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LAWS OF LARGE NUMBERS ON SIMPLY CONNECTED STEP 2-NILPOTENT LIE GROUPS

BY

DANIEL NEUENSCHWANDER (BIEL)

Abstract. The Strong Law of Large Numbers due to Marcinkiewicz and Zygmund is carried over to simply connected step 2-nilpotent Lie groups. Moreover, for such groups, we prove analogues of the classical theorems of Hsu–Robbins–Erdös, respectively Baum–Katz, giving information on the rate of convergence in Laws of Large Numbers.

1. Introduction. Simply connected step 2-nilpotent Lie groups are groups which arise as follows: Let $[\cdot,\cdot]\colon \boldsymbol{R}^d\times\boldsymbol{R}^d\to\boldsymbol{R}^d\,(d\geqslant 0)$ be a skew-symmetric bilinear map such that $[[\boldsymbol{R}^d,\boldsymbol{R}^d],\boldsymbol{R}^d]=\{0\}$. Then $G=\boldsymbol{R}^d$, equipped with the multiplication

$$x\cdot y = x+y+\tfrac{1}{2}[x,y],$$

is a group of the above-mentioned type. Clearly, $e=0$ and $x^{-1}=-x$. The best-known (noncommutative) examples are the *Heisenberg groups* $\boldsymbol{H}^d$ given by

$$\boldsymbol{H}^d=\boldsymbol{R}^{2d+1}=\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$$

and

$$[x,y]=(0,0,\langle x^{(1)},y^{(2)}\rangle-\langle x^{(2)},y^{(1)}\rangle)\in\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}=\boldsymbol{H}^d,$$

where

$$x=(x^{(1)},x^{(2)},x^{(3)}),\qquad y=(y^{(1)},y^{(2)},y^{(3)})\in\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}=\boldsymbol{H}^d.$$

The so-called *groups of type H*, which arise in the context of composition of quadratic forms, all belong to this class (cf. Kaplan [5]). See also Folland and Stein [3].

For $G=\boldsymbol{R}$, it was shown by Hsu–Robbins–Erdös that in the Law of Large Numbers complete convergence is equivalent to the finiteness of the second moment. Baum and Katz strengthened the Marcinkiewicz–Zygmund Law of Large Numbers in the sense that there is not only strong convergence, but convergence of certain series which implies complete convergence. Both theorems due to Hsu–Robbins–Erdös, respectively Baum–Katz, may be interpreted as results concerning the rate of convergence in the corresponding Laws of Large Numbers.

In this paper we will prove an analogue of the Marcinkiewicz–Zygmund Law of Large Numbers (which contains the Kolmogorov Strong Law of Large Numbers for random variables with expectation as a special case) for simply connected step 2-nilpotent Lie groups. Moreover, we carry over the theorems of Hsu–Robbins–Erdös, respectively Baum–Katz, to this context.

2. Preliminaries and notation. For real-valued functions f, g, the notation $f \lesssim g$ means that there is a constant $K > 0$ such that $f(x) \leqslant Kg(x)$ for all x.

Let G be a simply connected step 2-nilpotent Lie group. Let $V_2 = [G, G]$ be the center of G, and V_1 a complement of V_2, i.e.

$$G = V_1 \oplus V_2. \tag{1}$$

The notation $x = (x', x'') \in G$ will always be understood with respect to (1), i.e. $x' \in V_1$, $x'' \in V_2$. For $a > 0$, $x \in G$ put

$$\delta_a(x) = (ax', a^2 x'').$$

Clearly, δ_a is an automorphism of G. A *homogeneous gauge* on G is a continuous function $|\cdot|\colon G \to [0, \infty[$ satisfying

$$|0| = 0, \qquad |x| > 0 \quad (x \in G \backslash \{0\}),$$

$$|\delta_a(x)| = a|x| \quad (a > 0, x \in G)$$

(cf. Goodman [4]). By a compactness argument we have

$$|-x| \lesssim |x| \tag{2}$$

(cf. Goodman [4], Lemma 2) and

$$|x \cdot y| \lesssim |x| + |y| \quad (x, y \in G). \tag{3}$$

An example is

$$|x|_1 = (\|x'\|^4 + \|x''\|^2)^{1/4}.$$

By Goodman [4], Lemma 1, all homogeneous gauges $|\cdot|$ are equivalent (i.e. $|\cdot| \lesssim |\cdot|_1 \lesssim |\cdot|$).

