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EXTREME VALUES OF DERIVATIVES OF SMOOTHED FRACTIONAL BROWNIAN MOTIONS

BY

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Abstract. Let $B_H(\cdot)$ be a fractional Brownian motion on $\boldsymbol{R}$ with parameter $1/2 < H < 1$, and consider its smoothed version $b_n^{-H} \int K((t-s)/b_n) B_H(s)\, ds$, $t \in \boldsymbol{R}$, where the kernel $K(\cdot)$ is a density function and the $b_n > 0$ are some bandwidths. The derivative of this process arises naturally as a heuristic approximation of a nonparametric kernel regression estimator when the normal errors are long-range dependent. We show that, with suitable centering and norming, the distribution of the supremum and absolute supremum of this derivative over the interval $[0, 1]$ converges, as $n \to \infty$, to the Gumbel extreme-value distribution and its square, respectively. A version of the problem for finite differences is also considered, along with higher-order derivatives.

1. Introduction and statement of results. Consider a Gaussian process $B_H(t)$, $t \in \boldsymbol{R}$, with mean zero and with stationary increments, determined by the covariance function

$$\mathrm{E}\left(B_H(s) B_H(t)\right) = \tfrac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |s-t|^{2H}\right\}, \qquad s, t \in \boldsymbol{R}, \tag{1.1}$$

where H, the Hurst index, is a number such that $1/2 < H < 1$. The process $B_H(\cdot)$ is called a *fractional Brownian motion* and coincides with the ordinary standard Brownian motion when H is taken to be formally as $H = 1/2$. The process $B_H(\cdot)$ is self-similar with parameter H, i.e. for every fixed $c > 0$ the distributional equality

$$\{B_H(ct)\colon\ t \in \boldsymbol{R}\} \overset{\mathscr{D}}{=} \{c^H B_H(t)\colon\ t \in \boldsymbol{R}\}$$

holds. Fractional Brownian motions are the only Gaussian processes among self-similar processes that arise as distributional limits of normalized partial-sum processes based on long-range dependent random variables; cf. Taqqu [6], [7], and the references therein. It is proved by Mandelbrot and Van Ness [4] that, for elementary events outside a set of probability zero, any separable version of $B_H(\cdot)$ has continuous, nowhere differentiable sample functions.

For a fixed $H \in (1/2, 1)$, we consider the smoothed fractional Brownian motion obtained as a convolution of $B_H(\cdot)$ with a smooth probability density function $K(\cdot)$:

$$B_{H,n}(t) := \frac{1}{b_n^H} \int K\left(\frac{t-s}{b_n}\right) B_H(s)\, ds, \qquad t \in \boldsymbol{R}, \tag{1.2}$$

where b_n is a sequence of positive constants tending to zero. Any convergence relation is meant as $n \to \infty$ unless otherwise specified. Also, single and double integrals without specified boundaries are meant to be over the whole real line $\boldsymbol{R}$ or plane $\boldsymbol{R}^2$, respectively. It is easy to see that $B_{H,n}(\cdot)$ inherits the order of smoothness of $K(\cdot)$, i.e. if $K(\cdot)$ is continuously differentiable a given number of times, then so is $B_{H,n}(\cdot)$. Furthermore, since the sample functions of $B_H(\cdot)$ are continuous, $b_n^{H-1} B_{H,n}(\cdot)$ converges to $B_H(\cdot)$ uniformly over every finite interval for every bounded kernel K for which $xK(x) \to 0$ as $|x| \to \infty$; cf. Theorem 1A of Parzen [5].

