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INVARIANT MEASURES FOR STOCHASTIC HEAT EQUATIONS*

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Abstract. The paper is concerned with the asymptotic behaviour of solutions to the nonlinear stochastic heat equations, with spatially homogeneous noise, in the whole space. Sufficient conditions for the existence of invariant measures, in weighted spaces of locally square-integrable functions, are given.

For linear equations with multiplicative noise an invariant measure, supported by positive functions, is constructed. The existence of a stationary solution to the vector Burgers equations is obtained as an application of the general theory.

1. INTRODUCTION

This work is concerned with the asymptotic behaviour of solutions to the stochastic heat equation:

$$(1.1)\qquad \begin{cases} \partial_t X(t, \xi) = \frac{1}{2}\Delta_\xi X(t, \xi) + b\big(\xi, X(t, \xi)\big)\dot{\mathcal{W}}(\xi, t), & \xi \in \mathbf{R}^d,\ t > 0, \\ X(0, \xi) = x(\xi), \quad \xi \in \mathbf{R}^d, \end{cases}$$

where Δ_ξ is the Laplace operator on $\mathbf{R}^d$, b is a real function, $\mathcal{W}$ is a spatially homogeneous Wiener process (see [5] and [13]). We investigate the existence and supports of invariant measures for (1.1). The results of the present paper allow to apply the Cole–Hopf transformation to a class of solutions of (1.1), to construct stationary solutions of the vector, *stochastic Burgers equation*:

$$(1.2)\qquad \partial_t u(t, \xi) = \tfrac{1}{2}\Delta_\xi u(t, \xi) - \langle u(t, \xi), \nabla_\xi \rangle u(t, \xi) + \nabla \dot{\mathcal{W}}(t, \xi)$$

in $\mathbf{R}^d$, $d > 2$ (cf. [1] and [7]).

The paper consists of 4 sections and is organized as follows.

After the Introduction, we collect in Section 2 some results on L^2_ϱ-spaces and on the heat semigroup $S(t)$, $t \geqslant 0$, on those spaces needed in the sequel.

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We also recall an existence result for equation (1.1) from [13]. *Existence of solutions* to (1.1) has been studied by several authors (see, e.g., [5], [8], [11], and [13]). Here we use the approach from [13] which leads to Markov solutions on weighted spaces L^2_ϱ with weights ϱ.

Our main results on *existence of invariant measures* for equation (1.1) are contained in Section 3. Theorem 3.1 gives a general sufficient condition for the existence of an invariant measure for (1.1) on a weighted space L^2_ϱ. In its proof we use the so-called *compactness method* introduced, for infinite-dimensional systems, in [2]. However, equations studied here are essentially different from those treated in [2]. In particular, in contrast to the situation in [2], operators $S(t)$, $t > 0$, are *not compact* in any L^2_ϱ. Moreover, the diffusion mapping in (1.1) is, in general, *unbounded*. On the other hand, as in [2], we construct an invariant measure as a weak limit of a suitable subsequence of time averages of the laws of $X(t)$:

$$\frac{1}{T}\int_1^{T+1} \mathscr{L}(X(t))\,dt, \qquad T \geqslant 1.$$

Theorem 3.3 is concerned with a more *specific condition* for the existence of an invariant measure for equation (1.1) in terms of the coefficient b and of the correlation function of the noise. The theorem is a *generalization* of a result obtained by Dawson and Salehi in [5] for the linear case, under stronger assumptions on the correlation function of the noise. In [5] the chaos expansion technique was used and the invariant measure was constructed only on the space of tempered distributions. Here we show that the invariant measure is supported by L^2_ϱ, where $\varrho = (1+|\xi|^r)^{-1}$, $\xi \in \boldsymbol{R}^d$, $r > 2d$. In the special case of the constant diffusion function b we can recover some recent results from [4]. It turns out that in this latter case, if $d > 2$, our sufficient condition for the existence of an invariant measure is, in fact, also necessary.

The final Section 4 is devoted to more refined results for the *linear equation*:

$$\text{(1.3)} \qquad \begin{cases} \partial_t X(t, \xi) = \frac{1}{2}\Delta_\xi X(t, \xi) + X(t, \xi)\,\dot{\mathscr{W}}(\xi, t), & \xi \in \boldsymbol{R}^d,\ t > 0, \\ X(0, \xi) = x(\xi), & \xi \in \boldsymbol{R}^d. \end{cases}$$

In Section 4 we discuss also the *Burgers equation* (1.2). In the linear case (1.3), the measure $\delta_{\{\boldsymbol{0}\}}$, where $\boldsymbol{0}$ denotes the function equal to 0 on $\boldsymbol{R}^d$, is always invariant. It is therefore important to show the existence of invariant measures with supports different from $\{\boldsymbol{0}\}$. This is the object of Section 4.1. Next, in Section 4.2, we state a condition on the *spectral density* of the covariance operator under which there exists an invariant measure on a weighted Sobolev space H^n_ϱ. Here we use a novel technique of *interpolation spaces*. Then (see Section 4.3) the existence of a stationary solution taking values in the set of strictly positive functions on $\boldsymbol{R}^d$ is studied by using a result recently proved in [15]. The final subsection is devoted to the applicability of the *Cole-Hopf transform* $X \to -\nabla_\xi \log(X)$ (see [1]). In fact, one of our main motivations to study the questions discussed in the paper was to find sufficient conditions for the existence of a stationary solution of equation (1.3) to which the Cole–Hopf transform can be applied. Using the stochastic Feynman–Kac formula and some properties of the stochastic flows developed by Kunita [9] a construction of such a solution was recently proposed by Kifer [7]. Our approach, based on the theory of stochastic evolution equations, seems to be more natural and leads to different sufficient conditions (see Theorem 4.9). Moreover, it applies to non-linear equations that are beyond the scope of the stochastic Feynman–Kac formula.

The paper is a slightly modified version of preprint [16].

