PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 16, Fasc. 2 (1996), pp. 299–309

ON THE CENTRAL LIMIT THEOREM FOR INDEPENDENT RANDOM VARIABLES WITH ALMOST SURE CONVERGENCE

BY

BEATA RODZIK AND ZDZISŁAW RYCHLIK (LUBLIN)

Abstract. We obtain an almost sure convergence limit theorem for independent nonidentically distributed random variables. Let S_n, $n \geqslant 1$, be the partial sums of independent random variables with zero means and finite variances and let $a(x)$ be a real function. We present sufficient conditions under which in logarithmic means $a(S_n/(\mathrm{E}S_n^2)^{1/2})$ converges almost surely to $\int_{-\infty}^{\infty} a(x)\,d\Phi(x)$.

1. Introduction. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables, defined on some probability space $(\Omega, \mathscr{A}, P)$, such that $\mathrm{E}X_n = 0$ and $\mathrm{E}X_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$.

Let us put

$$S_0 = 0, \quad S_n = X_1 + \ldots + X_n, \quad V_n^2 = \mathrm{E}S_n^2.$$

It is well known that under some additional assumption

$$S_n/V_n \xrightarrow{\mathscr{D}} \Phi \quad \text{as } n \to \infty,$$

where Φ denotes the standard normal distribution. But for mathematical statistics it may be of some interest whether assertions are possible for almost every realization of the random variables X_n, $n \geqslant 1$. Namely, for $x \in \boldsymbol{R}$ we denote by δ_x the probability measure on $\boldsymbol{R}$ which assigns its total mass to x. Let us observe that the distribution function of S_n/V_n is just the average of the random measure $\delta_{S_n(\omega)/V_n}$ with respect to P, i.e., for every $A \in \mathscr{B}(\boldsymbol{R})$

$$P(S_n/V_n \in A) = \int \delta_{S_n(\omega)/V_n}(A)\,dP(\omega).$$

Of course, for every $\omega \in \Omega$, $\{\delta_{S_n(\omega)/V_n}, n \geqslant 1\}$ is a sequence of probability measures on the space $(\boldsymbol{R}, \mathscr{B}(\boldsymbol{R}))$. Moreover, under the assumptions of Theorem 2 of Rodzik and Rychlik [8], P-a.s.

$$(1.1) \qquad (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_{k+1}^2/V_k^2)\,\delta_{S_k(\omega)/V_k} \xrightarrow{\mathscr{D}} \Phi \quad \text{as } n \to \infty,$$

where $\xrightarrow{\mathscr{D}}$ denotes the weak convergence of measures on $(\boldsymbol{R}, \mathscr{B}(\boldsymbol{R}))$. Thus we form time averages with respect to a logarithmic scale and prove almost sure convergence for the resulting random measures.

In this paper we present sufficient conditions under which P-a.s.

$$(1.2) \qquad (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2/V_k^2)\, a\big(S_k(\omega)/V_k\big) \to \int_{-\infty}^{\infty} a(x)\, d\Phi(x) \quad \text{as } n \to \infty$$

for a real function $a(\cdot)$ which is almost everywhere continuous and $|a(x)| \leqslant \exp(\gamma x^2)$ for some $\gamma < 1/4$. Of course from (1.2) we easily get (1.1).

The almost sure version of the central limit theorem has been studied by many authors in the case where $\{X_n, n \geqslant 1\}$ is a sequence of independent or weakly dependent and identically distributed random variables. In this case, (1.1) and some extensions of (1.1) have been considered by Schatte [9], Brosamler [3], Lacey and Philipp [5], Atlagh and Weber [1], Berkes and Dehling [2], Peligrad and Shao [6]. The assertion (1.2), in the case of independent and identically distributed random variables, has been considered by Schatte [10]. Thus the main result presented in this paper extends Theorem 1 of [10] to the case of nonidentically distributed random variables. In the proofs we shall also follow the ideas of [10].