Let $p > 0$, $c \in G$, and let X be a G-valued random variable. Since

$$E|X \cdot c|^p \lesssim E(|X| + |c|)^p,$$

it follows that

$$E|X|^p < \infty \Rightarrow E|X \cdot c|^p < \infty.$$

A sequence $\{X_n\}_{n \geqslant 1}$ of random variables is said to *converge completely* to the random variable X if for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty.$$

By the Borel–Cantelli Lemma, complete convergence implies a.s. convergence.

3. The Marcinkiewicz-Zygmund Law of Large Numbers. The following lemma is well known:

LEMMA 1. *We have* $\|x'\|, \sqrt{\|x''\|} \lesssim |x|$.

Now we prove the following analogue of the classical Marcinkiewicz-Zygmund Strong Law of Large Numbers (see e.g. Chow and Teicher [2], Theorem 5.2.2):

THEOREM 1. *Let G be a simply connected step 2-nilpotent Lie group, and $|\cdot|$ an arbitrary homogeneous gauge on G. Assume $X_1, X_2, \ldots$ are i.i.d. G-valued random variables defined on some common probability space $(\Omega, \mathscr{B}, P)$. Then for any $p \in]0, 2[$*

$$\delta_{n^{-1/p}}\Big(\prod_{j=1}^{n} (X_j \cdot c)\Big) \xrightarrow{a.s.} 0 \qquad (n \to \infty) \tag{4}$$

for some $c \in G$ iff $E|X_1|^p < \infty$. If so, then in the case $1 \leqslant p < 2$ we have

$$c' = -EX_1',$$

while c'' and, in the case $0 < p < 1$, also c' can be chosen arbitrarily.

For the proof of the case $1 \leqslant p < 2$ we need the following lemma, which is similar to Kronecker's Lemma (cf. Chow and Teicher [2], Lemma 5.1.2).

LEMMA 2. *For any sequence $\{a_n\}_{n \geqslant 1} \subset \mathbf{R}^d$ and $\{b_n\}_{n \geqslant 1} \subset]0, \infty[$ such that $b_{n+1} \geqslant b_n$ $(n \geqslant 1)$, $b_n \to \infty$ $(n \to \infty)$, and*

$$\frac{1}{b_n} \sum_{j=1}^{n} \frac{a_j}{b_j} \to 0 \qquad (n \to \infty)$$

we have

$$\frac{1}{b_n^2} \sum_{j=1}^{n} a_j \to 0 \qquad (n \to \infty).$$

Proof. For $\varepsilon > 0$, choose $N \in \mathbf{N}$ such that

$$\frac{1}{b_n} \left\| \sum_{j=1}^{n} \frac{a_j}{b_j} \right\| \leqslant \frac{\varepsilon}{3} \qquad (n \geqslant N).$$