Our primary objective here is to consider asymptotic extremal properties of the derivative $B'_{H,n}(\cdot)$ of $B_{H,n}(\cdot)$, i.e. the asymptotic behavior of suprema of the process

$$V_n(t) := B'_{H,n}(t) = \frac{1}{b_n^{1+H}} \int K'\left(\frac{t-s}{b_n}\right) B_H(s)\, ds, \qquad t \in \boldsymbol{R}, \tag{1.3}$$

in the case when the second equality in (1.3) is valid. We also consider a second process that has asymptotic extremal properties similar to those of $V_n(\cdot)$, defined as

$$W_n(t) := \frac{1}{b_n^{2+H}} \int K\left(\frac{t-s}{b_n}\right) \frac{1}{n} \left\{ B_H\left(s+\frac{1}{n}\right) - B_H(s) \right\} ds, \qquad t \in \boldsymbol{R}. \tag{1.4}$$

The processes $V_n(\cdot)$ and $W_n(\cdot)$ appear as heuristic approximations to the random parts of the Priestley–Chao and the Gasser–Müller nonparametric regression estimators in a fixed-design model with long-range dependent normal errors, respectively. We refer to [2] for a description and an analysis of the related statistical problem and a discussion leading to the processes $V_n(\cdot)$ and $W_n(\cdot)$. (We note that the relationship between the main parameter α in [2] and the Hurst index H here is $\alpha = 2-2H$.) In this note, we determine the asymptotic distributions of the suprema of $V_n(\cdot)$ and $|V_n(\cdot)|$ over the interval $[0, 1]$, and the analogous results for $W_n(\cdot)$ are derived from those for $V_n(\cdot)$. While we trust that these results are interesting in their own right, they are certainly relevant concerning the description of the asymptotic distributions of maximal deviations of regression estimators under long-range dependence. We have shown in [2] that, with suitable centering and norming, the distributions of the one- and two-sided maximal deviations of both the Priestley–Chao and the Gasser–Müller estimators from the estimated function, calculated over an increasingly finer grid of points in [0, 1], converge to the Gumbel extreme-value distribution and its square, respectively. Here we prove that the same properties hold both for $V_n(\cdot)$ and $W_n(\cdot)$, the asymptotic analogues of the estimators, when the maxima over an increasingly finer grid of points in [0, 1] are replaced by the suprema over all the points in the interval [0, 1]. These results for the suprema and absolute suprema of $V_n(\cdot)$ and $W_n(\cdot)$ suggest that the same should hold for the one- and two-sided maximal deviations of the kernel regression estimators. However, it appears to be a difficult open problem to see whether this is indeed true.

Define

$$\sigma^2 = \sigma^2(H, K) := -\tfrac{1}{2}\iint K'(u)\,K'(v)\,|u-v|^{2H}\,dudv. \tag{1.5}$$

Under the condition of part (i) of the Theorem below, as pointed out at the beginning of the proof of Lemma 1 in the next section, σ^2 is the constant variance of the mean zero stationary Gaussian process

$$Z(y) = \int K'(u)\,B_H(y-u)\,du, \qquad y \in \boldsymbol{R}, \tag{1.6}$$

and hence it is a positive number. Since $0 < 2-2H < 1$, the condition also implies that

$$\lambda^2 = \lambda^2(H, K) := -\frac{H(2H-1)}{2}\iint \frac{K'(u)\,K'(v)}{|u-v|^{2-2H}}\,dudv \tag{1.7}$$

is finite. Furthermore, the condition ensures, by Lemma 1 below, that the covariance function $R(y) := \mathrm{E}\big(Z(0)\,Z(y)\big)$, $y \in \boldsymbol{R}$, of $Z(\cdot)$ is twice differentiable, and it turns out that for the second derivative $R''(\cdot)$ of $R(\cdot)$ we have $R''(0) = -\lambda^2$. Hence, by Lemma 3 of Section 9.3 in Cramér and Leadbetter [1], λ^2 is in fact the second spectral moment of $Z(\cdot)$, and hence positive. So, the integrals in (1.5) and (1.7) are negative.