2. Results. We shall now state the main results of the paper.

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables, defined on $(\Omega, \mathscr{A}, P)$, with $\mathrm{E}X_n = 0$ and $0 < \mathrm{E}X_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Define, for $n \geqslant 1$, $S_n = X_1 + \ldots + X_n$, $V_n^2 = \mathrm{E}S_n^2$. Let $a(x)$ be a real function which is a.e. continuous and for which $|a(x)| \leqslant \exp(\gamma x^2)$ for some $\gamma < 1/4$. Assume*

$$(2.1) \qquad (\max_{1 \leqslant k \leqslant n} \sigma_k^2)/V_n^2 \to 0 \quad \textit{as } n \to \infty,$$

and for some positive, nondecreasing real function f on $\boldsymbol{R}^+$, such that the function $f(x)/x$ is nonincreasing on $\boldsymbol{R}^+$,

$$(2.2) \qquad \sup_n (\log V_n^2)\big(f(V_n^2)/V_n^2\big)^{1/4} < \infty,$$

$$(2.3) \qquad \lim_{n \to \infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2/V_k^2)\big(f(V_k^2)/V_k^2\big)^{1/4} (\log V_k^2)^{(3\gamma \vee 2\gamma)+1} = 0,$$

and

$$(2.4) \qquad \sum_{n=1}^{\infty} \big(f(V_n^2)\big)^{-1} \mathrm{E}X_n^2 I\big(X_n^2 \geqslant f(V_n^2)\big) < \infty.$$

Then

$$(2.5) \qquad P\Big(\lim_{n \to \infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2/V_k^2)\, a(S_k/V_k) = \int_{-\infty}^{\infty} a(x)\, d\Phi(x)\Big) = 1.$$

Let us observe that, in general, (2.2) does not imply (2.3). For example, if $\gamma > 0$ and $V_n^2 = n$, $n \geqslant 2$, $f(x) = x/(\log x)^4$, then (2.2) holds. But

$$(\log N)^{-1} \sum_{n=2}^{N} n^{-1} (\log n)^{3\gamma} = (\log N)^{-1} \sum_{n=2}^{N} N^{-1} (n/N)^{-1} \left\{ \log \left(N \frac{n}{N} \right) \right\}^{3\gamma}$$

$$\leqslant (\log N)^{-1} \int_{1/N}^{1} x^{-1} (\log Nx)^{3\gamma} \, dx = (3\gamma+1)^{-1} (\log N)^{3\gamma+1} (\log N)^{-1} \to \infty$$

as $N \to \infty$.

On the other hand, in general, (2.3) does not imply (2.2) either. For example, if $-4 < \gamma < 0$ and $V_n^2 = n$, $n \geqslant 2$, $f(x) = x(\log x)^{-4-\gamma}$, then

$$(\log n)\big(f(n)/n\big)^{1/4} = (\log n)^{-\gamma/4} \to \infty \quad \text{as } n \to \infty,$$

and

$$(\log N)^{-1} \sum_{n=3}^{N} n^{-1} (\log n)^{7\gamma/8} \leqslant (\log N)^{-1} \int_{\log 2}^{\log N} x^{7\gamma/8} \, dx$$

$$= (7\gamma/8+1)^{-1} \big((\log N)^{7\gamma/8+1} - (\log 2)^{7\gamma/8+1}\big) (\log N)^{-1} \to 0 \quad \text{as } N \to \infty.$$

We also note that if $\gamma < 0$, then (2.2) implies (2.3). This is a consequence of the Toeplitz lemma.

Now let us observe that if (2.2) and (2.4) hold, then

$$X_n (2 \log\log V_n^2)^{1/2} / V_n \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{2.6}$$

Namely, we have

$$\sum_{n=1}^{\infty} P\big(V_n^{-1} |X_n| (2 \log\log V_n^2)^{1/2} \geqslant (2 \log\log V_n^2)^{1/2} \big(f(V_n^2)/V_n^2\big)^{1/2}\big)$$

$$= \sum_{n=1}^{\infty} P\big(X_n^2 \geqslant f(V_n^2)\big) \leqslant \sum_{n=1}^{\infty} \big(f(V_n^2)\big)^{-1} \mathrm{E} X_n^2 I\big(X_n^2 \geqslant f(V_n^2)\big) < \infty.$$

Since, by (2.2), $(2 \log\log V_n^2)^{1/2} \big(f(V_n^2)/V_n^2\big)^{1/2} \to 0$ as $n \to \infty$, it follows that (2.6) holds.

On the other hand, by (2.4) and Kronecker's lemma

$$\big(f(V_n^2)\big)^{-1} \sum_{k=1}^{n} \mathrm{E} X_k^2 I\big(X_k^2 \geqslant f(V_k^2)\big) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Let us put

$$L_n(\varepsilon) = V_n^{-2} \sum_{k=1}^{n} \mathrm{E} X_k^2 I(|X_k| > \varepsilon V_k).$$