Then, by summation by parts, we get

$$\frac{1}{b_n^2}\left\|\sum_{j=1}^{n} a_j\right\| = \frac{1}{b_n^2}\left\|\sum_{j=1}^{n} b_j\frac{a_j}{b_j}\right\| \leqslant \frac{1}{b_n}\left\|\sum_{j=1}^{n} \frac{a_j}{b_j}\right\| + \frac{1}{b_n^2}\left\|\sum_{j=1}^{n-1} (b_{j+1}-b_j)\sum_{i=1}^{j}\frac{a_i}{b_i}\right\|$$

$$\leqslant \frac{\varepsilon}{3}+\frac{1}{b_n^2}\left\|\sum_{j=1}^{N-1} (b_{j+1}-b_j)\sum_{i=1}^{j}\frac{a_i}{b_i}\right\| + \frac{\varepsilon}{3} \leqslant \varepsilon$$

for n large enough. ■

Proof of Theorem 1. 1. The "only if" part may be proved similarly to that in the classical situation (cf. Chow and Teicher [2], p. 122): Since

$$\delta_{n^{-1/p}}(X_n\cdot c) = \delta_{n^{-1/p}}\Big(\big(-\prod_{j=1}^{n-1}(X_j\cdot c)\big)\cdot\prod_{j=1}^{n}(X_j\cdot c)\Big)$$

$$= \delta_{(n/(n-1))^{-1/p}}\Big(\delta_{(n-1)^{-1/p}}\big(-\prod_{j=1}^{n-1}(X_j\cdot c)\big)\Big)\cdot\delta_{n^{-1/p}}\Big(\prod_{j=1}^{n}(X_j\cdot c)\Big)$$

$$\xrightarrow{\text{a.s.}} 0 \quad (n\to\infty)$$

by (4), it follows from the Borel–Cantelli Lemma that

$$\sum_{n=1}^{\infty} P(|X_1\cdot c| > n^{1/p}) < \infty,$$

which implies $E|X_1\cdot c|^p < \infty$ by Corollary 4.1.3 of Chow and Teicher [2]. So $E|X_1\cdot c|^p < \infty$.

2. Assume $0 < p < 1$, $E|X_1|^p < \infty$, and let $c\in G$ be arbitrary. By considering $X_j\cdot c$ instead of X_j, we may without loss of generality assume $c = 0$. By Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers we have

$$n^{-1/p}\Big(\sum_{j=1}^{n} X_j\Big)' \xrightarrow{\text{a.s.}} 0 \quad (n\to\infty), \tag{5}$$

so we have to prove

$$n^{-2/p}\Big(\prod_{j=1}^{n} X_j\Big)'' \xrightarrow{\text{a.s.}} 0 \quad (n\to\infty). \tag{6}$$

This is equivalent to

$$n^{-2/p}\Big(\sum_{j=1}^{n} X_j\Big)'' + \frac{1}{2}n^{-2/p}\Big(\sum_{1\leqslant i<j\leqslant n}[X_i, X_j]\Big)'' \xrightarrow{\text{a.s.}} 0 \quad (n\to\infty).$$

The first summand tends to 0 a.s. by Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers. So it remains to show

$$n^{-2/p}\Big(\sum_{1 \leq i < j \leq n} [X_i, X_j]\Big)'' \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty). \tag{7}$$

But for this we have

$$\Big\| n^{-2/p}\Big(\sum_{1 \leq i < j \leq n} [X_i, X_j]\Big)''\Big\| = O\Big(\Big(n^{-1/p} \sum_{j=1}^{n} \|X_j'\|\Big)^2\Big) \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty)$$

by Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers, which proves (7), and thus (6).

3. Assume $1 \leq p < 2$, $E|X_1|^p < \infty$, and again without loss of generality $c = 0$, $EX_1' = 0$. Let $c'' \in V_2$ be arbitrary. Hence, as above, in order to prove (4), it remains to show (7). Define the V_2-valued random variables

$$Z_n := \Big[n^{-1/p} \sum_{j=1}^{n-1} X_j, X_n\Big]'',$$

$$\bar{Z}_n := Z_n \cdot \mathbb{1}_{A_j}, \quad \text{where } A_j = \Big\{n^{-1/p} \Big\|\Big(\sum_{j=1}^{n-1} X_j\Big)'\Big\| \leq 1\Big\}.$$

