PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 17, Fasc. 2 (1997), pp. 377–385

ON GENERALIZED POISSON DISTRIBUTIONS

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Abstract. In this paper, we show that, for $\theta > 0$ and λ in [0, 1], the measure μ defined on nonnegative integers by

$$\mu(n) = \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta}$$

defines a probability distribution (called *Generalized Poisson Distribution* and abbreviated as GPD). Furthermore, we show that, for $\lambda > 1$, μ does not define a probability measure, and finally we prove that GPD is a particular case of the compound Poisson distribution.

1. Introduction. The Poisson distribution is one of the most important probability distributions. This has several generalizations, for example, the compound Poisson distribution. Less known is a two-parameter family of distributions, studied extensively by Consul [1] and called by him the *Generalized Poisson Distribution* (GPD). This is a two-parameter distribution induced by the measure μ concentrated on the nonnegative integers defined for $0 \leqslant \lambda \leqslant 1$, $\theta > 0$ by

$$\mu(n) = \frac{\theta(\theta+n\lambda)^{n-1} e^{-n\lambda-\theta}}{n!}.$$

This family is very close to the so-called *Borel–Tanner distribution* (Johnson et al. [5]). In [1] Consul studies this family and gives various applications. The proof that this is a distribution, however, is not entirely easy. Consul and Jain in [2] refer to Jenson [4] for a proof. The proof there uses Lagrange's expansion, which although valid generally does not seem to give the exact domain of validity. In this paper we prove this and in addition we show that for $\lambda > 1$ GPD is not a proper distribution and we find its exact value. Furthermore, we draw that GPD is a particular case of the compound Poisson distribution, where the compounding distribution is itself GPD.

2. μ is a probability measure. Our first goal is to prove that

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta} = 1 \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = e^{\theta}$$

for $\theta > 0$ and $0 \leqslant \lambda \leqslant 1$. Let us first prove the above identity for $\theta > 0$ and λ in $[0, x_0)$, where x_0 is such that $x_0 \exp\{1+x_0\} = 1$ (x_0 is approximately 0.28). For this purpose, consider the infinite series

$$I_1 = \sum_{n=0}^{\infty} \frac{(\theta+n\lambda)^n}{n!} e^{-n\lambda-\theta},$$

which can be written as

$$I_1 = \sum_{n=0}^{\infty} \frac{(\theta+n\lambda)^n}{n!} \left[\sum_{n=0}^{\infty} \frac{(-1)^k(\theta+n\lambda)^k}{k!} \right]$$

or

$$I_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\theta+n\lambda)^{n+k}(-1)^k}{n!k!}. \tag{1}$$

LEMMA 1. *The infinite series*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{(-1)^k(\theta+n\lambda)^{n+k}}{n!k!} \right|$$

converges uniformly on any bounded subset of Z (where Z is the set of complex numbers) with $|\theta| \leqslant M$, $M > 0$ and λ in $[0, x_0)$, where x_0 is such that $x_0 \exp\{1+x_0\} = 1$.

Proof. We have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{(-1)^k(\theta+n\lambda)^{n+k}}{n!k!} \right| \leqslant \sum_{n=0}^{\infty} \frac{(|\theta|+n\lambda)^n}{n!} e^{|\theta|+n\lambda}$$

$$= e^{|\theta|} \sum_{n=0}^{\infty} \frac{(|\theta|+n\lambda)^n}{n!} e^{n\lambda} = e^{|\theta|} + e^{|\theta|} \sum_{n=1}^{\infty} \frac{(|\theta|+n\lambda)^n}{n!} e^{n\lambda}$$

$$\leqslant e^{|\theta|} + e^{|\theta|} \sum_{n=1}^{\infty} \frac{e^{n\lambda}(n\lambda)^n(1+|\theta|/n\lambda)^n}{n!} \leqslant e^{|\theta|} + e^{|\theta|+\lambda/|\theta|} \sum_{n=1}^{\infty} \frac{\lambda^n n^n e^{n\lambda}}{n!};$$

to get the last inequality we have used the fact that $(1+|\theta|/n\lambda)^n \leqslant e^{\lambda/|\theta|}$. Now we use the Stirling formula [3] ($n^n/n! < e^n/\sqrt{2\pi n}$) in the last inequality to obtain

$$e^{|\theta|} + e^{|\theta|+\lambda/|\theta|} \sum_{n=1}^{\infty} \frac{\lambda^n n^n e^{n\lambda}}{n!} \leqslant e^{|\theta|} + e^{|\theta|+\lambda/|\theta|} \sum_{n=1}^{\infty} \frac{(\lambda e^{\lambda})^n e^n}{\sqrt{2\pi n}}$$

$$\leqslant e^{|\theta|} + e^{|\theta|+\lambda/|\theta|} \sum_{n=1}^{\infty} (\lambda e^{1+\lambda})^n.$$