Then, taking into account the assumptions concerning the function f, we get

$$L_n(\varepsilon) = \big(f(V_n^2)/V_n^2\big)\big(f(V_n^2)\big)^{-1} \sum_{k=1}^{n} \mathrm{E}X_k^2 I\big(X_k^2 > \varepsilon^2 f(V_k^2)\big(V_k^2/f(V_k^2)\big)\big)$$

$$\leqslant \big(f(\sigma_1^2)/\sigma_1^2\big)\big(f(V_n^2)\big)^{-1} \sum_{k=1}^{n} \mathrm{E}X_k^2 I\big(X_k^2 \geqslant f(V_k^2)\big(\varepsilon\sigma_1^2/f(\sigma_1^2)\big)\big).$$

Hence, by (2.7), there exists an $\varepsilon > 0$ such that $L_n(\varepsilon) \to 0$ as $n \to \infty$; in fact, for every $\varepsilon \geqslant f(\sigma_1^2)/\sigma_1^2$

(2.8) $$L_n(\varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Thus, by (2.6), (2.8) and Lemma 2 (ii), every sequence $\{X_n, n \geqslant 1\}$ satisfying the assumptions of Theorem 1 satisfies also the central limit theorem.

COROLLARY 1. *Under the assumptions of Theorem* 1 *with some* $0 < \gamma < 1/4$, *for every* $\varrho > 0$

$$P\Big(\lim_{n\to\infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2 S_k^{2\varrho}/V_k^{2(\varrho+1)}) = 2^{\varrho}(\pi)^{-1/2}\Gamma(\varrho+\tfrac{1}{2})\Big) = 1$$

and

$$P\Big(\lim_{n\to\infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2 S_k^2/V_k^4) = 1\Big) = 1.$$

THEOREM 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables, defined on* $(\Omega, \mathscr{A}, P)$, *with* $\mathrm{E}X_n = 0$, $0 < \mathrm{E}X_n^2 = \sigma^2 < \infty$, $n \geqslant 1$. *If, for some* $0 < r < 1$,

(2.9) $$\sum_{n=1}^{\infty} n^{-r} \mathrm{E}X_n^2 I(|X_n| \geqslant \sigma^r n^{r/2}) < \infty,$$

then for every real function $a(x)$ *which is a.e. continuous and* $|a(x)| \leqslant \exp(\gamma x^2)$, $\gamma < 1/4$,

(2.10) $$P\Big(\lim_{n\to\infty} (\log n)^{-1} \sum_{k=1}^{n} k^{-1} a(S_k/\sigma k^{1/2}) = \int_{-\infty}^{\infty} a(x)\, d\Phi(x)\Big) = 1.$$

COROLLARY 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables, defined on* $(\Omega, \mathscr{A}, P)$, *with* $\mathrm{E}X_n = 0$, $0 < \mathrm{E}X_n^2 = \sigma^2$ *and* $\mathrm{E}|X_n|^{2+\delta} = \beta_{2+\delta} < \infty$, $n \geqslant 1$, *for some* $\delta > 0$. *If* $a(x)$ *is a real function which is a.e. continuous and* $|a(x)| \leqslant \exp(\gamma x^2)$ *for some* $\gamma < 1/4$, *then* (2.10) *holds and for every* $\varrho > 0$

$$P\Big(\lim_{n\to\infty} (\log n)^{-1} \sum_{k=1}^{n} k^{-\varrho-1} S_k^{2\varrho} = (2\sigma^2)^{\varrho}(\pi)^{-1/2}\Gamma(\varrho+\tfrac{1}{2})\Big) = 1.$$

3. Auxiliary lemmas. The proof of Theorem 1 is based on a martingale form of the Skorokhod representation theorem and on the Tomkins law of the iterated logarithm.

We present these results for the convenience of the reader.

LEMMA 1 (Strassen [11], Theorem 4.4). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables such that, for all n,* $\mathrm{E}(X_n^2|X_1, \ldots, X_{n-1})$ *is defined and* $E(X_n|X_1, \ldots, X_{n-1}) = 0$ *P-a.s. Put*

$$S_n = \sum_{i \leqslant n} X_i \quad \text{and} \quad V_n = \sum_{i \leqslant n} \mathrm{E}(X_i^2|X_1, \ldots, X_{i-1}),$$

where, in order to avoid trivial complications, we assume $V_1 = \mathrm{E}X_1^2 > 0$. *Let f be a positive, nondecreasing real function f on* $\boldsymbol{R}^+$ *such that the function* $f(x)/x$ *is nonincreasing on* $\boldsymbol{R}^+$. *Assume that* $V_n \to \infty$ *P-a.s. as* $n \to \infty$ *and*

$$(3.1) \qquad \sum_{n=1}^{\infty} \left(f(V_n)\right)^{-1} \int_{x^2 > f(V_n)} x^2 dP(X_n \leqslant x \mid X_1, \ldots, X_{n-1}) < \infty \ \ P\text{-a.s.}$$