Without loss of generality, we may assume that there is a G-valued random variable X on $(\Omega, \mathscr{B}, P)$ which is distributed like X_1 and independent of $\{X_n\}_{n \geq 1}$. By Lemma 1, $E\|X'\| < \infty$, so since for every projection p onto some coordinate subspace of V_2

$$\begin{aligned} P\big(|p(\bar{Z}_n)| \geq x \,|\, X_1, X_2, \ldots, X_{n-1}\big) &\leq P(\|\bar{Z}_n\| \geq x \,|\, X_1, X_2, \ldots, X_{n-1}) \\ &\leq P(\|X_n'\| \geq Kx \,|\, X_1, X_2, \ldots, X_{n-1}) \\ &= P(\|X'\| \geq Kx \,|\, X_1, X_2, \ldots, X_{n-1}) \\ &\qquad (0 \leq x < \infty) \end{aligned}$$

for some fixed $K > 0$ a.s., the Theorem in Chatterji [1] yields

$$n^{-1/p} \sum_{j=1}^{n} (\bar{Z}_j - \alpha_j) \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty),$$

where α_n is a V_2-valued random variable on $(\Omega, \mathscr{B}, P)$ consisting of the components

$$E\big(p(\bar{Z}_n) \,|\, p(\bar{Z}_1), p(\bar{Z}_2), \ldots, p(\bar{Z}_{n-1})\big) \text{ a.s.}$$

By Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers, there is a.s. an N (random) such that a.s.

(8) $$Z_n = \bar{Z}_n \quad (n \geqslant N)$$

and

$$E(\bar{Z}_n | X_1, X_2, \ldots, X_{n-1}) = E(Z_n | X_1, X_2, \ldots, X_{n-1}) = 0 \quad (n \geqslant N)$$

(since $EX'_1 = 0$); hence a.s.

(9) $$\alpha_n = 0 \quad (n \geqslant N).$$

Thus, by (8) and (9),

$$n^{-1/p} \sum_{j=1}^{n} Z_j \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty).$$

By Lemma 2, putting

$$b_n := n^{1/p} \quad \text{and} \quad a_n := b_n Z_n = [\sum_{j=1}^{n-1} X_j, X_n]'',$$

we obtain (7).

4. The fact that c' is uniquely determined in the case $1 \leqslant p < 2$ follows from (5). ∎

4. Rates of convergence. We will use the following consequence of the Hölder Inequality:

LEMMA 3. *Assume $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in G$. Then for some $K > 0$ not depending on p, q*

$$\|\sum_{i=1}^{n} [x_i, y_i]\| \leqslant K \cdot \max_{1 \leqslant i \leqslant n} \|x'_i\| \cdot \sum_{i=1}^{n} \|y'_i\|.$$

Proof. By Hölder's Inequality,

$$\|\sum_{i=1}^{n} [x_i, y_i]\| \leqslant K (\sum_{i=1}^{n} \|x'_i\|^p)^{1/p} (\sum_{i=1}^{n} \|y'_i\|^q)^{1/q}$$

for $p, q > 1$, $1/p + 1/q = 1$. Now $q \to 1$ yields the assertion. ∎

First we carry over the theorem of Hsu–Robbins–Erdös (cf. Chow and Teicher [2], Corollary 10.4.2):

THEOREM 2. *Let G be a simply connected step 2-nilpotent Lie group, $|\cdot|$ a homogeneous gauge on G, and assume $\{X_n\}_{n \geqslant 1}$ are i.i.d. G-valued random variables. Then*

