit above is finite (the expression under the limit increases with the decrease of t).

4. THE KIGAMI LAPLACIAN AND THE DIRICHLET FORM

For $f \in L^2(\mathcal{F}, d\mu)$ we introduce the difference operators $H_{m,x}$ as follows: for $x \in \mathcal{F}_m^0$

$$H_{m,x}(f) = \sum_{y \in V_{m,x}} f(y) - 4f(x) = \sum_{y \in V_{m,x}} (f(y) - f(x)).$$

Put $\Delta_m f(x) = 5^m H_{m,x}(f)$. Then the *Laplace operator*, acting on functions from $C^0(\mathcal{F})$ (continuous with zero value on $\partial \mathcal{F}$), is defined as follows:

If there exists a function $g \in C(\mathcal{F})$ such that $\lim_{m \to \infty} \Delta_m f(x) = g(x)$ uniformly in $x \in \mathcal{F}_\infty^0$, then we set $\Delta f = g$.

The domain $D(\Delta)$ of the Laplacian consists of those functions from $L^2(\mathcal{F}, d\mu)$ for which the above uniform convergence holds.

Observe that the value of the Laplacian of a continuous function at a given point $x \in \mathcal{F}_\infty^0$ can be assigned if the limit above merely exists: in this case we just put

$$\Delta f(x) = \lim_{m \to \infty} \Delta_m f(x). \tag{5}$$

At the boundary points $x \in \partial \mathcal{F}$ we define the one-sided difference operators by

$$D_{m,x}(f) = \sum_{y \in V_{m,x}} f(y) - 2f(x) = \sum_{y \in V_{m,x}} (f(y) - f(x)).$$

The difference operators at the boundary give raise to the so-called normal derivative at the boundary points: for $x \in \partial\mathscr{F}$

$$\frac{df}{dn}(x) = \lim_{m\to\infty} -\frac{2}{3}\left(\frac{5}{3}\right)^m D_{m,x}(f),$$

provided that the limit exists.

Now we are ready to introduce the Dirichlet form. Let

$$\mathscr{E}^{(m)}(f, g) = \frac{2}{3}\left(\frac{5}{3}\right)^m \sum_{x,y\in\mathscr{F}_m} (f(x)-f(y))(g(x)-g(y))$$

$$= -\frac{2}{3}\left(\frac{5}{3}\right)^m \Big(\sum_{x\in\mathscr{F}_m^0} H_{m,x}(f)\cdot g(x) + \sum_{x\in\partial\mathscr{F}} D_{m,x}(f)\cdot g(x)\Big).$$

$\mathscr{E}^{(m)}(f,f)$ for $f\in L^2$ is known to be increasing when m increases (see Proposition 4.1 of [3]) so that we can define

$$\mathscr{E}(f,f) = \lim_{m\to\infty} \mathscr{E}^{(m)}(f,f), \tag{6}$$

and the domain of the Dirichlet form $D(\mathscr{E})$ consists of those functions from $L^2(\mathscr{F}, d\mu)$ for which the limit (6) is finite. As the last thing, we note after Kigami that, for $f, g\in L^2(\mathscr{F}, d\mu)$

$$\mathscr{E}^{(m)}(f, g) = \int_{\mathscr{F}_m^0} \Delta_m f(x)g(x)\,d\mu_m(x) + \frac{2}{3}\left(\frac{5}{3}\right)^m \sum_{x\in\partial\mathscr{F}} D_{m,x}(f)g(x).$$

If $f\in D(\Delta)$ and the normal derivatives of f at the boundary exist, then for any $g\in D(\mathscr{E})$

$$\mathscr{E}(f, g) = -\int_{\mathscr{F}} \Delta f(x)g(x)\,d\mu(x) + \sum_{x\in\partial\mathscr{F}} \frac{df}{dn}(x)g(x).$$

5. THE DIRICHLET FORM OF THE REFLECTED PROCESS

The main result of this paper is the following

THEOREM 1. *We have*

$$(\mathscr{E}^R, D(\mathscr{E}^R)) = (\tfrac{1}{4}\mathscr{E}, D(\mathscr{E})).$$

Proof. We will show that, for every $f\in C_c(\mathscr{F})$ (continuous functions with compact support inside $\mathscr{F}$), $R_\alpha^R f$ belongs to $D(\mathscr{E})$ and

$$(\tfrac{1}{4}\mathscr{E})_\alpha(R_\alpha^R f, v) = (f, v)_{L^2(\mathscr{F},d\mu)} \tag{7}$$

for an arbitrary function $v\in D(\mathscr{E})$. This will be enough in view of 1.3.10 of [2].