THEOREM. (i) *If $K(\cdot)$ is a differentiable density with a bounded derivative $K'(\cdot)$ and a support contained in $(-1, 1)$, then*

$$P\left\{\sqrt{2\log\frac{1}{b_n}}\,\frac{\sup_{0\leqslant t\leqslant 1} V_n(t)}{\sigma(H, K)} - \left[2\log\frac{1}{b_n} + \log\frac{\lambda(H, K)}{2\pi\sigma(H, K)}\right] \leqslant x\right\} \to \exp\{-e^{-x}\}, \qquad x \in \boldsymbol{R},$$

and

$$P\left\{\sqrt{2\log\frac{1}{b_n}}\,\frac{\sup_{0\leqslant t\leqslant 1} |V_n(t)|}{\sigma(H, K)} - \left[2\log\frac{1}{b_n} + \log\frac{\lambda(H, K)}{2\pi\sigma(H, K)}\right] \leqslant x\right\} \to \exp\{-2e^{-x}\}, \qquad x \in \boldsymbol{R}.$$

(ii) *Suppose, in addition, that the second derivative $K''(\cdot)$ of $K(\cdot)$ also exists and is bounded and that $nb_n^{1+H}/\sqrt{\log(1/b_n)} \to \infty$. Then the two statements in part* (i) *hold as well for $W_n(\cdot)$ in place of $V_n(\cdot)$.*

The result can be generalized for higher-order derivatives of $B_{H,n}(\cdot)$. Consider some $m \in \boldsymbol{N}$. Suppose that the kernel $K(\cdot)$ has a support contained in $(-1, 1)$, is m times differentiable and the m-th derivative $K^{(m)}(\cdot)$ of $K(\cdot)$ is bounded. Then the m-th derivative of $B_{H,n}(\cdot)$ can be written as

$$B_{H,n}^{(m)}(t) = \frac{1}{b_n^{H+m}} \int K^{(m)}\left(\frac{t-s}{b_n}\right) B_H(s)\, ds, \qquad t \in \boldsymbol{R}.$$

This is the asymptotic analogue of the random part of the Priestley–Chao kernel estimator of the $(m-1)$-st derivative of the regression function in the fixed-design model, considered in [2], with long-range dependent normal errors. If we define $V_{n,m}(t) := b_n^{m-1} B_{H,n}^{(m)}(t)$, $t \in \boldsymbol{R}$, so that $V_{n,1}(\cdot) = V_n(\cdot)$, then part (i) of the theorem remains true for $V_{n,m}$ replacing V_n, provided we also replace $\sigma(H, K) = \sigma_1(H, K)$ and $\lambda(H, K) = \lambda_1(H, K)$ by the square roots of

$$\sigma_m^2(H, K) := -\tfrac{1}{2} \iint K^{(m)}(u) K^{(m)}(v) |u-v|^{2H}\, du dv$$

and

$$\lambda_m^2(H, K) := -\frac{H(2H-1)}{2} \iint \frac{K^{(m)}(u) K^{(m)}(v)}{|u-v|^{2-2H}}\, du dv,$$

respectively. This follows by making straightforward changes in the proofs of Lemma 1 and the Theorem below.

2. Proofs. We need two lemmas. Consider first the process $Z(\cdot)$, defined in (1.6), and let $R(y) = \mathrm{E}(Z(0) Z(y))$, $y \in \boldsymbol{R}$, as above.

LEMMA 1. *If the conditions of part* (i) *of the Theorem are satisfied, then $Z(\cdot)$ is a stationary process, its covariance function $R(\cdot)$ is twice differentiable and, with $\lambda^2 = \lambda^2(H, K)$ given in* (1.7), *$R''(0) = -\lambda^2$.*