(10) $$\sum_{n=1}^{\infty} P\Big(\big|\delta_{n^{-1}}\big(\prod_{i=1}^{n} (X_i \cdot c)\big)\big| \geqslant \varrho\Big) < \infty \quad \textit{for every } \varrho > 0$$

iff

$$E|X_1|^2 < \infty, \quad c' = -EX'_1.$$

Proof. As in the proof of Theorem 1, we may assume without loss of generality that $c = 0$, $EX'_1 = 0$. We first prove the "if" direction: By Lemma 3,

$$\begin{aligned}|\delta_{n-1}(\prod_{j=1}^{n} X_j)| &\leqslant Kn^{-1}\|\sum_{j=1}^{n} X'_j\| + K\{n^{-2}\|\sum_{j=1}^{n} X''_j\|\}^{1/2}\\ &\quad + (K/\sqrt{2})\{n^{-1}\max_{1\leqslant j\leqslant n}\|X'_j\|\cdot n^{-1}\sum_{j=1}^{n}\|X'_j\|\}^{1/2}\\ &= KT_n^{(1)} + K\sqrt{T_n^{(2)}} + (K/\sqrt{2})\sqrt{T_n^{(3)}\cdot T_n^{(4)}}.\end{aligned}$$

Suppose $\sigma > 0$. We have

$$\sum_{n=1}^{\infty} P(T_n^{(1)} \geqslant \sigma) < \infty \tag{11}$$

by Corollary 10.4.2 of Chow and Teicher [2], and

$$\sum_{n=1}^{\infty} P(T_n^{(2)} \geqslant \sigma) < \infty \tag{12}$$

by Theorem 10.4.1 of Chow and Teicher [2] (with $\alpha = 2$, $p = \gamma = 1$; it is easy to see that the theorem is also valid in the case $EX \neq 0$, $\alpha > 1$). For $T_n^{(3)}$ we get

$$\begin{aligned}\sum_{n=1}^{\infty} P(T_n^{(3)} \geqslant \sigma) &\leqslant \sum_{n=1}^{\infty} nP(\|X'_1\| \geqslant \sigma n)\\ &\leqslant 1 + \int_1^{\infty} (t+1)P(\|X'_1\|^2 \geqslant \sigma^2 t^2)\,dt\\ &\leqslant 1 + H\int_1^{\infty} P(\|X'_1\|^2 \geqslant s)\,ds \leqslant 1 + HE\|X'_1\|^2 < \infty.\end{aligned} \tag{13}$$

Again by Corollary 10.4.2 of Chow and Teicher [2] we have

$$\sum_{n=1}^{\infty} P(T_n^{(4)} - E\|X'_1\| \geqslant \sigma) < \infty. \tag{14}$$

Now without loss of generality $E\|X'_1\| > 0$, for otherwise (12) proves the "if" direction. Since

$$\begin{aligned}P\Big(|\delta_{n-1}(\prod_{i=1}^{n} X_i)| \geqslant \varrho\Big) &\leqslant P\left(T_n^{(1)} \geqslant \frac{\varrho}{3K}\right) + P\left(T_n^{(2)} \geqslant \left(\frac{\varrho}{3K}\right)^2\right)\\ &\quad + P\left(T_n^{(3)} \geqslant \left(\frac{\varrho}{3K}\right)^2 \frac{1}{E\|X'_1\|}\right) + P(T_n^{(4)} - E\|X'_1\| \geqslant E\|X'_1\|),\end{aligned}$$

inequalities (11)–(14) yield the assertion.

As far as the "only if" part is concerned, first observe that, by (10) and the Borel–Cantelli Lemma,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i'+c') \xrightarrow{\text{a.s.}} 0;$$

hence by the classical Marcinkiewicz–Zygmund Law of Large Numbers it follows that $EX_1' = -c'$. Now we show that $E|X_1|^2 < \infty$. For this, it suffices to prove, by Corollary 10.4.2 of Chow and Teicher [2], that $E\|X_1''\| < \infty$. Observe that