The proof is split into several lemmas.

First we show that without difficulty we can calculate the pointwise value (see (5)) of $\Delta(R_\alpha^R f)(x)$ for $x\in\mathscr{F}_\infty^0$.

LEMMA 1. *Let $f \in C_c(\mathcal{F})$ and let $\alpha > 0$ be fixed. Let $x \in \mathcal{F}^0_\infty$. Then, for $m \geqslant i(x)$,*

$$\Delta_m(R^R_\alpha f)(x) = 4\big(\alpha R^R_\alpha f(x) - f(x)\big) + \varepsilon_x(m), \tag{8}$$

where $\varepsilon(m) = \sup_{x \in \mathcal{F}_m} |\varepsilon_x(m)|$ is a sequence of positive numbers tending to zero. In particular, it follows that, pointwise

$$\Delta(R^R_\alpha f)(x) = 4\big(\alpha R^R_\alpha f(x) - f(x)\big). \tag{9}$$

Before passing to the proof of this lemma, let us notice that we can naturally set the value of $\Delta(R^R_\alpha f)(x)$ at the points which are not from $\mathcal{F}^0_\infty$. As the right-hand side of (9) is continuous, we put

$$\Delta(R^R_\alpha f)(x) = \lim_{x_n \to x} \Delta(R^R_\alpha f)(x_n) = 4\big(\alpha R^R_\alpha f(x) - f(x)\big)$$

with (x_n) being a sequence of points from $\mathcal{F}^0_\infty$, approaching x.

Proof of the lemma. We shall make use of Dynkin's formula. For a given $m > 0$, let T_m denote the hitting time of the m-th grid, $\mathcal{G}_m$. For all $x \in \mathcal{F}^0_\infty$ the formula reads

$$R^R_\alpha f(x) = \mathrm{E}^0_x[\exp\{-\alpha T_m\} R^R_\alpha f(X_{T_m})] + \mathrm{E}^0_x\Big[\int_0^{T_m} e^{-\alpha s} f(X_s)\,ds\Big], \tag{10}$$

where E^0_x are the expectations corresponding to the law $(Q_x)_{x \in \mathcal{F}}$ of the reflected Brownian motion on $\mathcal{F}$.

Observe that, for $x \in \mathcal{F}^0_\infty$, P_x-almost surely T_m is the exit time from $B(x, 1/2^m)$ provided $m \geqslant i(x)$. As $(X_t)_{t \geqslant 0}$ and $(Z_t)_{t \geqslant 0}$ are equidistributed up to time T_0, we have

$$\mathrm{E}^0_x[T_m] = \mathrm{E}_x[T_m] = 1/5^m \quad \text{and} \quad 1 - \alpha/5^m \leqslant \mathrm{E}^0_x[\exp\{-\alpha T_m\}] \leqslant 1. \tag{11}$$

For $m \geqslant i(x)$, the formula (10) can be written as

$$\begin{aligned} &\tfrac{1}{4}\mathrm{E}^0_x[\exp\{-\alpha T_m\}]\, H_{m,x}(R^R_\alpha f) \\ &\qquad = \big(1 - \mathrm{E}^0_x[\exp\{-\alpha T_m\}]\big) R^R_\alpha f(x) - \mathrm{E}^0_x\Big[\int_0^{T_m} e^{-\alpha s} f(X_s)\,ds\Big] \end{aligned} \tag{12}$$

and

$$\begin{aligned} \Delta_m(R^R_\alpha f)(x) &= 5^m H_{m,x}(R^R_\alpha f) \\ &= 4\left(\frac{5^m\big(1 - \mathrm{E}^0_x[\exp\{-\alpha T_m\}]\big)}{\mathrm{E}^0_x[\exp\{-\alpha T_m\}]} R^R_\alpha f(x) \right. \\ &\qquad \left. - \frac{1}{\mathrm{E}^0_x[\exp\{-\alpha T_m\}]}\, 5^m \mathrm{E}^0_x\Big[\int_0^{T_m} e^{-\alpha s} f(X_s)\,ds\Big]\right). \end{aligned} \tag{13}$$