Let S be the (random) function on $\boldsymbol{R}^+ \cup \{0\}$ *obtained by interpolating* S_n *at* V_n *in such a way that* $S(0) = 0$ *and S is constant in each* $\langle V_n, V_{n+1})$ *(or, alternatively, is linear in each* $\langle V_n, V_{n+1}\rangle$*). Then without loss of generality there is a Brownian motion* $\{W(t), t \geqslant 0\}$ *such that, as* $t \to \infty$,

$$(3.2) \qquad S(t) = W(t) + o\left(\log t \left(tf(t)\right)^{1/4}\right) \ \ P\text{-a.s.}$$

LEMMA 2 (Tomkins [12], Theorem 3.1). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables such that* $\mathrm{E}X_n = 0$ *and* $\mathrm{E}X_n^2 < \infty$, $n \geqslant 1$. *Define, for* $n \geqslant 1$, $S_n = X_1 + \ldots + X_n$, $V_n^2 = \mathrm{E}S_n^2$, $t_n^2 = 2\log\log V_n^2$, *and the Lindeberg function*

$$L_n(\varepsilon) = V_n^{-2} \sum_{k=1}^{n} \mathrm{E}X_k^2 I(|X_k| > \varepsilon V_k), \qquad \varepsilon > 0.$$

Suppose that

$$(3.3) \qquad t_n X_n/V_n \to 0 \ \ a.s. \quad \text{and} \quad V_n \to \infty \quad \text{as } n \to \infty.$$

(i) *The functions*

$$L_-(\varepsilon) = \liminf_{n\to\infty} L_n(\varepsilon) \quad \text{and} \quad L_+(\varepsilon) = \limsup_{n\to\infty} L_n(\varepsilon)$$

are both constant functions, and

$$\left(1 - L_+(\varepsilon)\right)^{1/2} \leqslant \limsup_{n\to\infty} S_n/(V_n t_n) \leqslant \left(1 - L_-(\varepsilon)\right)^{1/2} \ \ a.s. \quad \text{for every } \varepsilon > 0.$$

(ii) *If* $\lim_{n\to\infty} L_n(\varepsilon) = 0$ *for some* $\varepsilon > 0$, *then the* CLT *holds.*

(iii) *Let* $\mathrm{E}X_n^2 = o(V_n^2)$. *If the* CLT *holds, then*

$$(3.4) \qquad \limsup_{n\to\infty} S_n/(t_n V_n) = 1 \ \ P\text{-a.s.}$$

4. Proofs of theorems. The symbol C, with or without subscripts, denotes a positive generic constant.

Proof of Theorem 1. Assume that $a(x) = I_{(-\infty,u)}(x)$ is the indicator function of the interval $(-\infty, u)$. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and normally distributed random variables with zero means and variance σ_n^2, $n \geqslant 1$. Then S_n/V_n is normally distributed with zero mean and variance one.

Let, for $j < n$, $S_{j,n} = S_n - S_j$ and

$$g_{jn} = \mathrm{E}\left\{\left(I_{(-\infty,u)}(S_j/V_j) - \Phi(u)\right)\left(I_{(-\infty,u)}(S_n/V_n) - \Phi(u)\right)\right\}, \tag{4.1}$$

where, and in what follows, $\Phi(u) = \Phi((-\infty, u))$. Then $S_{j,n}$ is independent of S_j and normally distributed with zero mean and variance $V_{j,n}^2 = V_n^2 - V_j^2$. Furthermore,

$$S_n/V_n = (S_j/V_j)(V_j/V_n) + (S_{j,n}/V_{j,n})(V_{j,n}/V_n), \tag{4.2}$$

and so, by (4.1) and (4.2),

$$\begin{aligned} g_{jn} &= \mathrm{E} I_{(-\infty,u)}(S_j/V_j) I_{(-\infty,u)}(S_n/V_n) - \Phi^2(u) \\ &= P\left(S_j < uV_j;\ (S_{j,n}/V_{j,n}) < (uV_n - S_j)/V_{j,n}\right) - \Phi^2(u) \\ &= (2\pi)^{-1/2} \int_{-\infty}^{u} \exp(-x^2/2)\left\{\Phi\left((uV_n - xV_j)/V_{j,n}\right) - \Phi(u)\right\} dx. \end{aligned} \tag{4.3}$$

