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CONVERGENCE OF WEIGHTED AVERAGES OF ASSOCIATED RANDOM VARIABLES

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Abstract. We study the almost sure convergence of weighted averages of associated and negatively associated random variables. Our theorems extend and generalize strong laws of large numbers for positively and negatively associated sequences. We also present applications of our results to almost sure central limit problem.

1. Introduction. Let $(X_n)_{n\in N}$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{A}, P)$. A finite family $\{X_1, \ldots, X_n\}$ of random variables is called *associated* if

$$\operatorname{Cov}\big(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)\big) \geqslant 0$$

for any real coordinatewise nondecreasing functions f, g on R^n such that this covariance exists. It is called *negatively associated* if for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on R^A and g on R^B

$$\operatorname{Cov}\big(f(X_k, k\in A), g(X_k, k\in B)\big) \leqslant 0.$$

An infinite family of random variables is *associated* (*negatively associated*) if every finite subfamily is associated (negatively associated). These concepts of dependence were introduced by Esary et al. [5] and Joag-Dev and Proschan [7]. Basic properties of associated and negatively associated random variables may be found in [5], [7], [9] and [10].

Let $(a_n)_{n\in N}$ be a sequence of positive numbers. We set $b_n = \sum_{k=1}^{n} a_k$ and assume that

$$a_n/b_n \to 0 \quad \text{and} \quad b_n \to \infty \quad \text{as } n \to \infty. \tag{1}$$

Let us also define $S_n = \sum_{k=1}^{n} X_k$ and $S_n^* = \sum_{k=1}^{n} a_k X_k$.

In this paper we are interested in almost sure convergence of a sequence $(S_n^* - ES_n^*)/b_n$ to zero as $n \to \infty$. This problem for positive random variables with uniformly bounded expectations was considered in [6]. In the case $a_n \equiv 1$, we have the strong law of large numbers which, for positively and negatively dependent random variables, was studied in [4], [8] and [9]. Our goal is to extend and generalize some of the results obtained in [4] and [8]. The condition (2) used in our main theorem is due to Etemadi [6], but in general our result cannot be obtained from his.

2. Results.

THEOREM 1. *Let $(X_n)_{n\in N}$ be a sequence of associated random variables with finite second moments, and $(a_n)_{n\in N}$ a sequence of positive numbers satisfying* (1). *Assume that*

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} a_i a_j \operatorname{Cov}(X_i, X_j)/b_j^2 < \infty. \tag{2}$$

Then as $n\to\infty$, $(S_n^*-\mathrm{E}S_n^*)/b_n \to 0$ almost surely.

Setting $a_k = 1$, $k\in N$, we get $b_n = n$ and $S_n = S_n^*$. Thus Theorem 1 yields the following corollary:

COROLLARY 1 (Theorem 2 of Birkel [4]). *Let $(X_n)_{n\in N}$ be a sequence of associated random variables with finite second moments. If*

$$\sum_{j=1}^{\infty} j^{-2}\operatorname{Cov}(X_j, S_j) < \infty, \tag{3}$$

then $(X_n)_{n\in N}$ fulfils the SLLN, *that is $(S_n-\mathrm{E}S_n)/n \to 0$ almost surely as $n\to\infty$.*

The following example shows that our Theorem 1 is more general than the mentioned result of Birkel [4].

EXAMPLE 1. Let $(\xi_n)_{n\in N}$ be a sequence of independent random variables with the same standard normal distribution. We set

$$X_n = I_{(-\infty,u)}\big((\xi_1+\ldots+\xi_n)/\sqrt{n}\big) \quad \text{for arbitrary } u\in R.$$

$(X_n)_{n\in N}$ is a sequence of associated random variables. In paragraph 2.1 of [12] it is proved that for $j<n$

$$\operatorname{Cov}(X_j, X_n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} \exp[-x^2/2]\left(\Phi\left(\frac{\sqrt{n}u-\sqrt{j}x}{\sqrt{n-j}}\right)-\Phi(u)\right)dx,$$

where Φ denotes the standard normal distribution. Let us observe that for $n/16 \leqslant j \leqslant n/4$ we have

$$\frac{\sqrt{n}-\sqrt{j}}{\sqrt{n-j}} \leqslant \frac{\sqrt{3}}{2}.$$

For $u<0$ and $x\leqslant u$, from the above it follows that

$$\frac{\sqrt{n}u-\sqrt{j}x}{\sqrt{n-j}} \geqslant \frac{(\sqrt{n}-\sqrt{j})u}{\sqrt{n-j}} \geqslant \frac{\sqrt{3}}{2}u.$$