As we will study the convergence of (13) when m goes to infinity, in view of (11) it is enough to see that

$$\lim_{m\to\infty} 5^m \mathrm{E}_x^0 \Big[\int_0^{T_m} e^{-\alpha s} f(X_s)\, ds \Big] = f(x) \tag{14}$$

and that the distance from the limit depends only on m. The behavior of the first summand in (13) is controlled by (11) and therefore depends on m only.

For the second part we observe that again

$$\mathrm{E}_x^0 \Big[\int_0^{T_m} e^{-\alpha s} f(X_s)\, ds \Big] = \mathrm{E}_x \Big[\int_0^{T_m} e^{-\alpha s} f(Z_s)\, ds \Big]. \tag{15}$$

Let $\mathscr{K} = \sup_{x\in\mathscr{F}} |f(x)|$ and let $\omega_f(\delta)$ be the modulus of continuity of f. We estimate now, for $m \geqslant i(x)$,

$$\Big|5^m \mathrm{E}_x \Big[\int_0^{T_m} e^{-\alpha s} f(Z_s)\, ds \Big] - f(x)\Big| \leqslant 5^m \mathrm{E}_x \Big[\int_0^{T_m} e^{-\alpha s} |f(Z_s) - f(x)|\, ds \Big]$$
$$+ 5^m \mathrm{E}_x \Big[\int_0^{T_m} |e^{-\alpha s} - 1| \cdot |f(x)|\, ds \Big] \stackrel{\mathrm{df}}{=} a_x(m) + b_x(m).$$

Now, before the moment T_m we did not travel farther than at a distance of $1/2^m$; thus $d(Z_s, x) \leqslant 1/2^m$ for $s < T_m$. Hence

$$|a_x(m)| \leqslant 5^m\, \mathrm{E}_x [T_m] \cdot \omega_f(1/2^m) = \omega_f(1/2^m).$$

The estimate of $b_x(m)$ is not hard either. For $x \in \mathscr{G}_m$ and $m \in \mathbf{Z}$ we use the inequality (Theorem 2.19 from [1])

$$P_x [T_m > t] \leqslant c_1 \exp\{-c_2 \cdot 5^m t\},$$

where c_1 and c_2 are positive constants.

Consequently,

$$|b_x(m)| \leqslant \mathscr{K} 5^m \mathrm{E}_x \Big[\int_0^{T_m} \alpha s\, ds \Big] = \mathscr{K} 5^m \mathrm{E}_x \Big[\frac{\alpha T_m^2}{2} \Big]$$
$$= \alpha \mathscr{K} 5^m \int_0^{\infty} s P_x [T_m > s]\, ds \leqslant c_1 \alpha \mathscr{K} 5^m \int_0^{\infty} s \exp\{-c_2 \cdot 5^m s\}\, ds = \frac{c_1 \alpha \mathscr{K}}{c_2 \cdot 5^m}.$$

The last expression goes to zero exponentially as m goes to infinity with the rate of convergence controlled only by m. Thus the lemma is established. ∎

Next, we show that the normal derivative of $R_\alpha^R f$ vanishes at the vertices.

LEMMA 2. *Suppose that $f \in C_c(\mathscr{F})$ and let $p \in \partial\mathscr{F}$. Then*

$$\frac{dR_\alpha^R f}{dn}(p) \text{ exists and is equal to zero.} \tag{16}$$

Proof. This lemma follows from certain symmetries of the process.

First note that the reflected Brownian motion on the gasket is symmetric with respect to the vertices of the triangle (which follows from the symmetries of the "free" process and from the way we have defined the projection π). Then, if ϕ denotes the rotation by 120° around the barycenter of $\mathscr{F}$, for an arbitrary $f \in C_c(\mathscr{F})$ we have

$$(R_\alpha^R f)(\phi x) = (R_\alpha^R f_\phi)(x), \quad \text{where } f_\phi = f \circ \phi.$$

Therefore, as (16) is to be shown for all $f \in C_c(\mathscr{F})$, it is enough to check this assertion for $p = 0$. This simplifies significantly the proof, as 0 plays quite a special role on the gasket.