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2. PRELIMINARY RESULTS

Before giving a precise meaning to equation (1.1) we have to recall several analytical concepts.

By S we denote the semigroup generated by $\frac{1}{2}\Delta$ on $L^2(\boldsymbol{R}^d)$ or on some extensions of $L^2(\boldsymbol{R}^d)$. We have, for all $t > 0$ and all $\varphi \in L^2(\boldsymbol{R}^d)$,

$$S(t)\varphi = G(t, \cdot) * \varphi \quad \text{and} \quad G(t, \xi) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-|\xi|^2}{2t}\right),$$

where $*$ denotes the convolution operator.

Since we work on the whole $\boldsymbol{R}^d$, we will study equation (1.1) in weighted spaces. An *admissible weight* ϱ is a positive bounded continuous function $\varrho \in L^1(\boldsymbol{R}^d)$ such that, for all $T > 0$, there exists a constant $C_\varrho(T)$ satisfying

$$G(t, \cdot) * \varrho \leqslant C_\varrho(T)\varrho \quad \text{for all } t \in [0, T]. \tag{2.1}$$

We denote by L^2_ϱ the weighted space $L^2(\boldsymbol{R}^d, \sqrt{\varrho(\xi)}\, d\xi)$ with scalar product

$$\langle \varphi, \psi \rangle_\varrho = \int_{\boldsymbol{R}^d} \varphi(\xi)\psi(\xi)\varrho(\xi)\, d\xi$$

and norm

$$|\varphi|_\varrho^2 = \int_{\boldsymbol{R}^d} \varphi^2(\xi)\varrho(\xi)\, d\xi.$$

Since ϱ is bounded, $L^2(\boldsymbol{R}^d) \subset L^2_\varrho$. In the following we will denote $L^2(\boldsymbol{R}^d)$ simply by L^2.

EXAMPLE. The functions

$$\varrho(\xi) = \exp(-r|\xi|) \text{ with } r > 0 \quad \text{and} \quad \varrho(\xi) = (1 + |\xi|^r)^{-1} \text{ with } r > d$$

are admissible weights.

The next proposition states that the heat semigroup is compact from a weighted space to another one. It is of great importance in our proof of the existence of invariant measure.

PROPOSITION 2.1. *For all admissible weights ϱ, the semigroup $S(\cdot)$ can be extended to a $\mathscr{C}_0$-semigroup on L^2_ϱ. Moreover, if $\hat{\varrho}$ is another admissible weights such that*

$$\int_{\mathbf{R}^d} \frac{\varrho(\xi)}{\hat{\varrho}(\xi)}\, d\xi < +\infty, \tag{2.2}$$

then, for all $t > 0$, $S(t)$ is compact from $L^2_{\hat{\varrho}}$ to L^2_ϱ.

Proof. For all $\varphi \in L^2$ and for all $t \in [0, T]$ we have

$$\begin{aligned}
|S(t)\varphi|_\varrho^2 &= \int_{\mathbf{R}^d} \varrho(\xi)\Big(\int_{\mathbf{R}^d} G(t, \xi-\eta)\varphi(\eta)\, d\eta\Big)^2 d\xi \\
&\leqslant \int_{\mathbf{R}^d} \varrho(\xi)\Big(\int_{\mathbf{R}^d} G(t, \xi-\eta)\, d\eta\Big)\Big(\int_{\mathbf{R}^d} G(t, \xi-\eta)\varphi^2(\eta)\, d\eta\Big) d\xi \\
&\leqslant \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} G(t, \xi-\eta)\varphi^2(\eta)\varrho(\xi)\, d\eta\, d\xi \leqslant C_\varrho(T)|\varphi|_\varrho^2.
\end{aligned}$$

Therefore the operator $S(t)$ can be extended to a bounded linear map from L^2_ϱ to itself. Moreover, since L^2 is dense in L^2_ϱ, it is easy to show that S is strongly continuous in L^2_ϱ.

We claim now that, if (2.2) holds, then $S(t)$, $t > 0$, is a Hilbert–Schmidt operator, and therefore compact from $L^2_{\hat{\varrho}}$ to L^2_ϱ.

Let $\{e_i\}$ be an orthonormal basis in L^2; then, as can be easily verified, $\{\hat{\varrho}^{-1/2} e_i\}$ is an orthonormal basis in $L^2_{\hat{\varrho}}$. Moreover,

$$\begin{aligned}
\sum_{i=1}^\infty \left|S(t)\Big(\frac{e_i}{\sqrt{\hat{\varrho}}}\Big)\right|_\varrho^2 &= \sum_{i=1}^\infty \int_{\mathbf{R}^d} \varrho(\xi)\Big(\int_{\mathbf{R}^d} G(t, \xi-\eta)\frac{e_1(\eta)}{\sqrt{\hat{\varrho}(\eta)}}\, d\eta\Big)^2 d\xi \\
&\leqslant \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} G^2(t, \xi-\eta)\hat{\varrho}^{-1}(\eta)\varrho(\xi)\, d\xi\, d\eta \\
&\leqslant (4\pi t)^{-d/2}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} G(t/2, \xi-\eta)\hat{\varrho}^{-1}(\eta)\varrho(\xi)\, d\xi\, d\eta \\
&\leqslant (4\pi t)^{-d/2} C_\varrho(t/2)\int_{\mathbf{R}^d} \hat{\varrho}^{-1}(\xi)\varrho(\xi)\, d\xi
\end{aligned}$$

and our claim follows by (2.2). ∎

Remark 2.2. Condition (2.2) holds for the classes of weights introduced in the Example. If $\varrho(\xi) = \exp(-r|\xi|)$ and $\hat{\varrho}(\xi) = \exp(-\hat{r}|\xi|)$, then (2.2) holds if and only if $r > \hat{r}$. Moreover, if $\varrho(\xi) = (1+|\xi|^r)^{-1}$ and $\hat{\varrho}(\xi) = (1+|\xi|^{\hat{r}})^{-1}$, then (2.2) holds if and only if $r > \hat{r}+d$.

