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A NOTE ON THE RATE OF CONVERGENCE FOR p-CONVOLUTIONS ON R^d

BY

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Abstract. This paper focuses on p-convolutions, a class of generalized convolutions of random vectors. It establishes the rate of convergence in uniform (Kolmogorov) metric for normalized n-fold p-convolution of random vectors to a generalized stable law. The method of proof relies on the application of probability metrics. Applications of generalized stable laws to financial data are also mentioned.

1. Introduction. The theory of generalized convolutions represents a unifying approach to many binary operations on probability measures. Urbanik [11]–[13] developed a theory of generalized convolutions of probability measures on the positive half-line (*Urbanik convolutions*). Urbanik's ideas inspired a series of results: a characterization of the domain of attraction of stable measures in Urbanik's sense (Bingham [1] and [2]), normed rings given by Urbanik convolutions (Volkovich [15], [16]), rate of convergence in the Central Limit Theorem for Urbanik convolutions (Panorska [6]). All of the aforementioned works dealt with generalized convolutions of probability measures on R^+. In this paper, for the first time, we consider an example of generalized convolution in R^d, *p-convolution*, and study the rate of convergence in the generalized Central Limit Theorem. We state our main result, Theorem 2.1, in Section 2, after preliminary definitions and notation. In Section 3, we include some auxiliary lemmas and prove Theorem 2.1 using probability metrics techniques (see Zolotarev [17] and Rachev [8] for details on probability metrics).

2. Notation and statement of the main result. Let X and Y be i.i.d. random vectors in R^d, and $\|\cdot\|$ be the Euclidean norm in R^d. Vatan [14] first pointed out the importance of studying the generalized sum

$$X \circ Y = (X^{\langle p\rangle} + Y^{\langle p\rangle})^{\langle 1/p\rangle}, \tag{1}$$

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where $X^{\langle p\rangle} = X\,\|X\|^{p-1}$ with $p > 0$. The generalized sum (1) defines a commutative and associative operation on random vectors called *p-convolution*. In what follows, we always assume that the operation on random vectors is fixed, i.e. that p is fixed. Since p-convolution generalizes summation of random vectors, we can define an analog of stable measure with respect to it. We say that a vector Y_α is *symmetric* (α, p)*-stable* if

$$Y_\alpha \stackrel{\mathrm{d}}{=} n^{-1/\alpha}(Y_\alpha^{(1)} \circ \ldots \circ Y_\alpha^{(n)}) \tag{2}$$

holds for any $n \geqslant 1$, where $Y_\alpha^{(i)}$, $i = 1, \ldots, n$, are i.i.d. copies of Y_α, and $\circ$ is defined in (1).

Remark 1. For any $X \in R^d$, $(X^{\langle p\rangle})^{\langle 1/p\rangle} = X$.

Remark 2. It follows that if Y_α is (α, p)-stable, then

$$Y_\alpha^{\langle p\rangle} \stackrel{\mathrm{d}}{=} n^{-p/\alpha}[(Y_\alpha^{(1)})^{\langle p\rangle} + \ldots + (Y_\alpha^{(n)})^{\langle p\rangle}]$$

is a symmetric (strictly) Paretian (α/p)-stable (for a complete treatment of Paretian stable random vectors see Janicki and Weron [3] or Samorodnitsky and Taqqu [9]). Thus, for any (α, p)-stable vector, we must have $0 < \alpha \leqslant 2p$.

Remark 3. Any (α, p)-stable Y_α belongs to the domain of attraction of a Paretian α-stable law, and is heavy-tailed: $P(|Y_\alpha| > x) \sim ax^{-\alpha}$ as $x \to \infty$, for some positive constant a.

The following characterization of p-stability is frequently used in our proofs:

LEMMA 2.1. *For any* (α, p)*-stable vector* Y_α *and* λ, λ_1 $\lambda_2 \geqslant 0$ *such that* $\lambda^\alpha = \lambda_1^\alpha + \lambda_2^\alpha$, *we have* $\lambda_1 Y_\alpha^{(1)} \circ \lambda_2 Y_\alpha^{(2)} \stackrel{\mathrm{d}}{=} \lambda Y_\alpha$, *where* $Y_\alpha^{(1)}$, $Y_\alpha^{(2)}$ *are i.i.d. copies of* Y_α.