Therefore

$$\operatorname{Cov}(X_j, X_n) \geqslant \Phi(u)\big(\Phi(\sqrt{3}\,u/2) - \Phi(u)\big) > 0$$

and

$$\operatorname{Cov}(X_j, S_j) \geqslant \sum_{i=[j/16]}^{[j/4]} \operatorname{Cov}(X_i, X_j) \geqslant C\cdot j$$

for some constant C. Consequently, (3) is not satisfied. But as in [12] we claim that

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} \frac{1}{ij}\operatorname{Cov}(X_i, X_j)/\log^2 j \leqslant C\cdot \sum_{j=1}^{\infty} \frac{1}{j\log^2 j} < \infty.$$

Thus (2) is satisfied and, for every $u \in R$, we have

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\left(I_{(-\infty,u)}\left(\frac{\xi_1 + \ldots + \xi_k}{\sqrt{k}}\right) - \Phi(u)\right) = 0 \text{ almost surely.}$$

Our condition (2) is the same as condition (b) used by Etemadi [6] but the following example demonstrates that our result cannot be obtained from Theorem 1 of [6].

EXAMPLE 2. Let $(\xi_n)_{n\in N}$ be a sequence of i.i.d. random variables such that $E\xi_1 = 1$ and $E\xi_1^2 = 2$ (e.g. exponential with parameter 1). For $n \in N$ let us put

$$X_n = \frac{1}{2^{n-1}}\xi_1 + \ldots + \frac{1}{2^{n-1}}\xi_{n-1} + n\xi_n.$$

It is easy to see that $(X_n)_{n\in N}$ is an associated sequence such that

$$EX_n = n + (n-1)2^{-n+1} \quad \text{and} \quad \operatorname{Var}(X_n) > n^2 \quad \text{for } n \in N.$$

Therefore neither Theorem 1 of [6] nor Theorem 2 of [4] is applicable in this case. But let us observe that for $i < j$ we have

$$\operatorname{Cov}(X_i, X_j) = \frac{1}{2^{j-1}}\left(i + \frac{i-1}{2^{i-1}}\right) \leqslant \frac{i}{2^{j-2}}.$$

Thus we get

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} \frac{1}{ij}\operatorname{Cov}(X_i, X_j)/\log^2 j \leqslant \sum_{j=1}^{\infty}\sum_{i=1}^{j} \frac{1}{2^{j-2}}\frac{1}{j\log^2 j} \leqslant \sum_{j=1}^{\infty} \frac{1}{2^{j-2}}\frac{1}{\log^2 j} < \infty.$$

We conclude from Theorem 1 and the above inequality that

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}(X_k - EX_k) \to 0 \text{ almost surely} \quad \text{as } n \to \infty.$$

Now let us state an analogue of Theorem 1 in the case of negatively associated random variables. This theorem extends some of the results obtained in [8].

THEOREM 2. *Let $(X_n)_{n\in N}$ be a sequence of negatively associated random variables with finite second moments, and $(a_n)_{n\in N}$ a sequence of positive numbers satisfying* (1). *Assume that*

$$\sum_{j=1}^{\infty} \frac{a_j^2}{b_j^2} \operatorname{Var}(X_j) < \infty. \tag{4}$$

Then as $n \to \infty$, $(S_n^* - \mathrm{E}S_n^*)/b_n \to 0$ almost surely.

3. Some applications. Theorem 1 provides a very useful tool for proving the so-called strong version of the central limit theorem (see [11], [12] and the references therein). In this section we state a result of this kind for associated random variables.

In what follows let $\sigma_n^2 = \mathrm{E}S_n^2$ and denote by Φ the standard normal distribution. We shall also need the following coefficient:

$$u(n) = \sup_{k\in N} \sum_{|j-k|\geq n} \operatorname{Cov}(X_j, X_k). \tag{5}$$

THEOREM 3. *Let $(X_n)_{n\in N}$ be a sequence of associated zero mean random variables such that S_n has bounded continuous density for every $n\in N$. Assume that*

$$u(n) = O(e^{-\lambda n}) \quad \text{for some } \lambda > 0, \tag{6}$$

$$\inf_{n\in N} \sigma_n^2/n > 0, \tag{7}$$

$$\sup_{n\in N} \mathrm{E}|X_n|^3 < \infty. \tag{8}$$

Then

$$P\left[\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(-\infty,u)}(S_k/\sigma_k) = \Phi(u)\right] = 1 \quad \text{for all } u\in(-\infty, \infty). \tag{9}$$

As in [11] (Remark 3) one can easily get the following equivalent form of Theorem 3.