By $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbf{P})$ we denote a fixed stochastic basis. We will assume that the $(\mathscr{F}_t)$-adapted Wiener process $\mathscr{W}$ in equation (1.1) has, in general, an unbounded covariance operator Q of convolution type. More precisely we assume that

$$Qx = \Gamma * x, \qquad x \in \mathscr{D}(Q),$$

where Γ is a positive definite distribution with even spectral density $\gamma \geqslant 0$ (see [6], Chapter II). This means that the Fourier transform of the function γ is identical with the distribution Γ and that

$$E\langle \mathscr{W}(t), \phi\rangle \langle \mathscr{W}(s), \psi\rangle = (t \wedge s)\langle \Gamma * \phi, \psi\rangle$$

for arbitrary rapidly decreasing functions ϕ and ψ.

Following [13] we write equation (1.1) in the integral form:

$$X^x(t) = S(t)x + \int_0^t S(t-s) M\big(B(X^x(s))\big)\, dW_s. \tag{2.3}$$

The mapping M in (2.3) is defined, for φ in L^2_ϱ and rapidly decreasing functions ψ, by

$$M(\varphi)\psi = \varphi\,(\widehat{\gamma^{1/2} * \psi}),$$

and the mapping $B\colon L^2_\varrho \to L^2_\varrho$ is given by

$$B(\phi)(\xi) = b\big(\xi, \phi(\xi)\big), \qquad \xi \in \boldsymbol{R}^d,$$

where $b\colon \boldsymbol{R}^d \times \boldsymbol{R} \to \boldsymbol{R}$ is a measurable function. Moreover, W is an $(\mathscr{F}_t)$-adapted Wiener process on L^2 with the covariance operator I.

In the sequel we will always require that the spectral density γ and the function b satisfy the following assumption:

HYPOTHESIS 2.1. (i) *There exists* $p \in [1, +\infty]$, *with* $(1-1/p)(d/2) < 1$, *such that* $\gamma \in L^p(\boldsymbol{R}^d)$.

(ii) *There exist constants* l *and* l_0 *such that*

$$|b(\xi, \zeta) - b(\xi, z)| \leqslant l|\zeta - z| \quad \textit{and} \quad |b(\xi, 0)| \leqslant l_0 \quad \textit{for all } \xi \in \boldsymbol{R}^d, \zeta \in \boldsymbol{R}.$$

The number l will be called the *Lipschitz constant* of b.

Notice that, under the above hypothesis, if $\phi \in L^2_\varrho$, then $B(\phi) \in L^2_\varrho$ and

$$|B(\phi) - B(\psi)|_\varrho \leqslant l|\psi - \phi|_\varrho \quad \text{and} \quad |B(0)|_\varrho \leqslant l_0 |\varrho|_{L^1(\boldsymbol{R}^d)} \quad \text{for all } \psi, \phi \in L^2_\varrho. \tag{2.4}$$

We can now state the existence and uniqueness result for equation (2.3) as proved in [13]. Let us recall that if $(K, |\cdot|_K)$ is a Banach space, we denote by $\mathscr{C}_{\mathscr{P}}(0, T, L^2(\Omega, K))$ the linear set of all $(\mathscr{F}_t)$-predictable processes $Y\colon \Omega \times [0, T] \to K$ such that the map $t \to E(|Y(t)|_K^2)$ is continuous. The set is endowed with the norm

$$|\phi|_{\mathscr{C}_{\mathscr{P}}(0,T,L^2(\Omega,K))} = \sup_{t\in[0,T]} E\big(|Y(t)|_K^2\big).$$

THEOREM 2.3. *Assume Hypothesis* 2.1 *is satisfied and let* ϱ *be an admissible weight. For all* $x \in L^2_\varrho$ *there exists a unique solution* $X^x(\cdot) \in \mathscr{C}_{\mathscr{P}}(0, T, L^2(\Omega, L^2_\varrho))$ *of equation* (2.3).

We finish this section with an analytical result from [13], needed in the following sections. It played a fundamental role in the proof of Theorem 2.3. If E and F are Hilbert spaces, we denote by $L_2(E, F)$ the space of Hilbert–Schmidt operators from E to F, and by $\|\cdot\|_{L_2(E,F)}$ the corresponding Hilbert–Schmidt norm.

PROPOSITION 2.4. *If Hypothesis 2.1 (i) holds, then, for all $\phi \in L^2_\varrho$, $S(t)M(\phi)$ is a Hilbert–Schmidt operator from L^2 to L^2_ϱ and*

$$\|S(t)M(\phi)\|_{L_2(L^2,L^2_\varrho)} \leqslant Ct^{-(1-1/p)(d/4)}|\phi|_\varrho.$$

Remark 2.5. Theorem 2.3 and Proposition 2.4 are proved in [13] only for exponential weights $\varrho(\xi) = \exp(-r|\xi|)$. However, the results as well as the arguments of the proofs extend to general weights studied here.

3. EXISTENCE OF INVARIANT MEASURES

Let us recall that the process $X^{\hat{x}}(\cdot)$ is bounded in probability in $L^2_{\hat{\varrho}}$ if

$$\forall \varepsilon > 0,\ \exists R > 0\colon\ \forall t > 0,\ P\{|X^{\hat{x}}(t)|_{\hat{\varrho}} \geqslant R\} < \varepsilon. \tag{3.1}$$

In this section we prove that if the process $X^{\hat{x}}(\cdot)$ is bounded in probability in $L^2_{\hat{\varrho}}$ for proper $\hat{\varrho}$, then there exists an invariant measure for equation (2.3) on the space L^2_ϱ. We also give conditions on the covariance operator of the Wiener process implying the boundedness in probability of $X^{\hat{x}}(\cdot)$.