Proof. In view of the assumptions on the kernel, we have $\int K'(s)\, ds = 0$. This and (1.1) together imply that

$$\mathrm{E}(Z(x) Z(y)) = -\tfrac{1}{2} \iint K'(u) K'(v) |x-y+v-u|^{2H}\, du dv = R(x-y) = R(y-x)$$

for all x, $y \in \boldsymbol{R}$, where, since $K'(\cdot)$ is bounded, the integral exists in the Lebesgue sense. Thus the process $Z(\cdot)$ is indeed stationary, and its variance is $\sigma^2 = R(0)$. (Another form of σ^2 comes from (2.5) below, where $r(0) = 1$.) Hence, by Lemma 3 of Section 9.3 in Cramér and Leadbetter [1], all the statements will follow if we show that

$$\frac{D(s)}{s^2} := \frac{2|R(s) - R(0) + \lambda^2 s^2|}{s^2} \to 0 \quad \text{as } s \to 0. \tag{2.1}$$

Set

$$I := [-1, 1]^2, \quad I_s^+ := \{(u, v)\colon\ v-u > 2|s|\} \cap I,$$
$$J_s^+ := \{(u, v)\colon\ 0 < v-u < 2|s|\} \cap I,$$
$$J_s^- := \{(u, v)\colon\ 0 < u-v < 2|s|\} \cap I,$$
$$I_s^- := \{(u, v)\colon\ u-v > 2|s|\} \cap I, \quad s \in \mathbf{R},$$

and put

$$A(s, u, v) := (s+v-u)^{2H} - (v-u)^{2H} - 2H(v-u)^{2H-1} s - \frac{2H(2H-1)}{2}(v-u)^{2H-2} s^2$$

for $-1 \leqslant u, v \leqslant 1$ and $s \in (-1, 1)$. Then, using the equality

$$\iint_{I_s^+} K'(u) K'(v) (v-u)^{2H-1}\, dudv = \iint_{I_s^-} K'(u) K'(v) (u-v)^{2H-1}\, dudv,$$

for all $s \in (-1, 1)$ we obtain

$$D(s) = \left| \iint_I K'(u) K'(v) \left\{ |s+v-u|^{2H} - |v-u|^{2H} - \frac{2H(2H-1)}{2} |v-u|^{2H-2} s^2 \right\} dudv \right|$$
$$\leqslant \left\{ \iint_{I_s^+} |K'(u) K'(v) A(s, u, v)|\, dudv + \iint_{I_s^-} |K'(u) K'(v) A(-s, v, u)|\, dudv \right\}$$
$$+ \iint_{J_s^- \cup J_s^+} |K'(u) K'(v)| \left\{ |3s|^{2H} + |2s|^{2H} + \frac{H(2H-1) s^2}{|v-u|^{2-2H}} \right\} dudv$$
$$=: D_1(s) + D_2(s).$$

Observe that since the planar Lebesgue measure of the set $J_s^- \cup J_s^+$ is not greater than $4\sqrt{2}|s|$, for all $s \in (-1, 1)$ we have

$$\frac{D_2(s)}{s^2} \leqslant 52\sqrt{2} [\sup_{-1 \leqslant u \leqslant 1} |K'(u)|]^2 |s|^{2H-1} + H(2H-1) \iint_{J_s^+ \cup J_s^-} \frac{|K'(v) K'(u)|}{|v-u|^{2-2H}} dudv.$$

Since $\iint |K'(u) K'(v)|\, |v-u|^{2H-2}\, dudv < \infty$ by the assumption, this upper bound goes to 0 as $s \to 0$. Hence (2.1) will follow if we show that

$$\frac{D_1(s)}{s^2} \to 0 \quad \text{as } s \to 0. \tag{2.2}$$

Since, for every $s \in (-1, 1)$, $|s|/|u-v| \leqslant 1/2$ on both I_s^- and I_s^+, we see by Newton's binomial expansion that $D_1(s)$ is equal to

$$\iint_{I_s^+} \frac{|K'(u) K'(v)|}{(v-u)^{-2H}} \left| \left(1 + \frac{s}{v-u}\right)^{2H} - 1 - 2H \frac{s}{v-u} - \frac{2H(2H-1)}{2} \left(\frac{s}{v-u}\right)^2 \right| dudv$$