But λ is in $(0, x_0)$ and the function xe^{1+x} is increasing on $(0, \infty)$, so $\lambda e^{1+\lambda} < x_0 \exp\{1+x_0\} = 1$. Hence $e^{|\theta|+\lambda/|\theta|}\sum_{n=0}^{\infty}(\lambda e^{1+\lambda})^n$ converges uniformly on $|\theta| \leqslant M$, and so does

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{(-1)^k(\theta+n\lambda)^{n+k}}{n!k!}\right|.$$

Thus we have proved the lemma.

Thanks to Lemma 1 we can change the order of summation in (1) in the following way: let $n+k = m$ to get

$$(2)\quad I_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^{m-n}\frac{(\theta+n\lambda)^m}{n!(m-n)!} = \sum_{m=0}^{\infty}\frac{\lambda^m(-1)^m}{m!}\left[\sum_{n=0}^{m}\binom{m}{n}(-1)^n\left(\frac{\theta}{\lambda}+n\right)^m\right].$$

LEMMA 2. *For any complex number* x

$$\sum_{n=0}^{m}\binom{m}{n}(-1)^n(x+n)^k = \begin{cases}0 & \text{for } 0 \leqslant k \leqslant m-1,\\ (-1)^m m! & \text{for } k = m.\end{cases}$$

Proof. Let us first prove that the lemma holds for $x = 0$, and the proof is by induction. Cearly,

$$\sum_{n=0}^{m}\binom{m}{n}(-1)^n(n)^k = 0 \quad \text{for } m = 0.$$

Suppose the claim is true for $m \leqslant N$. Now consider

$$\sum_{n=0}^{N+1}(-1)^n\binom{N+1}{n}n^k,$$

where $k \geqslant 1$ (for $k = 0$, the claim is true by Binomial Theorem). We have

$$\sum_{n=0}^{N+1}(-1)^n\binom{N+1}{n}n^k = (-1)(N+1)\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}(n+1)^{k-1}\right]$$

$$= (-1)(N+1)\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}\sum_{j=0}^{k-1}\binom{k-1}{j}n^{k-1-j}\right]$$

$$= (-1)(N+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{k-1-j}\right].$$

If $k \leqslant N$, then for $j = 0, 1, \ldots, k-1$, we have $k-1-j < N$, and so

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{k-1-j} = 0$$

by induction hypothesis. Hence

$$\sum_{n=0}^{N+1}(-1)^n\binom{N+1}{n}n^k=(-1)(N+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{k-1-j}\right]=0.$$

But if $k=N+1$, then

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{k-1-j}=\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{N-j}$$
$$=\begin{cases}0 & \text{for } j=1,2,\ldots,k-1,\\ (-1)^N N! & \text{for } j=0,\end{cases}$$

so

$$\sum_{n=0}^{N+1}(-1)^n\binom{N+1}{n}n^k=(-1)(N+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{k-1-j}\right]$$
$$=(-1)(N+1)\sum_{j=0}^{N}\binom{N}{j}\left[\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^{N-j}\right]$$
$$=(-1)(N+1)(-1)^n N!=(-1)^{N+1}(N+1)!.$$

Thus we have proved the claim for $x=0$. Now we can write

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}(x+n)^k=\sum_{n=0}^{m}(-1)\binom{m}{n}\left[\sum_{j=0}^{k}\binom{k}{j}x^j n^{k-j}\right],$$

which can be written as

$$\sum_{j=0}^{k}\binom{k}{j}x^j\left[\sum_{n=0}^{m}(-1)\binom{m}{n}n^{k-j}\right].$$

Using the lemma for $x=0$ to the inside sum, we get

$$\sum_{j=0}^{k}\binom{k}{j}x^j\left[\sum_{n=0}^{m}(-1)\binom{m}{n}n^{k-j}\right]=\begin{cases}0 & \text{for } k<m,\\ x^0(-1)^m m!=(-1)^m m! & \text{for } k=m.\end{cases}$$

Thus we have proved the lemma.