The following theorem is in the spirit of Theorem 4 in [2]. In contrast to [2], the diffusion mapping in equation (2.3) is unbounded and operators $S(t)$, $t > 0$, considered as maps from L^2_ϱ into itself, are not compact.

THEOREM 3.1. *Assume that Hypothesis 2.1 holds and that, for an admissible weight ϱ, there exist another admissible weight $\hat{\varrho}$ and an element $\hat{x}$ in $L^2_\varrho \cap L^2_{\hat{\varrho}}$ such that*

$$\int_{\mathbf{R}^d} \varrho(\xi)\big(\hat{\varrho}(\xi)\big)^{-1} d\xi < +\infty \ \textit{and}\ X^{\hat{x}}(\cdot) \textit{ is bounded in probability in } L^2_{\hat{\varrho}}.$$

Then there exists an invariant measure for equation (2.3) in L^2_ϱ.

Proof. To prove the theorem we show that for arbitrary $\varepsilon \in (0, 1)$ and all $R > 0$ there exists a compact set $\mathscr{K}$ in L^2_ϱ such that

$$P\{X^{\hat{x}}(t) \in \mathscr{K}\} \geqslant (1-\varepsilon)P\{|X^{\hat{x}}(t-1)|_{\hat{\varrho}} < R\} \quad \text{for all } t > 1. \tag{3.2}$$

The boundedness in probability of the process $X^{\hat{x}}(\cdot)$ in $L^2_{\hat{\varrho}}$ then implies that the family of laws $\mathscr{L}(X^{\hat{x}}(t))$, $t > 1$, is tight in L^2_ϱ. Therefore the family of laws

$$\frac{1}{T}\int_1^{T+1} \mathscr{L}(X^{\hat{x}}(t))\,dt, \qquad T > 0,$$

is tight as well and, by a general argument, the weak limit of any converging sequence

$$\frac{1}{T_j}\int_1^{T_j+1} \mathscr{L}(X^{\hat{x}}(t))\,dt, \qquad T_j \nearrow +\infty,$$

is invariant for equation (2.3) in L^2_ϱ.

As in [2] we divide the proof into three steps:

Step 1. Let $q > 2$ and $\alpha \in (q^{-1}, 2^{-1})$. Then operators F_α:

$$F_\alpha = \int_0^1 (1-s)^{\alpha-1} S(1-s) f(s)\,ds,$$

are compact from $L^q((0,1), L^2_{\hat{\varrho}})$ to L^2_ϱ.

Since $S(\varepsilon)$, $\varepsilon > 0$, is compact from $L^2_{\hat{\varrho}}$ to L^2_ϱ, it follows that operators

$$F^\varepsilon_\alpha f = \int_0^{1-\varepsilon} (1-s)^{\alpha-1} S(1-s) f(s)\,ds = S(\varepsilon)\int_\varepsilon^1 (1-s-\varepsilon)^{\alpha-1} S(1-s) f(s+\varepsilon)\,ds$$

are compact from $L^q((0,1), L^2_{\hat{\varrho}})$ to L^2_ϱ.

Direct estimates imply that $\lim_{\varepsilon\to 0} F^\varepsilon_\alpha = F_\alpha$ in the operator norm, so operators F_α are compact as well.

Step 2. Define, for all $x \in L^2_{\hat{\varrho}}$,

$$Y^x(s) = \int_0^s (s-\varsigma)^{-\alpha} S(s-\varsigma) M\big(B(X^x(\varsigma))\big)\,dW_\varsigma.$$

If

$$\frac{1}{q} < \alpha < \frac{1}{2}\left(1-\left(1-\frac{1}{p}\right)\frac{d}{2}\right),$$

then there exists k_1 such that

$$E\int_0^1 |Y^x(s)|^q_{\hat{\varrho}}\,ds \leqslant k_1 |x|^q_{\hat{\varrho}}.$$

To show this notice that by Proposition 2.4 we have

$$E\int_0^1 |Y^x(s)|^q_{\hat{\varrho}}\,ds \leqslant k\int_0^1\left(E\int_0^s (s-\varsigma)^{-2\alpha}\left\|S(s-\varsigma)M\big(B(X^x(\varsigma))\big)\right\|_{L_2(L^2,L^2_{\hat{\varrho}})}d\varsigma\right)^{q/2} ds$$

$$\leqslant kC^q E\int_0^1\left(\int_0^s (s-\varsigma)^{-(2\alpha+\sigma)}|B(X^x(\varsigma))|^2_\varrho\,d\varsigma\right)^{q/2} ds,$$

where $\sigma = (1-1/p)(d/2) < 1$.

Since $2\alpha+\sigma < 1$, by Young's inequality for convolutions we have

$$E\int_0^1 |Y^x(s)|^q_{\hat{\varrho}}\,ds \leqslant kC^q l^q |\hat{\varrho}|^q_{L^1(\mathbf{R}^d)}(1-2\alpha-\sigma)^{q/2} E\int_0^1 |X^x(\varsigma)|^q_\varrho\,d\varsigma.$$

Direct estimates (see also Lemma 3 in [2]) imply that for all $q > 1$ there exists k_2 such that

$$E|X^x(t)|_{\hat{\varrho}}^q \leqslant k_2(1+|x|_{\hat{\varrho}}^q),$$

and this completes the proof of Step 2.