Proof. Since $Y_\alpha^{\langle p\rangle}$ is Paretian (α/p)-stable, by Theorem 2.1.2 in [9] we have

$$\lambda_1^p (Y_\alpha^{(1)})^{\langle p\rangle} + \lambda_2^p (Y_\alpha^{(2)})^{\langle p\rangle} \stackrel{\mathrm{d}}{=} \lambda^p (Y_\alpha)^{\langle p\rangle}.$$

Taking $\langle 1/p\rangle$ "power" of both sides of the above equation and using Remark 1 we obtain

$$\lambda_1 Y_\alpha^{(1)} \circ \lambda_2 Y_\alpha^{(2)} \stackrel{\mathrm{d}}{=} \lambda Y_\alpha. \ ■$$

It is easy to see that (α, p)-stable random vectors are the only distributional limits of (scaled) n-fold p-convolutions of symmetric random vectors in R^d. Random vector X is said to *belong to the domain of attraction* of an (α, p)-stable vector Y_α if

$$c_n(X_1 \circ \ldots \circ X_n) \xrightarrow{\mathrm{d}} Y_\alpha \quad \text{as } n \to \infty, \tag{3}$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X. We shall study the rate of convergence in (3) when X_1 belongs to the *normal* domain of attraction of Y_α, that is when

$c_n = n^{-1/\alpha}$ in (3). Our main result, stated in Theorem 2.1 below, gives the rate of convergence in (3) in terms of the uniform (Kolmogorov) metric ϱ in $\mathscr{X}(R^d)$ – the space of R^d-valued random variables. By definition

$$\varrho(X, Y) := \sup_{C \in \mathscr{C}} |P(X \in C) - P(Y \in C)|,$$

where $\mathscr{C}$ is the family of all convex Borel sets in R^d. In addition, we shall utilize the total variation metric **Var**,

$$\mathbf{Var}(X, Y) := 2 \sup_{A \in \mathscr{B}(R^d)} |P(X \in A) - P(Y \in A)|,$$

and its *smoothed* version $\tilde{v}_{r,\alpha}$,

$$\tilde{v}_{r,\alpha}(X, Y) := \sup_{h \in R} |h|^r \mathbf{Var}(X \circ hY_\alpha, Y \circ hY_\alpha),$$

where $X, Y \in \mathscr{X}(R^d)$, and Y_α is a symmetric (α, p)-stable random vector independent of X and Y. For clarity, we shall use the following notation:

$$X^{\circ n} = X_1 \circ \ldots \circ X_n \quad \text{and} \quad Y_\alpha^{\circ n} = Y_\alpha^{(1)} \circ \ldots \circ Y_\alpha^{(n)},$$

where $X_1, \ldots, X_n$ and $Y_\alpha^{(1)}, \ldots, Y_\alpha^{(n)}$ are i.i.d. copies of X and Y_α, respectively. Finally, we define

$$\tilde{\tau}_{rp,\alpha} := \max\{\varrho(X, Y_\alpha), \tilde{v}_{p,\alpha}(X, Y_\alpha), [\tilde{v}_{rp,\alpha}(X, Y_\alpha)]^{1/(r-\alpha/p)}\}.$$

We shall state the main result now.

THEOREM 2.1. *If $rp > \alpha$, then the following estimate holds:*

$$\varrho(n^{-1/\alpha} X^{\circ n}, Y_\alpha) \leqslant C\tilde{v}_{rp,\alpha}(X, Y_\alpha) n^{1-rp/\alpha} + C\tilde{\tau}_{rp,\alpha} n^{-p/\alpha}$$

for some constant C.