$$+\iint_{I_s^-}\frac{|K'(u)K'(v)|}{(u-v)^{-2H}}\left|\left(1-\frac{s}{u-v}\right)^{2H}-1+2H\frac{s}{u-v}-\frac{2H(2H-1)}{2}\left(\frac{s}{u-v}\right)^2\right|dudv$$

$$=\iint_{I_s^+}\frac{|K'(u)K'(v)|}{(v-u)^{-2H}}\left|\sum_{k=3}^{\infty}\binom{2H}{k}\left(\frac{s}{v-u}\right)^k\right|dudv$$

$$+\iint_{I_s^-}\frac{|K'(u)K'(v)|}{(u-v)^{-2H}}\left|\sum_{k=3}^{\infty}\binom{2H}{k}\left(\frac{-s}{u-v}\right)^k\right|dudv.$$

Thus, for every $s\in(-1, 1)$,

$$\frac{D_1(s)}{s^2}=\iint_{I_s^+}\frac{|K'(u)K'(v)|}{(v-u)^{2-2H}}\left|\sum_{k=3}^{\infty}\binom{2H}{k}\left(\frac{s}{v-u}\right)^{k-2}\right|dudv$$

$$+\iint_{I_s^-}\frac{|K'(u)K'(v)|}{(u-v)^{2-2H}}\left|\sum_{k=3}^{\infty}\binom{2H}{k}\left(\frac{-s}{u-v}\right)^{k-2}\right|dudv=:B^+(s)+B^-(s).$$

Setting momentarily $\beta:=2H(2H-1)$ and $\alpha:=2-2H$, so that $0<\alpha<1$, for every x for which $0<|x|\leqslant 1/2$ we have

$$\left|\sum_{k=3}^{\infty}\binom{2H}{k}x^{k-2}\right|=\beta\left|\frac{2H-2}{3!}x+\frac{(2H-2)(2H-3)}{4!}x^2+\ldots\right|$$

$$=\beta\left|\frac{-\alpha}{3!}x+\frac{(-\alpha)(-\alpha-1)}{4!}x^2+\frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{5!}x^3+\ldots\right|$$

$$=\beta\left|\frac{\alpha}{3!}(-x)+\frac{\alpha(\alpha+1)}{4!}(-x)^2+\frac{\alpha(\alpha+1)(\alpha+2)}{5!}(-x)^3+\ldots\right|$$

$$\leqslant\beta\left(\frac{1!}{3!}|x|+\frac{2!}{4!}x^2+\frac{3!}{5!}|x|^3+\frac{4!}{6!}x^4+\ldots\right)$$

$$\leqslant\beta\sum_{k=1}^{\infty}|x|^k=\beta\left(\frac{1}{1-|x|}-1\right)\leqslant\beta=2H(2H-1).$$

Hence, setting

$$G_y(u, v):=\left|\sum_{k=3}^{\infty}\binom{2H}{k}\left(\frac{y}{|u-v|}\right)^{k-2}\right|,\qquad (u, v)\in I,\ u\neq v,\ y\in(-1, 1),$$

and denoting by χ_J the indicator of a set $J\subset \boldsymbol{R}^2$, we have

$$B^{\pm}(s)=\iint_I\frac{|K'(u)K'(v)|}{|u-v|^{2-2H}}G_{\pm s}(u, v)\chi_{I_s^{\pm}}(u, v)\,dudv,\qquad -1<s<1,$$

where $0\leqslant G_{\pm s}(u, v)\chi_{I_s^{\pm}}(u, v)\leqslant 2H(2H-1)$, $(u, v)\in I$, $s\in(-1, 1)$, and for each fixed $(u, v)\in I$ we have $G_{\pm s}(u, v)\chi_{I_s^{\pm}}(u, v)\to 0$ as $s\to 0$. Thus, by the bounded convergence theorem, $B^{\pm}(s)\to 0$ as $s\to 0$, implying (2.2), and hence the lemma. ∎