On the other hand, by the inequalities (3.3) and (3.4) of Petrov [7], p. 161, for every x and u we get

$$\begin{aligned} &\left|\Phi\left((uV_n - xV_j)/V_{j,n}\right) - \Phi(u)\right| \\ &\leqslant (2\pi e)^{-1/2}(V_n/V_{j,n} - 1) + (2\pi)^{-1/2}|x|(V_j/V_{j,n}) \leqslant (2\pi)^{-1/2}(1 + |x|)(V_j/V_{j,n}) \end{aligned} \tag{4.4}$$

since $V_n/V_{j,n} - 1 \leqslant V_j/V_{j,n}$. Hence, by (4.3) and (4.4),

$$|g_{jn}| \leqslant (2\pi)^{-1}(V_j/V_{j,n}) \int_{-\infty}^{u} (1 + |x|)\exp(-x^2/2)\, dx \leqslant C(V_j/V_{j,n}), \tag{4.5}$$

where C is an absolute constant.

It is evident that $|g_{jn}| \leqslant 1$. Hence, by (4.5)

$$\begin{aligned} \mathrm{E}\left\{\sum_{n=1}^{N} (\sigma_n^2/V_n^2)\left(I_{(-\infty,u)}(S_n/V_n) - \Phi(u)\right)\right\}^2 &\leqslant 2 \sum_{n=1}^{N} \sum_{j=1}^{n} (\sigma_n^2 \sigma_j^2/V_n^2 V_j^2)|g_{jn}| \\ &\leqslant 2C \sum_{n=1}^{N} (\sigma_n^2/V_n^2) \sum_{j \in A_n} \{\sigma_j^2/(V_{j,n} V_j)\} \\ &\quad + 2 \sum_{n=1}^{N} (\sigma_n^2/V_n^2) \sum_{j \in B_n} \{\sigma_j^2/V_j^2\}, \end{aligned}$$

where $A_n = \{j\colon V_j^2 < V_n^2/2\}$ and $B_n = \{j \leqslant n\colon V_j^2 \geqslant V_n^2/2\}$.

Note also that

$$\sigma_j^2/\{V_j^2(V_n^2-V_j^2)\}^{1/2} \leqslant \int_{V_{j-1}^2/V_n^2}^{V_j^2/V_n^2} \frac{dt}{\sqrt{t(1-t)}}$$

for every j such that $V_j^2 < V_n^2/2$. Hence

$$\sum_{j\in A_n} \{\sigma_j^2/(V_{j,n}V_j)\} \leqslant \int_0^{1/2} \frac{1}{\sqrt{t(1-t)}}\,dt = \frac{\pi}{2}.$$

Now,

$$\sum_{j\in B_n}^{n} (\sigma_j^2/V_j^2) \leqslant 2.$$

Consequently, combining the results from (4.3) down, we get

$$\text{(4.6)}\quad \mathrm{E}\{(\log V_N^2)^{-1} \sum_{n=1}^{N} (\sigma_n^2/V_n^2)(I_{(-\infty,u)}(S_n/V_n)-\Phi(u))\}^2$$

$$\leqslant C_1(\log V_N^2)^{-2} \sum_{n=1}^{N} (\sigma_n^2/V_n^2) \leqslant C_1(\log V_N^2)^{-2}(1+\sum_{n=2}^{N} \int_{V_{n-1}^2/V_N^2}^{V_n^2/V_N^2} (1/x)\,dx)$$

$$= C_1(\log V_N^2)^{-2}(1+\int_{\sigma_1^2/V_N^2}^{1} (1/x)\,dx) = C_1(\log V_N^2)^{-2}(1-\log\sigma_1^2+\log V_N^2)$$

$$\leqslant C_2(\log V_N^2)^{-1}.$$

Define an increasing sequence of integers $\{N_k, k\geqslant 1\}$ by $V_{N_k}^2 \leqslant 2^{k^2} < V_{N_k+1}^2$. Since $\sigma_n^2 = o(V_n^2)$, as $n\to\infty$, entails $V_{n+1}^2 \sim V_n^2$, necessarily

$$V_{N_k+1}^2 = V_{N_k}^2 + \sigma_{N_k+1}^2 = V_{N_k}^2 + o(V_{N_k}^2),$$

and so $V_{N_k}^2 \sim 2^{k^2}$ as $k\to\infty$.