3. Proof of the main result. We shall need the following four lemmas before we can prove Theorem 2.1.

LEMMA 3.1. *$\tilde{v}_{r,\alpha}$ is an ideal metric of order r with respect to operation $\circ$, that is $\tilde{v}_{r,\alpha}$ satisfies the following two conditions:*

(i) *Regularity*: $\tilde{v}_{r,\alpha}(X \circ Z, Y \circ Z) \leqslant \tilde{v}_{r,\alpha}(X, Y)$.

(ii) *Homogeneity*: $\tilde{v}_{r,\alpha}(cX, cY) = c^r \tilde{v}_{r,\alpha}(X, Y)$, $c > 0$.

Note. **Var** is an ideal metric (of order 0) for operation $\circ$.

Proof. Regularity of $\tilde{v}_{r,\alpha}$ follows from regularity of **Var**. To show homogeneity of order r, write

$$\begin{aligned}\tilde{v}_{r,\alpha}(cX, cY) &= \sup_{h \in R} |h|^r \mathbf{Var}[c(X \circ h/cY_\alpha), c(Y \circ h/cY_\alpha)] \\ &= \sup_{h>0} h^r \mathbf{Var}(X \circ h/cY_\alpha, Y \circ h/cY_\alpha) = c^r \tilde{v}_{r,\alpha}(X, Y).\end{aligned}$$

LEMMA 3.2. *Let $X, Y \in \mathscr{X}(R^d)$, and $\delta > 0$. Then for some positive constant C the following smoothing inequality holds*:

$$\varrho(X, Y) \leqslant C\varrho(X \circ \delta^{1/p} Y_\alpha, Y \circ \delta^{1/p} Y_\alpha) + C\delta. \tag{4}$$

Proof. First, we show that $\varrho(X, Y) = \varrho(X^{\langle p \rangle}, Y^{\langle p \rangle})$. By Remark 1, for any Borel set $C \in \mathscr{C}$, we have $\{X^{\langle p \rangle} \in C\}$ iff $\{X \in C^{\langle 1/p \rangle}\}$, where $C^{\langle 1/p \rangle} = \{c^{\langle 1/p \rangle}: c \in C\}$. Consequently,

$$\begin{aligned}\varrho(X^{\langle p \rangle}, Y^{\langle p \rangle}) &= \sup_{C \in \mathscr{C}} |P(X^{\langle p \rangle} \in C) - P(Y^{\langle p \rangle} \in C)| \\ &= \sup_{C \in \mathscr{C}} |P(X \in C^{\langle 1/p \rangle}) - P(Y \in C^{\langle 1/p \rangle})| = \varrho(X, Y).\end{aligned}$$

Then, recall the classical *smoothing inequality* for summation of random vectors (see Zolotarev [17] and Rachev [8]):

$$\varrho(X, Y) \leqslant C\varrho(X + \delta\Theta, Y + \delta\Theta) + C\delta,$$

where $X, Y \in \mathscr{X}(R^d)$ and Θ is a symmetric (Paretian) α-stable random vector, and write

$$\begin{aligned}\varrho(X, Y) &= \varrho(X^{\langle p \rangle}, Y^{\langle p \rangle}) \\ &\leqslant C\varrho\big(X^{\langle p \rangle} + (\delta^{1/p} Y_\alpha)^{\langle p \rangle}, Y^{\langle p \rangle} + (\delta^{1/p} Y_\alpha)^{\langle p \rangle}\big) + C\delta \\ &= C\varrho(X \circ \delta^{1/p} Y_\alpha, Y \circ \delta^{1/p} Y_\alpha) + C\delta. \ \blacksquare\end{aligned}$$