Step 3. We can now conclude the proof of the theorem. By the *factorization formula* (see e.g. Theorem 5.2.5 in [4]) we have

$$X^x(1) = S(1)x + \frac{\sin(\pi\alpha)}{\pi} F_\alpha Y^x(t).$$

If

$$\mathscr{K}(\theta) = \{S(1)y + F_\alpha h, \text{ where } |y|_\varrho \leqslant \theta \text{ and } |h|^q_{L^q(0,1,L^2_{\hat{\varrho}})} \leqslant \theta\},$$

then, for all $\theta > 0$, $\mathscr{K}(\theta)$ is relatively compact in L^2_ϱ. Moreover, by Step 1 and Chebyshev's inequality, if $\theta > |x|_{\hat{\varrho}}$, we have

$$P\{X^x(1) \notin \mathscr{K}(\theta)\} \leqslant P\left\{\int_0^1 |Y^x(\varsigma)|_{\hat{\varrho}}^q \, d\varsigma > \frac{\pi\theta}{\sin(\alpha\pi)}\right\}$$

$$\leqslant \sin^q(\alpha\pi)\pi^{-q}\theta^{-q} E\int_0^1 |Y^x(\varsigma)|_{\hat{\varrho}}^q \, d\varsigma \leqslant k_3\theta^{-q}(1+|x|_{\hat{\varrho}}^q),$$

where $k_3 = \sin^q(\alpha\pi)\pi^{-q}k_1$. Consequently, for all $\theta > R$ we have

$$P\{X^x(t) \in \mathscr{K}(\theta)\} \geqslant \left(1 - k_3\theta^{-q}(1+R^q)\right) P\{|X^x(t-1)|_{\hat{\varrho}} < R\}.$$

Since $X^{\hat{x}}$ is bounded in probability in $L^2_{\hat{\varrho}}$, for all $\varepsilon > 0$ we can choose R such that $P\{|X^{\hat{x}}(t)|_\varrho \geqslant R\} < \varepsilon$. It is enough now to fix $\theta > R$ such that $k_3\theta^{-q}(1+R^q) < \varepsilon$. This completes our proof. ■

Remark 3.2. If there exist $\varkappa$ such that, for some weight $\hat{\varrho}$ and state $\hat{x}$ in $L^2_{\hat{\varrho}}$,

$$E\left(|X^{\hat{x}}(t)|_{\hat{\varrho}}\right)^2 \leqslant \varkappa \quad \text{for all } t > 0, \tag{3.3}$$

then condition (3.1) holds. In fact, by Chebyshev's inequality we have

$$P\{|X^{\hat{x}}(t)|_{\hat{\varrho}} \geqslant R\} \leqslant R^{-2} E\left(|X^{\hat{x}}(t)|_{\hat{\varrho}}^2\right) \leqslant R^{-2}\varkappa.$$

The following theorem provides a condition, on the covariance of the noise, under which (3.3) holds for the constant function $\hat{x}(\xi) = \mathbf{1}(\xi) = 1$ for all $\xi \in \mathbf{R}^d$. The result is a *generalization* of Theorem 3.4 in [5] to *non-linear equations* and to general spectral densities.

THEOREM 3.3. *Assume Hypothesis* 2.1 *holds and define*

$$\tilde{\Gamma} = |\widehat{\gamma^{1/2}}| * |\widehat{\gamma^{1/2}}| \quad \text{and} \quad I = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \int_{\mathbf{R}^d} \tilde{\Gamma}(\zeta)|\zeta|^{2-d} \, d\zeta, \tag{3.4}$$

where Γ is the gamma function. If $d \geqslant 3$ and $\boldsymbol{l} < l^{-2}$, where l is the Lipschitz constant of b, then, for all admissible weights ϱ,

$$\sup_{t \geqslant 0} E\big(|X^1(t)|_\varrho\big)^2 < +\infty.$$

Proof. First notice that $\mathit{1} \in L^2_\varrho$ for all admissible weights ϱ. Let $X_0(t, \xi) = 0$ and

$$X_{n+1}(t) = \mathit{1} + \int_0^t S(t-s) M\big(B(X_n(s))\big) dW_s,$$

so that $X_n(t) \to X^1(t)$ in $L^2(\Omega, L^2_\varrho)$ for all admissible weights ϱ and all $t \geqslant 0$. Moreover, let $Z_n = X_n - X_{n-1}$. Fix a basis $\{e_i \colon i \in N\}$ of L^2 with $e_i \in \mathcal{S}(\boldsymbol{R}^d)$. Then for all $\xi \in \boldsymbol{R}^d$ we have

$$\sum_{i=1}^{\infty} \int_0^t \Big(\int_{\boldsymbol{R}^d} G(s, \xi-\eta) b(\eta, 0) (\widehat{\gamma^{1/2}} * e_i)(\eta) d\eta \Big)^2 ds$$

$$= \int_0^t \sum_{i=1}^{\infty} \Big(\int_{\boldsymbol{R}^d} \Big(\int_{\boldsymbol{R}^d} G(s, \xi-\eta) b(\eta, 0) \widehat{\gamma^{1/2}}(\zeta-\eta) d\eta \Big) e_i(\zeta) d\zeta \Big)^2 ds$$

$$= \int_0^t \Big(\int_{\boldsymbol{R}^d} G(s, \xi-\eta) b(\eta, 0) \widehat{\gamma^{1/2}}(\zeta-\eta) d\eta \Big)^2 d\zeta\, ds$$

$$\leqslant l_0^2 \int_0^{\infty} \int_{\boldsymbol{R}^d} \big((G(s, \cdot) * |\widehat{\gamma^{1/2}}|)(z) \big)^2 dz\, ds = \int_0^{\infty} \int_{\boldsymbol{R}^d} G(2s, z) \tilde{\Gamma}(z) dz\, ds.$$

Since for all $\zeta \in \boldsymbol{R}^d$ we have

$$\int_0^{\infty} G(2s, \zeta) ds = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} |\zeta|^{2-d},$$

and by (3.4) we get

$$\sum_{i=1}^{\infty} \int_0^t \Big(\int_{\boldsymbol{R}^d} G(s, \xi-\eta) (\widehat{\gamma^{1/2}} * e_1)(\eta) d\eta \Big)^2 ds \leqslant l_0^2 \boldsymbol{l}.$$