LEMMA 2. *If K satisfies the conditions of part* (ii) *of the Theorem, then*

$$D_n := \sup_{0\leqslant t\leqslant 1} |V_n(t)-W_n(t)| = \mathcal{O}_P\left(\frac{1}{nb_n^{1+H}}\right).$$

Proof. Using (1.3) and (1.4), an integral transformation and the self-similarity property of the fractional Brownian motion $B_H(\cdot)$, our difference D_n equals

$$\frac{1}{b_n^H}\sup_{0\leqslant t\leqslant 1}\left|\int\left[K'(u)B_H(t-ub_n)-K(u)\frac{1}{nb_n}\left\{B_H\left(t-ub_n+\frac{1}{n}\right)-B_H(t-ub_n)\right\}\right]du\right|$$

$$\stackrel{\mathcal{D}}{=}\sup_{0\leqslant t\leqslant 1}\left|\int\left[K'(u)B_H\left(\frac{t}{b_n}-u\right)-K(u)\frac{1}{nb_n}\left\{B_H\left(\frac{t}{b_n}-u+\frac{1}{nb_n}\right)-B_H\left(\frac{t}{b_n}-u\right)\right\}\right]du\right|$$

$$=\sup_{0\leqslant t\leqslant 1}\left|\int\left[K'(u)B_H\left(\frac{t}{b_n}-u\right)-K(u)\Delta_{1/(nb_n)}\left\{B_H\left(\frac{t}{b_n}-u\right)\right\}\right]du\right|$$

$$=\sup_{0\leqslant t\leqslant 1}\left|\int B_H\left(\frac{t}{b_n}-u\right)[K'(u)-\Delta_{1/(nb_n)}\{K(u)\}]\,du\right|,$$

where $\Delta_s\{f(x)\}:=s^{-1}(f(x+s)-f(x))$ for any function $f(\cdot)$ and $s>0$. By the assumption on K, we have $C:=\sup_{x\in(-1,1)}|K''(x)|<\infty$. Thus, using again the self-similarity transformation of $B_H(\cdot)$, we obtain

$$D_n\leqslant\sup_{-1-b_n^{-1}\leqslant v\leqslant 1+b_n^{-1}}|B_H(v)|\int|K'(u)-\Delta_{1/(nb_n)}\{K(u)\}|\,du$$

$$\stackrel{\mathcal{D}}{=}\frac{1}{b_n^H}\sup_{-1-b_n\leqslant vb_n\leqslant 1+b_n}|B_H(vb_n)|\int|K'(u)-K'(v_n(u))|\,du$$

$$\leqslant\frac{1}{b_n^H}\frac{2C}{nb_n}\sup_{-1-b_n\leqslant u\leqslant 1+b_n}|B_H(u)|.$$

Since $\sup_{-1-b_n\leqslant u\leqslant 1+b_n}|B_H(u)|=\mathcal{O}_P(1)$ by the sample-continuity of $B_H(\cdot)$, the lemma follows. ∎

Proof of the Theorem. (i) Note again that, as in the proof of Lemma 2,

$$V_n(\cdot)=\frac{1}{b_n^H}\int K'(u)B_H(t-ub_n)\,du\stackrel{\mathcal{D}}{=}\int K'(u)B_H\left(\frac{t}{b_n}-u\right)du,$$

whence

$$\sup_{0\leqslant t\leqslant 1}V_n(t)\stackrel{\mathcal{D}}{=}\sup_{0\leqslant y\leqslant b_n^{-1}}Z(y)\quad\text{and}\quad\sup_{0\leqslant t\leqslant 1}|V_n(t)|\stackrel{\mathcal{D}}{=}\sup_{0\leqslant y\leqslant b_n^{-1}}|Z(y)|,$$