LEMMA 3.3. *For any $X, Y, Z, R \in \mathscr{X}(R^d)$, $\lambda > 0$, and symmetric α-stable Y_α, the following two inequalities hold*:

$$\begin{aligned}\text{(i)} \quad \varrho(\lambda Y_\alpha \circ X \circ Y, \lambda Y_\alpha \circ X \circ Z) &\leqslant \varrho(X, R)\,\mathbf{Var}(\lambda Y_\alpha \circ Y, \lambda Y_\alpha \circ Z) \\ &\quad + \varrho(\lambda Y_\alpha \circ R \circ Y, \lambda Y_\alpha \circ R \circ Z),\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad \varrho(\lambda Y_\alpha \circ X \circ Y, \lambda Y_\alpha \circ R \circ Y) &\leqslant \varrho(X, R)\,\mathbf{Var}(\lambda Y_\alpha \circ Y, \lambda Y_\alpha \circ Z) \\ &\quad + \varrho(\lambda Y_\alpha \circ X \circ Z, \lambda Y_\alpha \circ R \circ Z).\end{aligned}$$

Proof. We will prove part (i), and skip the details of the analogous proof of (ii). We have

$$\begin{aligned}&\varrho(\lambda Y_\alpha \circ X \circ Y, \lambda Y_\alpha \circ X \circ Z) \\ &= \varrho\big((\lambda Y_\alpha)^{\langle p \rangle} + X^{\langle p \rangle} + Y^{\langle p \rangle}, (\lambda Y_\alpha)^{\langle p \rangle} + X^{\langle p \rangle} + Z^{\langle p \rangle}\big) \\ &= \sup_{A \in \mathscr{C}} \big|P\big((\lambda Y_\alpha)^{\langle p \rangle} + X^{\langle p \rangle} + Y^{\langle p \rangle} \in A\big) - P\big((\lambda Y_\alpha)^{\langle p \rangle} + X^{\langle p \rangle} + Z^{\langle p \rangle} \in A\big)\big| \\ &= \sup_{A \in \mathscr{C}} \big|\textstyle\int P(X^{\langle p \rangle} \in A - v)\, d(F_{(\lambda Y_\alpha)^{\langle p \rangle} + Y^{\langle p \rangle}} - F_{(\lambda Y_\alpha)^{\langle p \rangle} + Z^{\langle p \rangle}})(v)\big| = L,\end{aligned}$$

where F_X denotes the distribution function of a random vector X. Let us put

$$F_{(\lambda Y_\alpha)^{\langle p \rangle} + Y^{\langle p \rangle}} - F_{(\lambda Y_\alpha)^{\langle p \rangle} + Z^{\langle p \rangle}} = F_{(\alpha, Y)} - F_{(\alpha, Z)}.$$

Then L can be bounded as follows:

$$L \leqslant \sup_{A\in\mathscr{C}} \left|\int (P(X^{\langle p\rangle} \in A-v) - P(R^{\langle p\rangle} \in A-v))\, d(F_{(\alpha,Y)} - F_{(\alpha,Z)})(v)\right|$$
$$+ \sup_{A\in\mathscr{C}} \left|\int P(R^{\langle p\rangle} \in A-v)\, d(F_{(\alpha,Y)} - F_{(\alpha,Z)})(v)\right|$$
$$\leqslant \varrho(X^{\langle p\rangle}, R^{\langle p\rangle}) \int |d(F_{(\alpha,Y)} - F_{(\alpha,Z)})|$$
$$+ \varrho\big((\lambda Y_\alpha)^{\langle p\rangle} + R^{\langle p\rangle} + Y^{\langle p\rangle},\ (\lambda Y_\alpha)^{\langle p\rangle} + R^{\langle p\rangle} + Z^{\langle p\rangle}\big)$$
$$= \varrho(X, R)\operatorname{Var}(\lambda Y_\alpha \circ Y, \lambda Y_\alpha \circ Z) + \varrho(\lambda Y_\alpha \circ R \circ Y, \lambda Y_\alpha \circ R \circ Z). \quad ■$$

The following estimate is an analog of a classical result (Senatov [10]), and will be used several times in the proof of the main theorem.

LEMMA 3.4. *For $X, Y \in \mathscr{X}(R^d)$ and $\lambda, \lambda_1\ \lambda_2 \geqslant 0$ such that $\lambda^\alpha = \lambda_1^\alpha + \lambda_2^\alpha$, we have*

$$\operatorname{Var}(\lambda_1 Y_\alpha^{(1)} \circ X \circ \lambda_2 Y_\alpha^{(2)},\ \lambda_1 Y_\alpha^{(1)} \circ Y \circ \lambda_2 Y_\alpha^{(2)}) \leqslant \lambda^{-rp}\, \tilde{v}_{rp,\alpha}(X, Y),$$

where $Y_\alpha^{(1)}$, $Y_\alpha^{(2)}$ are i.i.d. copies of Y_α.