Next, let

$$\mathscr{G}_+ = \{(x, y) \in \mathscr{G}: x \geqslant 0\} \quad \text{and} \quad \mathscr{G}_- = \{(x, y) \in \mathscr{G}: x \leqslant 0\}.$$

Define the mapping

$$\mathscr{G}_+ \ni z \mapsto \sigma z \in \mathscr{G}_-$$

as the composition of the symmetry with respect to the y-axis and the symmetry with respect to the symmetry axis of the angle α (see Fig. 3). For an

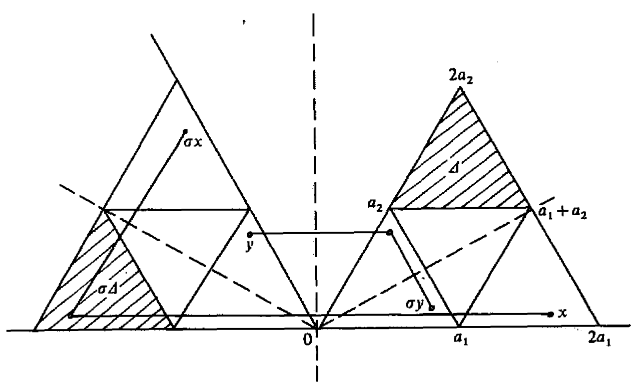

Fig. 3

analogous mapping $\mathscr{G}_- \mapsto \mathscr{G}_+$ we will also use the letter σ. Observe that the way we have defined σ gives

(17) $$\pi(\sigma y) = \pi y \quad \text{for all } y \in \mathscr{G}.$$

Due to the symmetry of the free Brownian motion we have

(18) $$\forall t > 0,\ \forall x, y \in \mathscr{G}_+,\ p(t, \sigma x, \sigma y) = p(t, x, y).$$

Let us now fix a function $f \in C_c(\mathcal{F})$ and a number $\alpha > 0$. Rewrite the expression defining $R^R_\alpha f$:

$$R^R_\alpha f(x) = c(x) \int_0^\infty e^{-\alpha t} \int_{\mathcal{F}} \sum_{y' \in \pi^{-1}(y)} p(t, x, y') f(y) \, d\mu(y) \, dt \tag{19}$$

$$= c(x) \int_0^\infty e^{-\alpha t} \sum_n \int_{\Delta_n \cup \sigma \Delta_n} p(t, x, y) f(\pi y) \, d\mu(y) \, dt$$

($c(x) = 1$ for $x \notin \mathcal{F}_0$ and $c(x) = 2$ for $x \in \mathcal{F}_0$), where the summation in the last sum runs over all the 0-triangles Δ_n which are included in $\mathcal{G}_+$ (0-triangle denotes a unit size triangle in the gasket with vertices from $\mathcal{G}_0$). The change of the order of summation is legitimate as the series is unconditionally convergent.

Using (17) we find that (19) is equal to

$$c(x) \int_0^\infty e^{-\alpha t} \sum_n \int_{\Delta_n} \big(p(t, x, y) + p(t, x, \sigma y)\big) f(\pi y) \, d\mu(y) \, dt.$$

This immediately gives

$$D_{m,0}(R^R_\alpha f) = 2 \int_0^\infty e^{-\alpha t} \sum_n \int_{\Delta_n} H_{m,0} p(t, \cdot, y) f(\pi y) \, d\mu(y) \, dt = H_{m,0}(R_\alpha h)$$

with $h\colon \mathcal{G} \mapsto \boldsymbol{R}$ defined by

$$h(y) = \begin{cases} f(\pi y) & \text{for } y \in \mathcal{G}_+, \\ 0 & \text{otherwise} \end{cases}$$

(this is slightly abusive as $h \notin L^2$, but it makes sense formally and the estimates below are correct).

All the time we allow ourselves to be careless about the behavior at the vertices, but the assumption that the support of f lies entirely within $\mathcal{F}$ makes it correct.