$$\text{(15)}\qquad \prod_{i=1}^{n}(X_i\cdot c)+\prod_{i=1}^{n}(X_{n+1-i}\cdot c)$$
$$= \sum_{i=1}^{n}(X_i\cdot c)+\frac{1}{2}\sum_{1\leqslant i<j\leqslant n}[X_i\cdot c, X_j\cdot c]+\sum_{i=1}^{n}(X_{n+1-i}\cdot c)$$
$$+\frac{1}{2}\sum_{1\leqslant i<j\leqslant n}[X_{n+1-i}\cdot c, X_{n+1-j}\cdot c]$$
$$= 2\sum_{i=1}^{n}(X_i\cdot c)+\frac{1}{2}\sum_{1\leqslant i<j\leqslant n}([X_i\cdot c, X_j\cdot c]+[X_j\cdot c, X_i\cdot c]) = 2\sum_{i=1}^{n}(X_i\cdot c).$$

Now we may proceed as in the proof of Corollary 10.4.2 in Chow and Teicher [2]: Put $h = \dim V_2$, let $\{\tilde{X}_n\}_{n\geqslant 1}$ be an independent copy of the process $\{X_n\}_{n\geqslant 1}$, and put $Y_n = (X_n\cdot c)-(\tilde{X}_n\cdot c)$. Then, by (15), the symmetry of Y_n, and Lévy's Inequality (cf. Chow and Teicher [2], Lemma 3.3.5), we get

$$\text{(16)}\qquad 1-P^n(\|Y_1''\| < \beta) = P\big(\max_{1\leqslant i\leqslant n}\|Y_i''\| \geqslant \beta\big) \leqslant P\big(\max_{1\leqslant i\leqslant n}\big\|\sum_{j=1}^{i} Y_j''\big\| \geqslant \beta/2\big)$$
$$\leqslant 2hP\big(\big\|\sum_{i=1}^{n} Y_i''\big\| \geqslant \beta/2h\big) \leqslant 4hP\big(\big\|\sum_{i=1}^{n}(X_i\cdot c)''\big\| \geqslant \beta/4h\big)$$
$$\leqslant 8hP\Big(\Big\|\Big(\prod_{i=1}^{n}(X_i\cdot c)\Big)''\Big\| \geqslant \beta/4h\Big) \leqslant 8hP\Big(\Big|\prod_{i=1}^{n}(X_i\cdot c)\Big| \geqslant L\sqrt{\beta/4h}\Big)$$

for some constant $L > 0$; hence for $\gamma = L/\sqrt{4h}$, by (10) and (16) we get

$$\infty > \sum_{n=1}^{\infty}\big(1-P^n(\|Y_1''\| < n^2)\big) = \sum_{n=1}^{\infty} P(\|Y_1''\| \geqslant n^2)\sum_{j=0}^{n-1} P^j(\|Y_1''\| < n^2)$$
$$\geqslant \sum_{n=1}^{\infty} nP(\|Y_1''\| \geqslant n^2)\left[\frac{1}{n}\sum_{j=0}^{n-1}\Big(1-8hP\Big(\Big|\prod_{i=1}^{j}(X_i\cdot c)\Big| \geqslant \gamma j\Big)\Big)\right].$$

The expression [...] tends to 1 as $n \to \infty$ by (10), and

$$\sum_{n=1}^{\infty} nP(\|Y_1''\| \geqslant n^2) \geqslant \int_1^{\infty} tP(\|Y_1''\| \geqslant (t+1)^2)\,dt$$

$$\geqslant \frac{1}{4}\int_1^{\infty} P(\|Y_1''\| \geqslant s)\,ds \geqslant \frac{1}{4}(E\|Y_1''\|-1),$$

so $E\|Y_1''\| < \infty$, and thus, by Lemma 10.1.1 of Chow and Teicher [2], $E\|(X_1\cdot c)''\| < \infty$. Since $EX_1' = 0$, it follows that

$$E(X_1\cdot c)'' = EX_1''+c'',$$

so $E\|X_1''\| < \infty$. Hence we have $E|X_1|^2 < \infty$. ∎

Now we formulate an analogue of the Baum–Katz Theorem (cf. Chow and Teicher [2], Theorem 5.2.7):