Now, using Lemma 2, we can write (2) in the form

$$I_1=\sum_{m=0}^{\infty}\lambda^m=\frac{1}{1-\lambda},$$

which implies that

$$\sum_{n=0}^{\infty}\frac{(\theta+n\lambda)^n}{n!}e^{-n\lambda-\theta}=\frac{1}{1-\lambda}, \tag{3}$$

and this can be written as

$$\sum_{n=0}^{\infty} \frac{(\theta+n\lambda)^{n-1}(\theta+n\lambda)}{n!} e^{-n\lambda} = \frac{e^{\theta}}{1-\lambda}$$

or

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} + \sum_{n=0}^{\infty} \frac{n\lambda(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = \frac{e^{\theta}}{1-\lambda},$$

where $\sum_{n=0}^{\infty} [\theta(\theta+n\lambda)^{n-1}/n!] e^{-n\lambda}$ converges for all λ in $[0, 1]$ and for any complex number θ. This can be shown in a way similar to the one we used to prove Lemma 1 but we are going to avoid this for the sake of brevity. From the above equation we get

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} + \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\theta+\lambda+n\lambda)^{n}}{n!} e^{-n\lambda} = \frac{e^{\theta}}{1-\lambda}.$$

Now using (3) we obtain

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} + \lambda e^{-\lambda} \frac{e^{\theta+\lambda}}{1-\lambda} = \frac{e^{\theta}}{1-\lambda}$$

or

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = \frac{e^{\theta}}{1-\lambda} - \frac{\lambda e^{\theta}}{1-\lambda} = e^{\theta}$$

for $\theta > 0$ and λ in $[0, x_0)$ (actually we have proved the above identity for all θ in Z because Lemmas 1 and 2 hold for all complex numbers).

Now let us prove that

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = e^{\theta}$$

for $\theta > 0$ and λ in $[0, 1]$. Let G be the simply connected region in Z enclosed by the curves

$$f(x) = \frac{1-xe^{1-x}}{2e^{1-x}} \quad \text{and} \quad g(x) = -\frac{1-xe^{1-x}}{2e^{1-x}},$$

where x is in $[0, 1]$. Now

$$x+f(x) = x-g(x) = \frac{xe^{1-x}+1}{2e^{1-x}}$$

and

$$(x+f(x))e^{1-x} = (x-g(x))e^{1-x} = \frac{xe^{1-x}+1}{2} \leqslant 1.$$

So if z is in G, then

$$|ze^{1-x}| < (x+\mathrm{Im}(z))\,e^{1-x} = \frac{xe^{1-x}+1}{2} < 1, \qquad \text{where } x = \mathrm{Re}(z).$$

Now for z in G consider the infinite series

$$I_2 = \sum_{n=0}^{\infty} \frac{\theta(\theta+n|z|)^{n-1}}{n!}(|e^{-nz}|)$$

which can be written as

$$I_2 = \sum_{n=0}^{\infty} \frac{\theta(\theta+n|z|)^{n-1}}{n!}e^{-nx},$$

$$I_2 = \sum_{n=0}^{\infty} \frac{\theta(\theta/|z|+n)^{n-1}}{|zn!|}(|ze^{-x}|)^n = \sum_{n=0}^{\infty} \frac{\theta(\theta/n|z|+1)^{n-1}}{|zn!|}n^{n-1}(|ze^{-x}|)^n$$

$$\leqslant \frac{\theta}{|z|}\sum_{n=0}^{\infty} e^{\theta/|z|}\frac{n^{n-1}}{n!}(|ze^{-x}|)^n.$$

Now, by using the Stirling formula [3] ($n^{n-1}/n! < e^n/n\sqrt{2\pi n}$) in the last inequality, we get

$$I_2 \leqslant 1+\frac{\theta}{|z|}e^{\theta/|z|}\sum_{n=1}^{\infty}\frac{e^n(|ze^{-x}|^n)}{\sqrt{2\pi n}}$$

or

$$I_2 \leqslant 1+\frac{\theta}{|z|}e^{\theta/|z|}\sum_{n=1}^{\infty}(|z|\,e^{1-x})^n. \tag{4}$$