Proof. By Lemma 2.1, we have $\lambda_1 Y_\alpha^{(1)} \circ \lambda_2 Y_\alpha^{(2)} \stackrel{d}{=} \lambda Y_\alpha$, and thus

$$\operatorname{Var}(\lambda_1 Y_\alpha^{(1)} \circ X \circ \lambda_2 Y_\alpha^{(2)},\ \lambda_1 Y_\alpha^{(1)} \circ Y \circ \lambda_2 Y_\alpha^{(2)}) = \operatorname{Var}(X \circ \lambda Y_\alpha,\ Y \circ \lambda Y_\alpha)$$
$$\leqslant \lambda^{-rp} \sup_{\lambda>0} \lambda^{rp} \operatorname{Var}(X \circ \lambda Y_\alpha,\ Y \circ \lambda Y_\alpha) = \lambda^{-rp}\, \tilde{v}_{rp,\alpha}(X, Y). \quad ■$$

Now we prove the main result.

Proof of Theorem 2.1. Note that C depends on r and α, but it does not depend on the distributions of X and Y_α. We will follow the main idea of Senatov [10] and Rachev [8], pp. 264–281. We will proceed by induction. For $n = 1, 2$, the assertion of the theorem is easy to see. Let us assume that, for all $j < n$,

$$\varrho(j^{1/\alpha} X^{\circ j}, Y_\alpha) \leqslant C\tilde{v}_{rp,\alpha}(X, Y_\alpha) j^{1-rp/\alpha} + C\tilde{\tau}_{rp,\alpha} j^{-p/\alpha}. \tag{5}$$

Set $\delta := A\tilde{\tau}_{rp,\alpha} n^{-p/\alpha}$ and $m = [n/2]$, where $[\cdot]$ denotes the greatest integer function. Using the stability property of Y_α, Lemma 3.4, and the triangle inequality (m times), we obtain

$$\varrho(n^{-1/\alpha} X^{\circ n}, Y_\alpha) \leqslant C\varrho(\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha} X^{\circ n},\ \delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha}\, Y_\alpha^{\circ n}) + C\delta$$
$$\leqslant \varrho(\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha} X^{\circ n},\ \delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha}\, X^{\circ(n-1)} \circ n^{-1/\alpha}\, Y_\alpha^{\circ n})$$
$$+ \sum_{j=1}^{m} \varrho(\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha} X^{\circ(n-j-1)} \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha}\, Y_\alpha^{(j)},$$
$$\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha} X^{\circ(n-j-1)} \circ n^{-1/\alpha}\, Y_\alpha^{(j+1)})$$
$$+ \varrho(\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha} X^{\circ(n-m-1)} \circ n^{-1/\alpha}\, Y_\alpha^{(m+1)},\ \delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha}\, Y_\alpha^{\circ n}) + C\delta$$
$$= J_1 + J_2 + J_3 + C\delta.$$

We estimate J_1 using Lemma 3.3 (i) with

$$X = n^{-1/\alpha} X^{\circ(n-1)}, \quad Y = n^{-1/\alpha} X_n, \quad Z = n^{-1/\alpha} Y_\alpha^n, \quad \text{and} \quad R = n^{-1/\alpha} Y_\alpha^{(n-1)}.$$

Then we get

$$J_1 \leqslant \varrho(n^{-1/\alpha} X^{\circ(n-1)}, n^{-1/\alpha} Y_\alpha^{\circ(n-1)}) \times \mathbf{Var}(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^n)$$
$$+\varrho(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^{\circ(n-1)} \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_\alpha \circ Y_\alpha^{\circ(n)} = I_1 + I_2.$$