where $Z(\cdot)$ is the mean-zero Gaussian process defined in (1.6). Hence, setting

$$a_n:=\sqrt{2\log(1/b_n)}\quad\text{and}\quad c_n:=2\log(1/b_n)+\log(\lambda/[2\pi\sigma]),$$

where λ and σ are given in (1.7) and (1.5), and finally $Y(\cdot) := Z(\cdot)/\sigma$, it suffices to show that

$$P\{a_n \sup_{0 \leqslant t \leqslant b_n^{-1}} Y(t) - c_n \leqslant x\} \to \exp\{-e^{-x}\}, \quad x \in \boldsymbol{R}, \tag{2.3}$$

and

$$P\{a_n \sup_{0 \leqslant t \leqslant b_n^{-1}} |Y(t)| - c_n \leqslant x\} \to \exp\{-2e^{-x}\}, \quad x \in \boldsymbol{R}. \tag{2.4}$$

For (2.3) we check the conditions of a general result concerning the maxima of a Gaussian process over increasing intervals. By Lemma 1, the mean-zero Gaussian process $Y(\cdot)$ is stationary, has variance 1, and its covariance function $r(t) := \mathrm{E}(Y(0)\,Y(t)) = R(t)/\sigma^2$, $t \in \boldsymbol{R}$, is twice differentiable with $r''(0) = -\lambda^2/\sigma^2$. Furthermore, upon integrating by parts we see that

$$r(t) = \frac{H(2H-1)}{\sigma^2} \iint \frac{K(u)\,K(v)}{|t+v-u|^{2-2H}}\,du\,dv, \quad t \in \boldsymbol{R}. \tag{2.5}$$

This implies that $r(t)\log t \to 0$ as $t \to \infty$. Thus (2.3) follows from Theorem 8.2.7 in Leadbetter et al. [3].

For all n large enough, the left-hand side of (2.4) is

$$P\{\sup_{0 \leqslant t \leqslant b_n^{-1}} Y(t) \leqslant a_n^{-1}(x+c_n),\ \inf_{0 \leqslant t \leqslant b_n^{-1}} Y(t) \geqslant a_n^{-1}(-x-c_n)\}.$$

In view of Corollary 11.1.6 in [3] concerning the asymptotic independence of suprema and infima, the limit of the last probability is the same as the limit of the product

$$P\{\sup_{0 \leqslant t \leqslant b_n^{-1}} Y(t) \leqslant a_n^{-1}(x+c_n)\}\, P\{\sup_{0 \leqslant t \leqslant b_n^{-1}} [-Y(t)] \leqslant a_n^{-1}(x+c_n)\}.$$

But, since $Y(\cdot) \overset{\mathscr{D}}{=} -Y(\cdot)$, the second factor here is the same as the first. Thus (2.4) follows from (2.3).

(ii) The assertion here follows directly from part (i) and Lemma 2. ∎

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REFERENCES

[1] H. Cramér and M. R. Leadbetter, *Stationary and Related Stochastic Processes*, Wiley, New York 1967.

[2] S. Csörgő and J. Mielniczuk, *Nonparametric regression under long-range dependent normal errors*, Ann. Statist. 23 (1995), pp. 1000–1014.

[3] M. R. Leadbetter, G. Lindgren and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer, New York 1983.
[4] B. Mandelbrot and J. W. Van Ness, *Fractional Brownian motion, fractional noises and applications*, SIAM Rev. 10 (1968), pp. 422–437.
[5] E. Parzen, *On estimation of a probability density function and mode*, Ann. Math. Statist. 33 (1962), pp. 1065–1076.
[6] M. S. Taqqu, *Weak convergence to fractional Brownian motion and to the Rosenblatt processes*, Z. Wahrsch. verw. Gebiete 31 (1975), pp. 287–302.
[7] — *Convergence of integrated processes of arbitrary Hermite rank*, ibidem 50 (1979), pp. 53–83.

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