Therefore we can fix $\xi \in \boldsymbol{R}^d$ in the definition of $Z_1 = X_1$ letting

$$Z_1(t, \xi) = \sum_{i=1}^{\infty} \int_0^t \Big(\int_{\boldsymbol{R}^d} G(s, \xi-\eta) b(\eta, 0) (\widehat{\gamma^{1/2}} * e_1)(\eta) d\eta \Big) d\beta_s^i,$$

and we obtain the following estimate:

$$E\big((Z_1(t, \xi))^2\big) \leqslant l_0^2 \boldsymbol{l} \quad \text{for all } \xi \in \boldsymbol{R}^d.$$

Consequently, we also have

$$E\big(|Z_1(t, \xi) Z_1(t, \eta)|\big) \leqslant l_0^2 \boldsymbol{l} \quad \text{for all } \xi, \eta \in \boldsymbol{R}^d.$$

In a similar way, since $|b(\eta, X_1(s,\eta)) - b(\eta, X_0(s,\eta))| \leqslant l|Z_1(s,\eta)|$, we have

$$E\sum_{i=1}^{\infty}\int_0^t\left(\int_{R^d} G(t,\xi-\eta)\left[b(\eta, X_1(s,\eta))-b(\eta, X_0(s,\eta))\right](\widehat{\gamma^{1/2}}*e_1)(\eta)\,d\eta\right)^2 ds$$

$$\leqslant ll^2\int_0^t\int_{R^d}\int_{R^d}\int_{R^d} G(t-s,\xi-\zeta-v)\,G(t-s,\xi-\zeta-w)$$

$$\times E|Z_1(s,\zeta+v)|\,|Z_1(s,\zeta+w)|\,\widehat{\gamma^{1/2}}(v)\widehat{\gamma^{1/2}}(w)\,dv\,dw\,d\zeta$$

$$\leqslant ll^2 l_0^2\int_0^t\int_{R^d}\int_{R^d}\int_{R^d} G(t-s,\xi-\zeta-v)\,G(t-s,\xi-\zeta-w)\,|\widehat{\gamma^{1/2}}(v)|\,|\widehat{\gamma^{1/2}}(w)|\,dv\,dw\,d\zeta$$

$$= ll^2 l_0^2\int_0^{\infty}\int_{R^d} G(2t,\zeta)\tilde{\Gamma}(\zeta)\,d\zeta\,dt \leqslant l^2 l^2 l_0^2.$$

Therefore we can fix $\xi\in R^d$ first in the definition of Z_2, and then in the definition of X_2, letting

$$Z_2(t,\xi)$$

$$= \sum_{i=1}^{\infty}\int_0^t\left(\int_{R^d} G(s,\xi-\eta)\left[b(\eta, X_1(s,\eta))-b(\eta, X_0(s,\eta))\right](\widehat{\gamma^{1/2}}*e_1)(\eta)\,d\eta\right)d\beta_s^i,$$

and we get

$$E\left((Z_2(t,\xi))^2\right) \leqslant l^2 l^2 l_0^2.$$

In a similar way, for all $n = 1, 2, \ldots$ we have

$$E\left((Z_n(t,\xi))^2\right) \leqslant l^n l^{2(n-1)} l_0^2.$$

Therefore, for any fixed weight ϱ, we have

$$(E|Z_n(t)|_\varrho^2)^{1/2} \leqslant l^{n/2} l^{n-1} l_0 |\varrho|_{L^1(R^d)}$$

and

$$(E|X^1(t)|_\varrho^2)^{1/2} \leqslant \sum_{n=1}^{\infty}(E|Z_n(t)|_\varrho^2)^{1/2} \leqslant l_0\sqrt{l}\,|\varrho|_{L_1}\sum_{n=2}^{\infty}(\sqrt{l}l)^n.$$

This proves our claim because $\sqrt{l}l < 1$. ∎

COROLLARY 3.4. *Let* $\varrho(\xi) = (1+|\xi|^r)^{-1}$. *If the assumptions of Theorem* 3.3 *hold, then equation* (2.3) *admits a non-trivial invariant measure in* L_ϱ^2 *for all* $r > 2d$.

Proof. It is enough to apply Theorem 3.1 for $\varrho(\xi) = (1+|\xi|^r)^{-1}$ and $\varrho(\xi) = (1+|\xi|^{\hat{r}})^{-1}$ with $d < \hat{r} < r-d$. ∎

Remark 3.5. If $\widehat{\gamma^{1/2}} \geqslant 0$, then $\tilde{\Gamma}$ is identical with the Fourier transform Γ of γ introduced in Section 2.

Remark 3.6. Assume that $\widehat{\gamma^{1/2}} \geqslant 0$, $d \geqslant 3$ and $b(\xi, \zeta) = 1$ for all $\xi \in \boldsymbol{R}^d$ and for all $\zeta \in \boldsymbol{R}$; then the assumptions of Theorem 3.3 reduce to

$$\int_{\boldsymbol{R}^d} \Gamma(\zeta)|\zeta|^{2-d} d\zeta < +\infty. \tag{3.5}$$

By the equality $\widehat{|\zeta|^{2-d}} = c|\zeta|^{-2}$ (see [14], p. 128), condition (3.5) takes the form

$$\int_{\boldsymbol{R}^d} \gamma(\zeta)|\zeta|^{-2} d\zeta < +\infty,$$

that is exactly the necessary and sufficient condition stated in [4], p. 196.

4. LINEAR EQUATIONS

In this section we prove more precise results for the linear equation (1.3). Equation (2.3) becomes now

$$X^x(t) = S(t)x + \int_0^t S(t-s) M\big((X^x(s))\big) dW_s. \tag{4.1}$$

4.1. Non-trivial invariant measures. Equation (4.1) has at least one invariant measure: the Dirac measure $\delta_{\{\boldsymbol{0}\}}$ concentrated on the constant function $\boldsymbol{0}$. So the question arises whether the procedure developed in the previous section leads to a measure different from $\delta_{\{\boldsymbol{0}\}}$. The next theorem states that this is the case if the invariant measure is constructed by using the solution to equation (4.1) starting from the function $\boldsymbol{1}$.