COROLLARY 2. *Under the assumptions of Theorem* 3

$$P\left[\lim_{n\to\infty} \sup_{-\infty<u<\infty} \left|\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(-\infty,u)}(S_k/\sigma_k) - \Phi(u)\right| = 0\right] = 1. \tag{10}$$

4. Proofs.

LEMMA 1. *Assume that $X_1, \ldots, X_n$ are associated zero mean random variables with finite second moments. Then, for every $\varepsilon > 0$,*

$$P[\max(|S_1|, \ldots, |S_n|) \geq \varepsilon] \leq 8\varepsilon^{-2} \mathrm{E}S_n^2.$$

Proof. Applying Corollary 5, formula (17), of [10] we get

$$P[\max(0, S_1, S_2, \ldots, S_n) \geq \varepsilon] \leq \varepsilon^{-2} \mathrm{E}[\max(0, S_1, S_2, \ldots, S_n)]^2 \leq \varepsilon^{-2} \mathrm{E}S_n^2.$$

Replacing random variables $X_1, \ldots, X_n$ by $-X_1, \ldots, -X_n$ which are also associated we obtain

$$P[\max(0, -S_1, S_2, \ldots, -S_n) \geqslant \varepsilon] \leqslant \varepsilon^{-2} \mathrm{E} S_n^2.$$

Hence

$$P[\max(|S_1|, \ldots, |S_n|) \geqslant \varepsilon] \leqslant P[\max(0, S_1, S_2, \ldots, S_n) \geqslant \varepsilon/2]$$
$$+P[\max(0, -S_1, -S_2, \ldots, -S_n) \geqslant \varepsilon/2] \leqslant 8\varepsilon^{-2} \mathrm{E} S_n^2.$$

Proof of Theorem 1. Without loss of generality we assume that $\mathrm{E}X_k = 0$, $k \in N$. Since $b_n \to \infty$, for each $k \in N$ we may define n_k such that

$$b_{n_k} \leqslant 2^k < b_{n_k+1}.$$

Let us observe that

$$1 - \frac{a_{n_k+1}}{b_{n_k+1}} < \frac{b_{n_k}}{2^k} \leqslant 1$$

and by (1) we have $b_{n_k}/2^k \to 1$ as $k \to \infty$. Therefore there exists a constant $M > 0$ such that, for every $k \in N$,

$$\frac{1}{M} \leqslant \frac{b_{n_k}}{2^k} \leqslant M.$$

It is easy to see that if $n_k \geqslant j$, then $2^{-k} \leqslant b_j^{-1}$.

Let $\varepsilon > 0$ be given; then we have

$$\sum_{k=1}^{\infty} P\left[\frac{1}{b_{n_k}} \Big|\sum_{i=1}^{n_k} a_i X_i\Big| \geqslant \varepsilon\right] \leqslant \varepsilon^{-2} \sum_{k=1}^{\infty} \frac{1}{b_{n_k}^2} \mathrm{Var}(S_{n_k}^*)$$
$$\leqslant 2M^2 \varepsilon^{-2} \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{j=1}^{n_k} \sum_{i=1}^{j} a_i a_j \mathrm{E} X_i X_j$$
$$= 2M^2 \varepsilon^{-2} \sum_{j=1}^{\infty} \Big(\sum_{k: n_k \geqslant j} 4^{-k}\Big) \sum_{i=1}^{j} a_i a_j \mathrm{Cov}(X_i, X_j)$$
$$\leqslant \frac{8M^2}{3\varepsilon^2} \sum_{j=1}^{\infty} \frac{1}{b_j^2} \sum_{i=1}^{j} a_i a_j \mathrm{Cov}(X_i, X_j) < \infty.$$

The Borel–Cantelli lemma implies that $b_{n_k}^{-1} S_{n_k}^* \to 0$ almost surely as $k \to \infty$. Thus it suffices to prove that

$$b_{n_k}^{-1} \max_{n_k < i \leqslant n_n+1} |S_i^* - S_{n_k}^*| \to 0 \text{ almost surely} \quad \text{as } k \to \infty.$$