But for all z in G we have $|z|\,e^{1-x} < 1$. Consequently, from (4) we conclude that the sequence of analytic functions

$$f_k(z) = \sum_{n=0}^{k} \frac{\theta(\theta+nz)^{n-1}}{n!}e^{-nz}$$

converges uniformly on G to an analytic function f, where

$$f(z) = \sum_{n=0}^{\infty} \frac{\theta(\theta+nz)^{n-1}}{n!}e^{-nz}.$$

But we know that for z in $(0, x_0)$

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+nz)^{n-1}}{n!}e^{-nz} = e^{\theta};$$

hence f (being an analytic function) must be a constant function on G, so

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = e^{\theta} \tag{5}$$

for all $\theta > 0$ (actually for all θ in Z) and $0 \leqslant \lambda < 1$. But the series

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda}$$

converges uniformly on compact subsets of Z with $|\theta| \leqslant M$ for some positive real number M and $\lambda \geqslant 0$ (this can easily be shown by using the Stirling formula [3] in a similar way to that we proved Lemma 1). So f is a continuous function of θ and $\lambda > 0$. Consequently, by continuity of f we get $f = e^{\theta}$ for $\lambda = 1$ as well, and we have achieved our first goal.

3. For $\lambda > 1$ GPD is not a distribution. Now our next goal is to prove that for all $\theta > 0$ and $\lambda > 1$

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta} < 1.$$

Proof. Let $\theta > 0$ and $\lambda > 1$. Then there exists a θ_0 in R such that $\theta = \theta_0 \lambda$, so we have

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda} = \sum_{n=0}^{\infty} \frac{\theta_0 \lambda(\theta_0 \lambda+n\lambda)^{n-1}}{n!} e^{-n\lambda} \tag{6}$$

$$= \sum_{n=0}^{\infty} \frac{\theta_0(\theta_0+n)^{n-1}}{n!} (\lambda e^{-\lambda})^n.$$

But the function $f(x) = xe^{-x}$ is one-to-one and onto from $(0, 1)$ to $(0, e^{-1})$, so $f(x) = xe^{-x}$ is one-to-one and onto from $(1, \infty)$ to $(0, e^{-1})$. Consequently, there is a unique μ in $(0, 1)$ such that $\mu e^{-\mu} = \lambda e^{-\lambda}$, so (6) can be written as

$$\sum_{n=0}^{\infty} \frac{\theta_0(\theta_0+n)^{n-1}}{n!} (\lambda e^{-\lambda})^n = \sum_{n=0}^{\infty} \frac{\theta_0(\theta_0+n)^{n-1}}{n!} (\mu e^{-\mu})$$

$$= \sum_{n=0}^{\infty} \frac{\mu\theta_0(\theta_0 \mu+n\mu)^{n-1}}{n!} e^{-n\mu},$$

which equals $\exp\{\mu\theta_0\}$ by (5). Therefore we can write

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} \exp\{-n\lambda-\theta\} = \exp\{\mu\theta_0-\theta\}.$$

But $\theta_0 = \theta/\lambda$, so the above equation becomes

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta} = e^{\theta(\mu/\lambda-1)},$$

which is less than 1 because $\mu < \lambda$. Consequently, for $\lambda > 1$ and $\theta > 0$ we have shown that

$$\sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta} < 1.$$

Furthermore, we have found its value in terms of μ, λ and θ, where $\mu e^{-\mu} = \lambda e^{-\lambda}$.

4. GPD as a compound Poisson distribution. Now we are going to prove that the GPD is a particular case of the compound Poisson distribution.

Let us define a function $\phi_\theta(z)$ in the following way:

$$\phi_\theta(z) = \sum_{n=0}^{\infty} \frac{\theta(\theta+n\lambda)^{n-1}}{n!} e^{-n\lambda-\theta} z^n,$$

where θ is a nonnegative real number and λ is in $[0, 1]$. This can be shown with the help of Stirling's formula [3] and the type of reasoning we did in proving Lemma 1 that for all values of λ $(0 \leqslant \lambda \leqslant 1)$ the function ϕ_θ is analytic in $D = \{z; |z| < e^{-1}/\lambda e^{-\lambda}\}$ (being a uniform limit of analytic functions) and is continuous on the closure of D.