To estimate J_2 we use Lemma 3.3 (ii) with

$$X = n^{-1/\alpha} X^{\circ(n-j-1)}, \quad R = n^{-1/\alpha} Y_\alpha^{\circ(n-j-1)}, \quad Y = n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_\alpha^{\circ j},$$

and

$$Z = n^{-1/\alpha} Y_\alpha^{(n-j)} \circ n^{-1/\alpha} Y_\alpha^{\circ(j)}.$$

Then we obtain

$$J_2 \leqslant \sum_{j=1}^{m} \varrho(n^{-1/\alpha} X^{\circ(n-j-1)}, n^{-1/\alpha} Y_\alpha^{\circ(n-j-1)})$$
$$\times \mathbf{Var}(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_\alpha^{\circ(j)}, \delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^{\circ(j+1)})$$
$$+\sum_{j=1}^{m} \varrho(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^{\circ(n-j-1)} \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_\alpha^{\circ(j)},$$
$$\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^{\circ(n)})$$
$$= I_3 + I_4.$$

We estimate J_3 using Lemma 3.4 with $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda = 1$, the fact that $\varrho \leqslant \frac{1}{2}\mathbf{Var}$, and the ideality properties of $\tilde{\nu}_{rp,\alpha}$:

$$J_3 \leqslant \varrho(m^{-1/\alpha} X^{\circ(n-m-1)} \circ m^{-1/\alpha} Y_\alpha^{\circ(m+1)}, m^{-1/\alpha} Y_\alpha^{\circ(n-m-1)} \circ m^{-1/\alpha} Y_\alpha^{\circ(m+1)})$$
$$\leqslant \mathbf{Var}(m^{-1/\alpha} X^{\circ(n-m-1)} \circ m^{-1/\alpha} Y_\alpha^{\circ(m)}, m^{-1/\alpha} Y_\alpha^{\circ(n-m-1)} \circ m^{-1/\alpha} Y_\alpha^{(m)})$$
$$\leqslant \mathbf{Var}(m^{-1/\alpha} X^{(n-m-1)} \circ Y_\alpha, m^{-1/\alpha} Y_\alpha^{\circ(n-m-1)} \circ Y_\alpha)$$
$$\leqslant m^{-rp/\alpha}(n-m-1)\tilde{\nu}_{rp,\alpha}(X, Y_\alpha) \leqslant D_1 n^{1-rp/\alpha} \tilde{\nu}_{rp,\alpha}(X, Y_\alpha)$$

for n large enough and some positive D_1. We use Lemma 3.4 with $\lambda_1 = \delta^{1/p}$, $\lambda_2 = (j/n)^\alpha$, and the ideality properties of $\tilde{\nu}_{rp,\alpha}$ to estimate the term involving $\mathbf{Var}$ in I_3:

$$\mathbf{Var}(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} X_{n-j} \circ Y_\alpha^{\circ(j)}, \delta^{1/p} Y_\alpha \circ n^{-1/\alpha} Y_\alpha^{(n-j)} \circ n^{-1/\alpha} Y_\alpha^{\circ(j)})$$
$$\leqslant 1/(\delta^{\alpha/p} + i/n)^{rp/\alpha} \tilde{\nu}_{rp,\alpha}(n^{-1/\alpha} X, n^{-1/\alpha} Y_\alpha) = n^{-rp/\alpha}(\delta^{\alpha/p} + i/n)^{rp/\alpha} \tilde{\nu}_{rp,\alpha}(X, Y_\alpha).$$