Let us note that $b_{n_k}^{-1} \leqslant 2M^2 b_{n_{k+1}}$; therefore, applying Lemma 1 to random variables $a_{n_k+1} X_{n_k+1}, \ldots, a_{n_{k+1}} X_{n_{k+1}}$, we get

$$\sum_{k=1}^{\infty} P[b_{n_k}^{-1} \max_{n_k < i \leqslant n_k+1} |S_i^* - S_{n_k}^*| \geqslant \varepsilon] \leqslant 8\varepsilon^{-2} \sum_{k=1}^{\infty} b_{n_k}^{-2} \mathrm{E}(S_{n_{k+1}}^* - S_{n_k}^*)^2$$
$$\leqslant 16M^2 \varepsilon^{-2} \sum_{k=1}^{\infty} b_{n_k}^{-2} \mathrm{Var}(S_{n_{k+1}}^*) < \infty.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 goes the lines of the proof of Theorem 1 and is based on Lemma 4 of [8] instead of Lemma 1, so we omit details.

We need the following version of Lemma 2.2 in [1].

LEMMA 2. *Suppose X and Y are associated random variables with bounded continuous densities. Then there exists a constant $C > 0$ such that*

$$\sup_{x,y}(P[X \leqslant x,\ Y \leqslant y]-P[X \leqslant x]P[Y \leqslant y]) \leqslant C(\mathrm{Cov}(X,\ Y))^{1/3}.$$

Proof of Theorem 3. We shall apply Theorem 1, so we start with estimating the covariance of $I_{(-\infty,u)}(S_i/\sigma_i)$ and $I_{(-\infty,u)}(S_j/\sigma_j)$ for $i<j$. In the consecutive inequalities, $C_i > 0$ denotes an absolute constant.

We have

$$\begin{aligned}(11)\quad & \mathrm{Cov}\big(I_{(-\infty,u)}(S_i/\sigma_i),\ I_{(-\infty,u)}(S_j/\sigma_j)\big)\\ &= P[S_i/\sigma_i \leqslant u,\ S_j/\sigma_j \leqslant u]-P[S_i/\sigma_i \leqslant u]P[S_j/\sigma_j \leqslant u]\\ &\leqslant C_1\big(\mathrm{Cov}(S_i/\sigma_i,\ S_j/\sigma_j)\big)^{1/3}\\ &= C_1\left(\frac{\sigma_i^2+\mathrm{Cov}(S_i,\ S_j-S_i)}{\sigma_i\sigma_j}\right)^{1/3} \leqslant C_2\left(\frac{\sigma_i^2+u(1)+\ldots+u(j-i)}{\sigma_i\sigma_j}\right)^{1/3}.\end{aligned}$$

Let us observe that by (6) there exists a constant C_3 independent of i and j such that for any $i<j$ we get

$$u(1)+\ldots+u(j-i) \leqslant C_3.$$

Assumptions (7) and (8) together with Theorem 1 of [2] yield

$$C_4\cdot n \leqslant \sigma_n^2 \leqslant C_5\cdot n \quad \text{for every } n\in N.$$

Therefore we get

$$(12)\qquad \mathrm{Cov}\big(I_{(-\infty,u)}(S_i/\sigma_i),\ I_{(-\infty,u)}(S_j/\sigma_j)\big) \leqslant C_6\,(i/j)^{1/3}.$$

From (12) it follows that

$$\begin{aligned}(13)\quad & \sum_{j=1}^{\infty}\frac{1}{j\log^2 j}\sum_{i=1}^{j}\frac{1}{i}\mathrm{Cov}\big(I_{(-\infty,u)}(S_i/\sigma_i),\ I_{(-\infty,u)}(S_j/\sigma_j)\big)\\ &\qquad\leqslant C_6\sum_{j=1}^{\infty}\frac{1}{j^{4/3}\log^2 j}\sum_{i=1}^{j} i^{-2/3} < \infty.\end{aligned}$$

By Theorem 1 we have

$$(14)\qquad \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(I_{(-\infty,u)}(S_k/\sigma_k)-P[S_k/\sigma_k \leqslant u]\big) \to 0 \text{ almost surely as } n\to\infty.$$

Now observe that our assumptions, by Theorem 2.1 of [3], imply

(15) $$\sup_{-\infty < u < \infty} |P[S_n/\sigma_n \leqslant u] - \Phi(u)| \leqslant C_7 \cdot n^{-1/2} \log^2 n.$$

Thus

(16) $$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} |P[S_k/\sigma_k \leqslant u] - \Phi(u)| \leqslant \frac{C_8}{\log n} \sum_{k=1}^{n} \frac{\log^2 k}{k^{3/2}} \to 0 \quad \text{as } n \to \infty.$$

From (14) and (16) we obtain the assertion.

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