It is known that GPD is infinitely divisible in parameter θ with λ fixed in $[0, 1]$ (Consul [1]). Consequently, ϕ_θ has the following property:

$$\phi_{\theta_1}(e^{it})\phi_{\theta_2}(e^{it}) = \phi_{\theta_1+\theta_2}(e^{it})$$

for all t in R. Since ϕ_θ is analytic in D which contains the unit circle, we have

$$\phi_{\theta_1}(z)\phi_{\theta_2}(z) = \phi_{\theta_1+\theta_2}(z) \tag{7}$$

for all z in D (for $\lambda = 1$, D does not contain the unit circle, but we avoid this simple part of the proof for the sake of brevity). The property (7) states that $\phi_\theta(z) \neq 0$ for all z in D. Indeed, if for some θ, $\phi_\theta(z) = 0$ for some value of z, then this will imply that $\phi_0(z) = 0$ for that value of z, but $\phi_0(z) = e^{-\theta}$ for all values of z. Consequently, we have a nonvanishing analytic function ϕ_0 on a simply connected domain, so there exists an analytic function ψ_θ such that $\phi_\theta(z) = \exp\{\psi_\theta(z)\}$. Now, by using (7) it can easily be deduced that the function $\psi_\theta = \theta\psi$ for some function ψ (analytic in D). Thus we have $\phi_\theta(z) = e^{\theta\psi}$.

Consider the mapping $U(t) = te^{-\lambda(t-1)}$ from $[0, 1]$ to $[0, 1]$, where $0 \leqslant \lambda \leqslant 1$. Clearly, $U'(t) = \lambda e^{-\lambda(t-1)}(1/\lambda - t) > 0$ for $t \in (0, 1)$, so $U(t)$ is strictly increasing on $(0, 1)$. Also $U(0) = 0$, $U(1) = 1$, and hence U maps $[0, 1]$ onto $[0, 1]$ in a one-to-one fashion. Now, for $0 \leqslant u \leqslant 1$, $\phi_\theta(u)$ is real, and there exists a t in

[0, 1] such that $u = te^{-\lambda(t-1)}$, so we have

$$\sum_{n=0}^{\infty} \frac{u^n \theta(\theta+n\lambda)^{n-1} e^{-n\lambda-\theta}}{n!} \phi_\theta(u) = \sum_{n=0}^{\infty} (te^{-\lambda(t-1)})^n \theta(\theta+n\lambda)^{n-1} e^{-n\lambda-\theta} \frac{1}{n!}$$
$$= e^{\theta t} e^{-\theta} \left[\sum_{n=0}^{\infty} \frac{\theta t e^{-\theta t} (\theta t + n\lambda t)^{n-1} e^{-n\lambda t}}{n!} \right],$$

where

$$\sum_{n=0}^{\infty} \frac{\theta t e^{-\theta t} (\theta t + n\lambda t)^{n-1} e^{-n\lambda t}}{n!} = 1.$$

Therefore, $\phi_\theta(u) = e^{\theta(t-1)}$, where $u = te^{-\lambda(t-1)}$, but we also know that $\phi_\theta(u) = e^{\theta\psi(u)}$. Thus, by comparing we get $t = \psi(u)+1$, so $u = (\psi(u)+1)e^{-\lambda(\psi(u))}$. Now both sides are analytic in z (u replaced by z), so $z = (\psi(z)+1)e^{(-\lambda)\psi(z)}$ for each z in D. Now for $z = e^{i\alpha}$ we get

$$e^{i\alpha} = (\psi(\alpha)+1)e^{-\lambda\psi(\alpha)} \tag{8}$$

(for simplicity, we denote $\psi(e^{i\alpha})$ by $\psi(\alpha)$). Now $e^{\lambda\psi(\alpha)}$ and $e^{i\alpha}$ are characteristic functions, and so is $e^{\lambda\psi(\alpha)}e^{i\alpha}$. Therefore, from (8) we conclude that $\psi(\alpha)+1$ is a characteristic function. Now put $\psi(\alpha)+1 = \mu$ in (8) to get

$$\mu e^{-i\alpha} = e^{\lambda(\mu-1)} = e^{-\lambda(1-\mu)},$$

where $\mu e^{-i\alpha}$ is the characteristic function of GPD(λ, λ), and μ is the characteristic function for GPD$(\lambda, \lambda) * \delta_1$. Thus we have shown that GPD is a compound Poisson distribution, where the compounding distribution is GPD(λ, λ) shifted one unit to the right.

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Received on 14.10.1996