Hence we infer by combining Chebyshev's inequality, (4.6) and the Borel–Cantelli lemma that P-a.s.

$$\text{(4.7)}\quad (\log V_{N_k}^2)^{-1} \sum_{n=1}^{N_k} (\sigma_n^2/V_n^2)(I_{(-\infty,u)}(S_n/V_n)-\Phi(u)) \to 0 \quad \text{as } k\to\infty.$$

On the other hand, for $N_k < N < N_{k+1}$ we have $(\log V_N^2)^{-1} \leqslant (\log V_{N_k}^2)^{-1}$ and

$$(\log V_N^2)^{-1} \Big| \sum_{n=N_k+1}^{N} (\sigma_n^2/V_n^2)(I_{(-\infty,u)}(S_n/V_n)-\Phi(u))\Big|$$

$$\leqslant (\log V_{N_k}^2)^{-1} \sum_{n=N_k+1}^{N_{k+1}} (\sigma_n^2/V_n^2) \leqslant (\log V_{N_k}^2)^{-1} \sum_{n=N_k+1}^{N_{k+1}} \int_{V_{n-1}^2/V_{N_{k+1}}^2}^{V_n^2/V_{N_{k+1}}^2} (1/x)\,dx$$

$$= (\log V_{N_k}^2)^{-1}\{\log V_{N_{k+1}}^2 - \log V_{N_k}^2\} \leqslant Ck^{-1} \to 0 \quad \text{as } k\to\infty.$$

Consequently, by (4.7), we get P-a.s.

$$(4.8)\qquad (\log V_N^2)^{-1}\sum_{n=1}^{N}(\sigma_n^2/V_n^2)\big(I_{(-\infty,u)}(S_n/V_n)-\Phi(u)\big)\to 0 \quad \text{as } n\to\infty.$$

Thus the case where $a(x)=I_{(-\infty,u)}(x)$ and X_n, $n\geqslant 1$, are normally distributed is considered.

If X_n, $n\geqslant 1$, are not normally distributed, then by Lemma 1

$$(4.9)\qquad S_n-W(V_n^2)=\varepsilon_n(\omega)\big(V_n^2 f(V_n^2)\big)^{1/4}(\log V_n^2)\ P\text{-a.s.} \quad \text{as } n\to\infty,$$

where $\{W(t),\, t\geqslant 0\}$ is a standard Brownian motion and $\varepsilon_n(\omega)\to 0$ as $n\to\infty$ for almost all $\omega\in\Omega$.

Let

$$\eta_n=\eta_n(\omega)=\sup_{k\geqslant n}|\varepsilon_k(\omega)|(\log V_k^2)\big(f(V_k^2)/V_k^2\big)^{1/4}.$$

Then, by (4.9) and (2.2),

$$(4.10)\qquad I_{(-\infty,u-\eta_n)}\big(W(V_n^2)/V_n\big)\leqslant I_{(-\infty,u)}(S_n/V_n)\leqslant I_{(-\infty,u+\eta_n)}\big(W(V_n^2)/V_n\big).$$

Let $\varepsilon>0$ be given. Using (4.10) and (4.8), it is easy to see that

$$\begin{aligned}(\log V_N^2)^{-1}&\sum_{n=1}^{N}(\sigma_n^2/V_n^2)I_{(-\infty,u)}(S_n/V_n)\\ &\leqslant(\log V_N^2)^{-1}\Big\{\sum_{n=1}^{M}(\sigma_n^2/V_n^2)\sum_{n=M+1}^{N}(\sigma_n^2/V_n^2)I_{(-\infty,u+\eta_M)}\big(W(V_n^2)/V_n\big)\Big\}\\ &\leqslant(1+\log V_M^2-\log\sigma_1^2)(\log V_N^2)^{-1}+\Phi(u+\eta_M)+\varepsilon\leqslant\Phi(u)+2\varepsilon\end{aligned}$$

for sufficiently large N and suitable $M=M(N)$. Similarly, the left-hand sum can be bounded below, so that (4.8) is established for X_n, $n\geqslant 1$, not necessary normally distributed, too.

Let now $a(x)=\exp(\gamma x^2)$, $\gamma<1/4$, and let $\{X_n,\, n\geqslant 1\}$ be a sequence of independent and normally distributed random variables with zero means and variance σ_n^2, $n\geqslant 1$. Then

$$\mathrm{E}a(S_n/V_n)=(1-2\gamma)^{-1/2}.$$

We set

$$h_{jn}=\mathrm{E}\big\{\big(a(S_j/V_j)-(1-2\gamma)^{-1/2}\big)\big(a(S_n/V_n)-(1-2\gamma)^{-1/2}\big)\big\}.$$