We now want to see that $5^m H_{m,0}(R_\alpha h) = \Delta_m(R_\alpha h)(0)$ can be bounded independently of m. This in turn will give the equalities

$$\frac{dR^R_\alpha f}{dn}(0) = -\lim_{m \to \infty} \frac{2}{3}\left(\frac{5}{3}\right)^m D_{m,0}(R^R_\alpha f) = -\frac{2}{3} \lim_{m \to \infty} \left(\frac{5}{3}\right)^m H_{m,0}(R_\alpha h) = 0$$

and the lemma will be proved.

To see that $\Delta_m(R_\alpha h)(0)$ can be bounded we use again Dynkin's formula. As in the proof of the previous lemma we get, using (11),

$$|H_{m,0}(R_\alpha h)| \leqslant \frac{|(1-\mathrm{E}_0[\exp\{-\alpha T_m\}])R_\alpha h(0)|+|\mathrm{E}_0[\int_0^{T_m} e^{-\alpha s} f(X_s)]|}{\frac{1}{4}\mathrm{E}_0[\exp\{-\alpha T_m\}]}$$

$$\leqslant \frac{\alpha \cdot 5^{-m}\cdot \alpha^{-1}\|f\|_\infty + 5^{-m}\|f\|_\infty}{\frac{1}{4}(1-\frac{1}{5}\alpha)} \stackrel{\mathrm{df}}{=} \frac{1}{5^m} c(f, \alpha),$$

which completes the proof of this lemma. ∎

Next, even though we are not able to establish that $R_\alpha^R f \in D(\Delta)$ $(f \in C_c(\mathscr{F}))$ (the validity of (12) is the major obstacle — (12) is valid only for $m \geqslant i(x)$), we can see that $R_\alpha^R f$ belongs to the domain of the underlying Dirichlet form.

LEMMA 3. *For $f \in C_c(\mathscr{F})$, $R_\alpha^R f$ belongs to $D(\mathscr{E})$.*

Proof. We must see that $\mathscr{E}^{(m)}(R_\alpha^R f, R_\alpha^R f) \leqslant C$ and that the bound C depends only on f and α, not on m ($\mathscr{E}^{(m)}$ was defined by (6)). This is a straightforward check which only uses the Dynkin formula and (11). ∎

Now we know that the statement of Theorem 1 is correct, i.e., that $\mathscr{E}_\alpha(R_\alpha^R f, v)$ makes sense for all $f \in C_c(\mathscr{F})$ and $v \in D(\mathscr{E})$. Moreover, one can show

LEMMA 4. *Let f and v be as in the assumptions of* (7). *Then*

$$\mathscr{E}(R_\alpha^R f, v) = -\int_{\mathscr{F}} \Delta(R_\alpha^R f)(x)\, v(x)\, d\mu(x)$$

(*here $\Delta(R_\alpha^R f)(x)$ denotes the pointwise value of the Laplacian, as calculated in Lemma* 1).

Remark 1. Lemma 4 shows, in particular, that $R_\alpha^R f \in D(\bar{\Delta})$ ($\bar{\Delta}$ denotes Friedrich's extension of Δ).

Proof. Recall that

$$\mathscr{E}^{(m)}(R_\alpha^R f, v) = -\frac{2}{3}\left(\frac{5}{3}\right)^m \sum_{x \in \mathscr{F}_m^0} H_{m,x}(R_\alpha^R f)\, v(x) - \frac{2}{3}\left(\frac{5}{3}\right)^m \sum_{x \in \partial\mathscr{F}} D_{m,x}(R_\alpha^R f)\, v(x).$$

Lemma 2 allows us to forget about the second summand, but, for the time being, denote it by $\varepsilon_1(m)$.

For the first part, we have

$$\frac{2}{3}\left(\frac{5}{3}\right)^m \sum_{x \in \mathscr{F}_m^0} H_{m,x}(R_\alpha^R f)\, v(x) = \int_{\mathscr{F}} \Delta_m(R_\alpha^R f)(x)\, v(x)\, d\mu_m(x) \tag{20}$$

with μ_m defined by (1). Now, $\Delta_m(R_\alpha^R f)(x)$ is extended to be equal to zero outside $\mathscr{F}_m^0$.