THEOREM 4.1. *Assume that Hypothesis* 2.1 (i) *holds. Moreover, suppose that there exists an admissible weight ϱ and $T_j \nearrow +\infty$ such that the sequence of measures on L_ϱ:*

$$\mu_j = \frac{1}{T_j} \int_1^{T_j+1} \mathscr{L}(X^1(t))\, dt, \qquad n = 1, 2, \ldots,$$

converges weakly in L_ϱ to a measure μ and that

$$\sup_{j \in N} \int_{\boldsymbol{R}^d} |x|_\varrho^2 \mu_j(dx) < +\infty.$$

Then $\mu(\{\boldsymbol{0}\}) < 1$.

Proof. Since $S(t)\boldsymbol{1} = \boldsymbol{1}$, equation (4.1) for $x = \boldsymbol{1}$ becomes

$$X^1(t) = \boldsymbol{1} + \int_0^t S(t-s) M\big((X^x(s))\big) dW_s.$$

Taking expectation we get $E(X^1(t)) = \boldsymbol{1}$. Thus

$$\int_{L_\varrho^2} x\mu_j(dx) = \frac{1}{T_j} \int_1^{T_j+1} E(X^1(t))\, dt = \boldsymbol{1}.$$

Moreover, there exists $\tilde{\varkappa}$ such that

$$\int_{L^2_\varrho} |x|^2_\varrho \, \mu_j(dx) \leqslant \tilde{\varkappa}.$$

Now we fix $\lambda > 0$ and define

$$\psi_\lambda(x) = \begin{cases} \langle x, \mathit{1}\rangle_\varrho & \text{if } |\langle x, \mathit{1}\rangle_\varrho| \leqslant \lambda, \\ \lambda & \text{if } \langle x, \mathit{1}\rangle_\varrho > \lambda, \\ -\lambda & \text{if } \langle x, \mathit{1}\rangle_\varrho < -\lambda. \end{cases}$$

Then

$$\begin{aligned} \int_{L^2_\varrho} \psi_\lambda(x)\, \mu_j(dx) &\geqslant \int_{L^2_\varrho} \langle x, \mathit{1}\rangle_\varrho \, \mu_j(dx) - 2 \int_{\{|\langle x, \mathit{1}\rangle_\varrho| > \lambda\}} |\langle x, \mathit{1}\rangle_\varrho| \, \mu_j(dx) \\ &\geqslant |\mathit{1}|^2_\varrho - 2(P\{|\langle x, \mathit{1}\rangle_\varrho| > \lambda\})^{1/2} \Big(\int_{L^2_\varrho} \langle x, \mathit{1}\rangle^2_\varrho \, \mu_j(dx)\Big)^{1/2} \\ &\geqslant |\mathit{1}|^2_\varrho - 2\lambda^{-1} \int_{L^2_\varrho} \langle x, \mathit{1}\rangle^2_\varrho \, \mu_j(dx) \geqslant (1 - 2\lambda^{-1}\tilde{\varkappa}) |\mathit{1}|^2_\varrho. \end{aligned}$$

Since ψ_λ is continuous and bounded on L^2_ϱ, we can pass to the limit with $j \to +\infty$, obtaining

$$\int_{L^2_\varrho} \psi_\lambda(x)\, \mu(dx) \geqslant (1 - 2\lambda^{-1}\tilde{\varkappa}) |\mathit{1}|^2_\varrho.$$

But $\psi_\lambda(\mathit{0}) = 0$, so by choosing $\lambda > 2\varkappa$ we conclude that $\mu(\{\mathit{0}\}) < 1$. ∎

Remark 4.2. Assume that Hypothesis 2.1 (i) is satisfied. Let ϱ, $\hat{\varrho}$ be two admissible weights such that (2.2) holds and

$$\sup_{t > 0} E(|X^1(t)|^2_\varrho) < +\infty, \quad \sup_{t > 0} E(|X^1(t)|^2_{\hat{\varrho}}) < +\infty. \tag{4.2}$$

Then, by Theorem 3.1, equation (4.1) has an invariant measure μ on L^2_ϱ and Theorem 4.1 implies that $\mu\{\mathit{0}\} < 1$. Sufficient conditions under which (4.2) holds are given in Theorem 3.3.

Remark 4.3. Theorem 4.1 holds in the case of non-linear function b, with the same proof. However, the problem is relevant only when $b(\xi, 0) = 0$ for all $\xi \in \boldsymbol{R}^d$.

4.2. Regularity. We show that, under suitable assumptions on γ, any invariant measure μ in L^2_ϱ for equation (4.1) is concentrated on a weighted H^n space. We obtain this result showing first that the solution of equation (4.1) at time $t > 0$ is more regular than at time $t = 0$.

We have to introduce some additional notation. Let H^n_ϱ be the completion of the space of regular functions with compact support in $\boldsymbol{R}^d$ with respect to the norm

$$|\phi|_{n,\varrho} = \Big(\sum_{|\alpha| < n} \int_{\boldsymbol{R}^d} |D^\alpha \phi(\xi)|^2 \varrho(\xi)\, d\xi \Big)^{1/2}.$$

The following is the main result of this section:

THEOREM 4.4. *If* μ *is an invariant measure for equation* (4.1) *in* L^2_ϱ *and*

$$(4.3) \qquad \int_{\mathbb{R}^d} (1+|\xi|^{2n})^p \gamma^p(\xi)\, d\xi < \infty \quad \text{with } p>1 \text{ and } \left(1-\frac{1}{p}\right)\frac{d}{2} < 1,$$

then $\mu(H^n_\varrho) = 1$.