Also, by inductive hypothesis, we estimate the term involving ϱ in I_3:

$$\begin{aligned}
&\varrho\left(n^{-1/\alpha}X^{\circ(n-j-1)},\, n^{-1/\alpha}\, Y_\alpha^{\circ(n-j-1)}\right)\\
&\qquad = \varrho\left((n-j-1)^{-1/\alpha}\, X^{\circ(n-j-1)},\, (n-j-1)^{-1/\alpha}\, Y_\alpha^{\circ(n-j-1)}\right)\\
&\qquad = \varrho\left((n-j-1)^{-1/\alpha}X^{\circ(n-j-1)},\, Y_\alpha\right)\\
&\qquad \leqslant B\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)(n-j-1)^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}(n-j-1)^{-p/\alpha}\right)\\
&\qquad \leqslant B\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)(n-m-1)^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}(n-m-1)^{-p/\alpha}\right)\\
&\qquad \leqslant BD_2\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)\, n^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}\, n^{-p/\alpha}\right)
\end{aligned}$$

for sufficiently large n and some positive D_2. Note that $\delta \leqslant A\tilde{\tau}_{rp,\alpha}n^{-p/\alpha}$, so that

$$\begin{aligned}
I_3 &\leqslant B\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)\, n^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}\, n^{-p/\alpha}\right)\\
&\quad \times \sum_{j=1}^{\infty} \tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)/\left(A^{\alpha/p}\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)\right)^{(\alpha/p)/(r-\alpha/p)}+j\right)^{-rp/\alpha}\\
&\leqslant D_3\, BA^{(\alpha/p)-r}\left(\tilde{\nu}_{rp,\alpha}\,(X,\, Y_\alpha)\, n^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}\, n^{-p/\alpha}\right)
\end{aligned}$$

for some positive D_3 and large n. Now we will estimate I_2+I_4. We use Lemma 3.4 with $\lambda_1 = \left((n-j-1)/(n-1)\right)^{1/\alpha}$, $\lambda_2 = \left(j/(n-1)\right)^{1/\alpha}$, $\lambda = 1$, $2\varrho \leqslant \mathbf{Var}$, and the ideality of $\tilde{\nu}_{rp,\alpha}$:

$$\begin{aligned}
I_2+I_4 &= \sum_{j=0}^{m} \varrho\left(\delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha}\, Y_\alpha^{\circ(n-j-1)} \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha}\, Y_\alpha^{(j)},\, \delta^{1/p}\, Y_\alpha \circ n^{-1/\alpha}\, Y_\alpha^{\circ(n)}\right)\\
&= \sum_{j=0}^{m} \varrho\left((n-1)^{-1/\alpha}\, Y_\alpha^{\circ(n-j-1)} \circ (n-1)^{-1/\alpha}\, X_{n-j} \circ (n-1)^{-1/\alpha}\, Y_\alpha^{\circ(j)},\right.\\
&\qquad \left.(n-1)^{-1/\alpha}\, Y_\alpha^{\circ(n-j-1)} \circ (n-1)^{-1/\alpha}\, Y_\alpha^{n-j} \circ (n-1)^{-1/\alpha}\, Y_\alpha^{\circ(j)}\right)\\
&\leqslant \frac{1}{2}\sum_{j=0}^{m} \mathbf{Var}\left(\left(\frac{n-j-1}{n-1}\right)^{1/\alpha} Y_\alpha \circ (n-1)^{1/\alpha}\, X_{n-j} \circ \left(\frac{j}{n-1}\right)^{1/\alpha} Y_\alpha,\right.\\
&\qquad \left.\left(\frac{n-j-1}{n-1}\right)^{1/\alpha} Y_\alpha \circ (n-1)^{-1/\alpha}\, Y_\alpha^{n-j} \circ \left(\frac{j}{n-1}\right)^{1/\alpha} Y_\alpha\right)\\
&\leqslant \sum_{j=0}^{m} \tilde{\nu}_{rp,\alpha}\left((n-1)^{-1/\alpha}\, X_{n-j},\, (n-1)^{-1/\alpha}\, Y_\alpha^{n-j}\right)\\
&\leqslant m(n-1)^{-rp/\alpha}\, \tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha) \leqslant D_4\, n^{1-rp/\alpha}\, \tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)
\end{aligned}$$

for some constant D_4 and large n. To estimate I_1, we use Lemma 3.4 with $\lambda_1 = \lambda = \delta^{1/p}$, $\lambda = 0$, the inductive hypothesis (5), and the ideality of $\tilde{\nu}_{rp,\alpha}$:

$$\begin{aligned}
\varrho\left(n^{-1/\alpha}\, X^{\circ(n-1)},\, n^{-1/\alpha}\, Y_\alpha^{\circ(n-1)}\right) &\leqslant B\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)(n-1)^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}(n-1)^{-p/\alpha}\right)\\
&\leqslant BD_5\left(\tilde{\nu}_{rp,\alpha}(X,\, Y_\alpha)\, n^{1-rp/\alpha}+\tilde{\tau}_{rp,\alpha}\, n^{-p/\alpha}\right)
\end{aligned}$$

for some constant D_5 and large n. Further,

$$\mathrm{Var}(\delta^{1/p} Y_\alpha \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_\alpha \circ \delta^{1/p} Y_\alpha^{(n)}) \leqslant 1/\delta \tilde{v}_{p,\alpha}(X, Y_\alpha) n^{-p/\alpha}.$$

Since $\tilde{v}_{p,\alpha}(X, Y_\alpha) n^{-p/\alpha} < \delta/A$, we obtain

$$I_1 \leqslant BD_5/A\left(\tilde{v}_{rp,\alpha}(X, Y_\alpha) n^{1-rp/\alpha} + \tilde{\tau}_{rp,\alpha} n^{-p/\alpha}\right).$$

Combining all the above estimates, we get

$$\begin{aligned} \varrho(n^{-1/\alpha} X^{\circ n}, Y_\alpha) &\leqslant I_1 + I_2 + I_3 + I_4 \\ &\leqslant CB(D_5/A + D_3 A^{\alpha/p - r})\left(\tilde{v}_{rp,\alpha}(X, Y_\alpha) n^{1-rp/\alpha} + \tilde{\tau}_{rp,\alpha} n^{-p/\alpha}\right) \\ &\quad + C(D_1 + D_4) n^{1-rp/\alpha} \tilde{v}_{rp,\alpha}(X, Y_\alpha) + CA\tilde{\tau}_{rp,\alpha} n^{-p/\alpha}. \end{aligned}$$

It is clear that we can choose a new constant C, so that

$$\varrho(n^{-1/\alpha} X^{\circ n}, Y_\alpha) \leqslant C\left(\tilde{v}_{rp,\alpha}(X, Y_\alpha) n^{1-rp/\alpha} + \tilde{\tau}_{rp,\alpha} n^{-p/\alpha}\right). \blacksquare$$

4. A note on applications. Interesting applications of this theory come from the area of "heavy tail" modeling, in particular in mathematical finance, as discussed by Mittnik and Rachev [5]. The problem of finding the probability distribution of financial asset returns has occupied a prominent place in statistical and financial literature (for a comprehensive review of the literature see Mittnik and Rachev [5] and the references therein). The (α, p)-stable laws were successfully used in modeling the distribution of financial data. Panorska [7] fitted (α, p)-stable models to foreign exchange rates and interest rates, and showed that they fit the data better than Paretian stable and normal models.

The empirical distribution of financial data is very often heavy-tailed and peaky at the center (i.e. leptokurtic), and thus to model it successfully we need distributions with similar properties. It turns out that variables stable with respect to p-convolutions have both of the above properties (for details on modeling and properties of (α, p)-stable models see Panorska [7]). Apart from that, they are *stable*, and thus have domains of attraction, which makes them even more attractive models. The (α, p)-stable models have several advantages over the Paretian stable ones. It occurs that their densities can be unimodal or bimodal, depending on α and p. Thus, we can fix the tail order α, and by varying p we obtain a unimodal or bimodal model. Another advantage of the (α, p)-stable laws is that by choosing appropriate p we can get a model with much steeper center peak than that allowed by Paretian stable distributions (financial data is often very peaky at the center). Finally, since α is not bounded above by 2, we can model data with tails lighter than those of Paretian stable distributions (see Loretan and Phillips [4]).

The (α, p)-stable models successfully compete with Paretian stable ones in modeling one-dimensional financial asset returns. Extensions to multivariate portfolios' modeling would be a natural next step in this line of research.

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