Let us now study the limit of (20) as $m \to \infty$, which exists by Lemma 3.

Recall that for $m \geqslant i(x)$ (see (8))

$$|\Delta_m(R_\alpha^R f)(x) - \Delta(R_\alpha^R f)(x)| \leqslant \varepsilon(m) \to 0 \quad \text{as } m \to \infty.$$

This way, as μ_m's are concentrated on appropriate $\mathscr{F}_m$'s and $\mu_m(\mathscr{F}_m) = 1$, equality (20) reads

$$\int_{\mathscr{F}} \Delta(R_\alpha^R f)(x) v(x) d\mu_m(x) + \tilde{\varepsilon}(m)$$

and

$$|\tilde{\varepsilon}(m)| \leqslant \varepsilon(m), \qquad \mathscr{E}^{(m)}(R_\alpha^R f, v) = -\int_{\mathscr{F}} \Delta(R_\alpha^R f)(x) v(x) d\mu_m(x) + \tilde{\varepsilon}(m) + \varepsilon_1(m).$$

But we know the value of $\Delta(R_\alpha^R f)(x)$ at all the points $x \in \mathscr{F}$. It changes continuously with x. Thus, the weak convergence $\mu_m \Rightarrow \mu$ and the convergence of the ε's to zero give

$$\mathscr{E}(R_\alpha^R f, v) = -\int_{\mathscr{F}} \Delta(R_\alpha^R f)(x) v(x) d\mu(x).$$

The lemma is established. ∎

Conclusion of the proof of Theorem 1. All the work is now done. Using Lemmas 1 and 4 we obtain

$$\begin{aligned}
(21)\qquad (\tfrac{1}{4}\mathscr{E})_\alpha(R_\alpha^R f, v) &= \tfrac{1}{4}\mathscr{E}(R_\alpha^R f, v) + \alpha(R_\alpha^R f, v)_{L^2(\mathscr{F}, d\mu)} \\
&= -(\tfrac{1}{4}\Delta(R_\alpha^R f), v)_{L^2(\mathscr{F}, d\mu)} + \alpha(R_\alpha^R f, v)_{L^2(\mathscr{F}, d\mu)} \\
&= ((\alpha - \tfrac{1}{4}\Delta)(R_\alpha^R f), v)_{L^2(\mathscr{F}, d\mu)} = (f, v)_{L^2(\mathscr{F}, d\mu)},
\end{aligned}$$

which completes the proof. ∎

6. CONCLUDING REMARKS

1. Observe that the only tool that we were using was the Dynkin formula — we did not appeal to the uniqueness of the "free" Brownian motion on the infinite gasket, as proved in [1]. In fact, one can go along similar lines to show that $(\frac{1}{4}\mathscr{E}, D(\mathscr{E})_0)$ is the Dirichlet form for the Brownian motion absorbed at the vertices $(D(\mathscr{E})_0 = \{f \in D(\mathscr{E}): f|_{\partial\mathscr{F}} = 0\})$ and that $(\frac{1}{4}\mathscr{E}^\infty, D(\mathscr{E}))$ is the Dirichlet form of the free process (the precise definition of $\mathscr{E}^\infty$ is yet to be made — Fukushima and Shima gave the construction of the Dirichlet form on the half-gasket, one has still to reflect it through zero). For the the last statement, the proof that would use the uniqueness theorem of Barlow and Perkins would require a careful verification of some symmetry properties for the diffusion associated with the underlying Dirichlet form — thus the approach developed in this paper seems useful.

2. All the work is done for the planar Brownian motion. For higher dimension, the construction of the reflected Brownian motion can be carried out similarly. The identification of its Dirichlet form and the one defined by Kigami [5] also holds.

3. We believe that a similar construction of the reflected Brownian motion can be carried out on more general nested fractals. However, one should be a bit careful: our "folding" makes sense only if we start with an infinite nested fractal rather than with a finite one. One can convince oneself that a similar "folding" can be performed on some other fractals, e.g. Kumagai's Pentakun (see [6]). To construct the appropriate folding on a general nested fractal and to identify its Dirichlet form with the one as in Kusuoka's paper [7] will be the subject of future research.

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