Then, taking into account (4.2), we conclude that

$$(4.11)\qquad h_{jn}=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{\gamma x^2+\gamma[xV_j+yV_{j,n}]^2/V_n^2-(x^2+y^2)/2\}\,dx\,dy \\ -(1-2\gamma)^{-1}$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{\frac{\gamma}{V_n^2}(x^2V_j^2 + 2xyV_jV_{j,n} - y^2V_j^2) + (\gamma - \tfrac{1}{2})(x^2+y^2)\right\} dxdy$$

$$- (1-2\gamma)^{-1}$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\exp\left\{\frac{\gamma}{V_n^2}(x^2 V_j^2 + 2xyV_jV_{j,n} - y^2V_j^2)\right\} - 1\right]$$

$$\times \exp\{(\gamma - \tfrac{1}{2})(x^2+y^2)\}\, dxdy.$$

Since $(xV_{j,n} - yV_j)^2 \geqslant 0$, we have

$$x^2V_j^2 + 2xyV_jV_{j,n} - y^2V_j^2 \leqslant x^2V_n^2.$$

On the other hand, $|\exp(x) - 1| \leqslant |x|(\exp x + 1)$. Hence, for $j < n$

$$|h_{jn}| \leqslant (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{|\gamma|}{V_n^2}\right)(x^2V_j^2 + 2|xy|\, V_jV_{j,n} + y^2V_j^2)$$

$$\times \exp\{(2\gamma - 1/2)(x^2+y^2)\}\, dxdy$$

$$\leqslant C(V_j^2/V_n^2 + V_jV_{j,n}/V_n^2),$$

so that for every $1 \leqslant j \leqslant n$, $n \geqslant 1$, $|h_{jn}| \leqslant C$. Furthermore, by the same inequality we get $|h_{jn}| \leqslant C(V_j/V_{j,n})$.

Thus, as in the case $a(x) = I_{(-\infty,u)}(x)$, we get P-a.s.

$$(\log V_{N_k}^2)^{-1} \sum_{n=1}^{N_k} (\sigma_n^2/V_n^2)\{\exp(\gamma S_n^2/V_n^2) - (1-2\gamma)^{-1/2}\} \to 0 \quad \text{as } k \to \infty,$$

where $\{N_k, k \geqslant 1\}$ is the sequence defined above. Moreover, by the law of the iterated logarithm, Theorem 1.106 of Freedman [4], we have P-a.s.

$$\exp(\gamma S_n^2/V_n^2) \leqslant (\log V_n^2)^{\gamma + 1/4} \tag{4.12}$$

for sufficiently large n. Hence for $N_k < N < N_{k+1}$ we have $(\log V_N^2)^{-1} \leqslant (\log V_{N_k}^2)^{-1}$ and

$$(\log V_{N_k}^2)^{-1} \sum_{n=N_k+1}^{N_{k+1}} (\sigma_n^2/V_n^2)\,|\exp(\gamma S_n^2/V_n^2) - (1-2\gamma)^{-1/2}|$$

$$\leqslant C(\log V_{N_k}^2)^{\gamma - 3/4} \sum_{n=N_k+1}^{N_{k+1}} (\sigma_n^2/V_n^2)$$

$$\leqslant C(\log V_{N_k}^2)^{\gamma - 3/4}(\log V_{N_{k+1}}^2 - \log V_{N_k}^2) \leqslant C_1 k^{2\gamma - 1/2}.$$

Thus P-a.s.

$$(\log V_N^2)^{-1} \sum_{n=1}^{N} (\sigma_n^2/V_n^2)\{\exp(\gamma S_n^2/V_n^2) - (1-2\gamma)^{-1/2}\} \to 0 \quad \text{as } N \to \infty.$$

If X_n, $n \geqslant 1$, are random variables not normally distributed, then

$$(4.13)\quad |\exp(\gamma S_n^2/V_n^2) - \exp(\gamma W^2(V_n^2)/V_n^2)|$$
$$\leqslant (2|\gamma|/V_n^2)|S_n - W(V_n^2)| \max\{|S_n|\exp(\gamma S_n^2/V_n^2),\ |W(V_n^2)|\exp(\gamma W^2(V_n^2)/V_n^2)\}.$$

But, under the assumptions of Theorem 1, (2.6) and (2.8) hold, so that by Lemma 2 (ii) and (iii)

$$(4.14)\quad \limsup_{n\to\infty} \frac{|S_n|}{(2V_n^2 \log\log V_n^2)^{1/2}} = 1 \quad P\text{-a.s.}$$

Thus, by (4.9), Theorem 1.106 of Freedman [4] and (4.14), the right-hand side of inequality (4.13) can be bounded by

$$4|\gamma\varepsilon_n|\left(f(V_n^2)/V_n^2\right)^{1/4} (\log V_n^2)^{(3\gamma \vee 2\gamma)+1}$$

for sufficiently large n. Since (2.3) holds and $\varepsilon_n \to 0$ P-a.s., as $n \to \infty$, so that (2.5) also holds for $a(x) = \exp(\gamma x^2)$, $\gamma < 1/4$.