THEOREM 3. *Let G be a simply connected step 2-nilpotent Lie group, $|\cdot|$ a homogeneous gauge on G, and assume that $\{X_n\}_{n\geqslant 1}$ are i.i.d. G-valued random variables. Suppose $0 < p < 2$, $E|X_1|^p < \infty$, and let $c \in G$ be such that $c' = -EX_1'$ in the case $1 \leqslant p < 2$. Then if $\alpha p \geqslant 1$, we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant i\leqslant n} \big|\delta_{n^{-\alpha}}\big(\prod_{j=1}^{i} (X_j\cdot c)\big)\big| \geqslant \varrho\Big) < \infty$$

for every $\varrho > 0$.

Proof. The proof is the same as in the classical case (cf. Chow and Teicher [2], p. 130): Again, without loss of generality, $c = 0$, $EX_1' = 0$. By Theorem 1,

$$\delta_{n^{-1/p}}\big(\prod_{i=1}^{n} X_i\big) \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty),$$

so

$$\max_{1\leqslant i\leqslant n} \big|\delta_{n^{-1/p}}\big(\prod_{j=1}^{i} X_j\big)\big| \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty).$$

Thus, by (2) and (3), we obtain

$$(17)\qquad \max_{n+1\leqslant i\leqslant 2n} \big|\delta_{n^{-1/p}}\big(\prod_{j=n+1}^{i} X_j\big)\big| = \max_{n+1\leqslant i\leqslant 2n} \big|\delta_{n^{-1/p}}\big(\big(-\prod_{j=1}^{n} X_j\big)\cdot \prod_{j=1}^{i} X_j\big)\big|$$

$$\lesssim \big|\delta_{n^{-1/p}}\big(\prod_{j=1}^{n} X_j\big)\big| + \max_{1\leqslant i\leqslant 2n} \big|\delta_{(2n)^{-1/p}}\big(\prod_{j=1}^{i} X_j\big)\big| \xrightarrow{\text{a.s.}} 0 \quad (n \to \infty).$$

Case 1: $\alpha p = 1$. Put

$$\varrho' = 2^{-2\alpha}\varrho.$$

Since the random variables

$$\{\max_{2^n+1\leqslant i\leqslant 2^{n+1}}|\prod_{j=2^n+1}^{i} X_j|\}_{n\geqslant 1}$$

are independent, it follows from (17) and the Borel–Cantelli Lemma that

$$\infty > \sum_{n=1}^{\infty} P(\max_{2^n+1\leqslant i\leqslant 2^{n+1}}|\prod_{j=2^n+1}^{i} X_j| \geqslant 2^{\alpha n}\varrho')$$

$$= \sum_{n=1}^{\infty} P(\max_{1\leqslant i\leqslant 2^n}|\prod_{j=1}^{i} X_j| \geqslant 2^{\alpha n}\varrho') \geqslant \int_0^{\infty} P(\max_{1\leqslant i\leqslant \lfloor 2^t\rfloor}|\prod_{j=1}^{i} X_j| \geqslant 2^{\alpha(t+1)}\varrho')\,dt$$

$$\geqslant (\log 2)^{-1}\int_1^{\infty} x^{-1} P(\max_{1\leqslant i\leqslant \lfloor x\rfloor}|\prod_{j=1}^{i} X_j| \geqslant 2^{\alpha}\varrho' x^{\alpha})\,dx$$

$$\geqslant (\log 2)^{-1}\sum_{n=1}^{\infty}\frac{1}{2n} P(\max_{1\leqslant i\leqslant n}|\prod_{j=1}^{i} X_j| \geqslant 2^{\alpha}\varrho'(2n)^{\alpha})$$

$$\geqslant (2\log 2)^{-1}\sum_{n=1}^{\infty}\frac{1}{n} P(\max_{1\leqslant i\leqslant n}|\prod_{j=1}^{i} X_j| \geqslant \varrho n^{\alpha}).$$