Let now $a(x)$ be a function satisfying the assumptions of Theorem 1. Then, similarly to Schatte [10], we introduce an auxiliary function $a_1(x)$ which vanishes for $|x| > K$ and is in each of the intervals

$$-K + 2iK/L \leqslant x < -K + 2(i+1)K/L, \qquad i = 0, 1, 2, \ldots, L-1,$$

equal to the supremum of $a(x) - \exp(\gamma x^2)$ in these intervals. Let $a_2(x) = a_1(x) + \exp(\gamma x^2)$ and choose first K and then L large enough so that

$$\int_{-\infty}^{\infty} a_2(x)\, d\Phi(x) \leqslant \int_{-\infty}^{\infty} a(x)\, d\Phi(x) + \varepsilon/2.$$

This is possible since $a(x)$ is continuous a.e. and, therefore, Riemann–Stieltjes integrable with respect to $\Phi(x)$. Obviously, $a(x) \leqslant a_2(x)$ for every real number x. The function $a_2(x)$ is a finite linear combination of the special functions already considered in the proof. Thus

$$(\log V_N^2)^{-1} \sum_{n=1}^{N} (\sigma_n^2/V_n^2)\, a(S_n/V_n) \leqslant (\log V_N^2)^{-1} \sum_{n=1}^{N} (\sigma_n^2/V_n^2)\, a_2(S_n/V_n)$$
$$\leqslant \int_{-\infty}^{\infty} a_2(x)\, d\Phi(x) + \varepsilon/2 \leqslant \int_{-\infty}^{\infty} a(x)\, d\Phi(x) + \varepsilon$$

for sufficiently large N and almost all ω. Replacing $a(x)$ by $-a(x)$ we obtain the assertion of Theorem 1.

Proof of Theorem 2. Let us observe that, under assumptions of Theorem 2, $V_n^2 = \sigma^2 n$, $n \geqslant 1$. On the other hand, for every $0 < r < 1$, the function $f(x) = |x|^r$ satisfies the assumptions of Theorem 1. Thus Theorem 2 is a consequence of Theorem 1.

Proof of Corollary 2. Let us observe that for every $\delta > 0$ there exists an r such that $0 < r < 1$ and (2.9) holds. In fact, for every $n \geqslant 1$

$$EX_n^2 I(|X_n| \geqslant \sigma^r n^{r/2}) \leqslant \sigma^{-r\delta} n^{-r\delta/2} \beta_{2+\delta}.$$

Thus it is enough to take $2/(2+\delta) < r < 1$, so that Corollary 2 is a consequence of Theorem 2.

REFERENCES

[1] M. Atlagh and M. Weber, *An almost sure central limit theorem relative to subsequences*, C. R. Acad. Sci. Paris, Sér. I, Math. 315 (1992), pp. 203–206.

[2] I. Berkes and H. Dehling, *On the almost sure (a.s.) central limit theorem for random variables with infinite variance*, J. Theoret. Probab. 7 (1994), pp. 667–680.

[3] G. A. Brosamler, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. 104 (1988), pp. 561–574.

[4] D. Freedman, *Brownian Motion and Diffusion*, Holden-Day, 1971.

[5] M. T. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Lett. 9 (1990), pp. 201–205.

[6] M. Peligrad and Q.-M. Shao, *A note on the almost sure central limit theorem for weakly dependent random variables*, ibidem 22 (1995), pp. 131–136.

[7] V. V. Petrov, *Limit Theorems for Sums of Independent Random Variables* (in Russian), Nauka, Moscow 1987.

[8] B. Rodzik and Z. Rychlik, *An almost sure central limit theorem for independent random variables*, Ann. Inst. H. Poincaré 30 (1994), pp. 1–11.

[9] P. Schatte, *On strong versions of the central limit theorem*, Math. Nachr. 137 (1988), pp. 249–256.

[10] — *On the central limit theorem with almost sure convergence*, Probab. Math. Statist. 11 (1991), pp. 237–246.

[11] V. Strassen, *Almost sure behaviour of sums of independent random variables and martingales*, Proc. Fifth Berkeley Symp. Math. Statist. Probab. 2, Univ. of California Press, 1967, pp. 315–343.

[12] R. J. Tomkins, *Refinements of Kolmogorov's law of the iterated logarithm*, Statist. Probab. Lett. 14 (1992), pp. 321–325.

Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland

Chair of Mathematics
Catholic University of Lublin
al. Racławickie 1
20-950 Lublin, Poland

Received on 14.9.1995