Case 2: $\alpha p > 1$. Put

$$\varrho' = 2^{-\alpha^2 p/(\alpha p-1)}\varrho.$$

Since for $n\geqslant 1$

$$(n+1)^{\alpha p/(\alpha p-1)} \geqslant n^{\alpha p/(\alpha p-1)} + \frac{\alpha p}{\alpha p-1} n^{1/(\alpha p-1)} \geqslant n^{\alpha p/(\alpha p-1)} + n^{1/(\alpha p-1)},$$

the random variables

$$\{\max_{\substack{n^{\alpha p/(\alpha p-1)}+1\leqslant i\\ \leqslant n^{\alpha p/(\alpha p-1)}+n^{1/(\alpha p-1)}}}|\prod_{j=\lfloor n^{\alpha p/(\alpha p-1)}\rfloor+1}^{i} X_j|\}_{n\geqslant 1}$$

are independent and, by (17),

$$\max_{\substack{n^{\alpha p/(\alpha p-1)}+1\leqslant i\\ \leqslant n^{\alpha p/(\alpha p-1)}+n^{1/(\alpha p-1)}}} n^{-\alpha/(\alpha p-1)}|\prod_{j=\lfloor n^{\alpha p/(\alpha p-1)}\rfloor+1}^{i} X_j|$$

$$\leqslant \max_{n^{\alpha p/(\alpha p-1)}+1\leqslant 2n^{\alpha p/(\alpha p-1)}} n^{-\alpha/(\alpha p-1)}|\prod_{j=\lfloor n^{\alpha p/(\alpha p-1)}\rfloor+1}^{i} X_j| \xrightarrow{\text{a.s.}} 0 \qquad (n\to\infty),$$

so, by the Borel–Cantelli Lemma,

$$\infty > \sum_{n=1}^{\infty} P\Big(\max_{\substack{n^{\alpha p/(\alpha p-1)}+1 \leqslant i \\ \leqslant n^{\alpha p/(\alpha p-1)}+n^{1/(\alpha p-1)}}} \Big| \prod_{j=\lfloor n^{\alpha p/(\alpha p-1)}\rfloor+1}^{i} X_j\Big| \geqslant n^{\alpha/(\alpha p-1)}\varrho'\Big)$$

$$= \sum_{n=1}^{\infty} P\Big(\max_{1\leqslant i\leqslant n^{1/(\alpha p-1)}} \Big|\prod_{j=1}^{i} X_j\Big| \geqslant n^{\alpha/(\alpha p-1)}\varrho'\Big)$$

$$\geqslant \int_{1}^{\infty} P\Big(\max_{1\leqslant i\leqslant \lfloor t^{1/(\alpha p-1)}\rfloor} \Big|\prod_{j=1}^{i} X_j\Big| \geqslant (t+1)^{\alpha/(\alpha p-1)}\varrho'\Big)\,dt$$

$$\geqslant (\alpha p-1)\int_{1}^{\infty} x^{\alpha p-2} P\Big(\max_{1\leqslant i\leqslant \lfloor x\rfloor} \Big|\prod_{j=1}^{i} X_j\Big| \geqslant 2^{\alpha/(\alpha p-1)}\varrho' x^{\alpha}\Big)\,dx$$

$$\geqslant A\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant i\leqslant n} \Big|\prod_{j=1}^{i} X_j\Big| \geqslant 2^{\alpha/(\alpha p-1)}\varrho' (2n)^{\alpha}\Big)$$

$$= A\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant i\leqslant n} \Big|\prod_{j=1}^{i} X_j\Big| \geqslant \varrho n^{\alpha}\Big)$$

for some constant $A > 0$. ■

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Université de Lausanne
Ecole des Hautes Etudes Commerciales
Institut de Sciences Actuarielles
CH-1015 Lausanne

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