Since, by definition,

$$\mu(H^n_\varrho) = \int_{L^2_\varrho} I_{H^n_\varrho}(X^x(t))\, \mu(dx),$$

the theorem is an immediate consequence of the following *regularization result*:

THEOREM 4.5. *If* (4.3) *holds, then, for all* x *in* L^2_ϱ *and all* $t>0$, $X^x(t) \in L^2(\Omega, \mathscr{F}_0, \boldsymbol{P}, H^n_\varrho)$. *In particular,* $X^x(t) \in H^n_\varrho$ $\boldsymbol{P}$*-almost surely.*

Proof. The proof is based on the interpolation theory (see [10]). For all $\alpha \in]0, 1[$ we denote by $V^{\alpha,n}_\varrho$ the real interpolation space $(H^n_\varrho, L^2_\varrho)_{\alpha,2}$ (see [10], p. 15). Moreover, by definition, $V^{0,n}_\varrho = L^2_\varrho$ and $V^{1,n}_\varrho = H^n_\varrho$.

Step 1. First we prove that if an initial state is in some $V^{\alpha,n}_\varrho$, then the solution of equation (4.1) is in $\mathscr{C}(0, T, L^2(\Omega, V^{\alpha,n}_\varrho))$. The result is proved in [13] for spaces H^n_ϱ. By Proposition 2.4 the map $\phi \to S(t)M(\phi)$ is bounded from L^2_ϱ to $L_2(L^2, L^2_\varrho)$ and

$$\|S(t)M(\phi)\|_{L_2(L^2,L^2_\varrho)} \leqslant Ct^{-(1-1/p)(d/4)} |\phi|_\varrho.$$

We also know (see [13]) that if (4.3) holds, then the map $\phi \to S(t)M(\phi)$ is bounded from H^n_ϱ to $L_2(L^2, H^n_\varrho)$ and

$$\|S(t)M(\phi)\|_{L_2(L^2,H^n_\varrho)} \leqslant Ct^{-(1-1/p)(d/4)} |\phi|_{H^n_\varrho}.$$

From the definition of the interpolation spaces (see [10], p. 15) we easily obtain

$$\big(L_2(L^2, H^n_\varrho), L_2(L^2, L^2_\varrho)\big)_{\alpha,2} \subset L_2(L^2, V^{\alpha,n}_\varrho).$$

By the interpolation property we infer that if (4.3) holds, then $\phi \mapsto S(t)M(\phi)$ is a bounded map from $V^{\alpha,n}_\varrho$ to $L_2(L^2, V^{\alpha,n}_\varrho)$, and

$$\|S(t)M(\phi)\|_{L_2(L^2,V^{\alpha,n}_\varrho)} \leqslant Ct^{-(1-1/p)(d/4)} |\phi|_{V^{\alpha,n}_\varrho}.$$

Now the claim follows from a standard fixed point argument in the space $\mathscr{C}_{\mathscr{P}}(0, \tau, L^2(\Omega, V^{\alpha,n}_\varrho))$.

Step 2. Since $|S(t)|_{\mathscr{L}(L^2_\varrho,H^n_\varrho)} \leqslant Ct^{-n/2}$, by interpolation we get, for all $\alpha \geqslant 0$, $\delta \geqslant 0$ with $\alpha+\delta \leqslant 1$,

$$|S(t)|_{\mathscr{L}(V^{\alpha,n}_\varrho, V^{(\alpha+\delta),n}_\varrho)} \leqslant Ct^{(-n\delta)/2}.$$

Consequently,

$$\begin{aligned}\|S(t)\,M(\phi)\|^2_{(L^2,V_\varrho^{(\alpha+\delta),n})} &= \|S(t/2)\,S(t/2)\,M(\phi)\|^2_{(L^2,V_\varrho^{(\alpha+\delta),n})}\\ &\leqslant |S(t/2)|^2_{\mathscr{L}(V_\varrho^{\alpha,n},V_\varrho^{(\alpha+\delta),n})}\,\|S(t/2)\,M(\phi)\|^2_{(L^2,V_\varrho^{\alpha,n})}\\ &\leqslant C_1^2\,t^{-2n\delta-(1-1/p)(d/2)}\,|\phi|^2_{V_\varrho^{\alpha,n}},\end{aligned}$$

where $C_1 = C^2\,2^{n\delta+(1-1/p)(d/4)}$.

Step 3. Now we claim that if

$$\delta < \frac{1}{2n}\left[1-\left(1-\frac{1}{p}\right)\frac{d}{2}\right] \quad \text{and} \quad \alpha\in[0,\,1-\delta],$$

then for all x in $V_\varrho^{\alpha,n}$ and all $t>0$

$$X^x(t)\in L^2(\Omega,\,\mathscr{F}_t,\,\boldsymbol{P},\,V_\varrho^{(\alpha+\delta),n}).$$

It is enough to show that

$$\int_0^t S(t-s)\,M\big(X^x(s)\big)\,dW_s\in L^2(\Omega,\,\mathscr{F}_t,\,\boldsymbol{P},\,V_\varrho^{(\alpha+\delta),n}).$$

But, if $2n\delta+(1-1/p)(d/2)<1$, then

$$\begin{aligned}\boldsymbol{E}\int_0^t \big\|S(t-s)\,M\big(X^x(s)\big)\big\|_{L_2(L^2,V_\varrho^{(\alpha+\delta),n})}&\\ \leqslant |X^x|_{\mathscr{C}_{\mathscr{P}}(0,T,L^2(\Omega,V_\varrho^{\alpha,n}))}\int_0^T C_1^2\,\varsigma^{-2n\delta-(1-1/p)(d/2)}\,d\varsigma &< +\infty,\end{aligned}$$

and the statement holds.

Step 4. We are now in a position to conclude the proof. Fix $t>0$ and choose $m\in\boldsymbol{N}$ such that $m>2n\left[1-(1-1/p)\,(d/2)\right]^{-1}$. Applying Step 3, in the interval $[0,\,t/m]$ with $\alpha=0$ and $\delta=m^{-1}$ we get

$$X^x(t/m)\in L^2(\Omega,\,\mathscr{F}_t,\,\boldsymbol{P},\,V_\varrho^{m^{-1},n}).$$

Then applying again Step 3, but now in the interval $[t/m,\,2t/m]$ and with $\alpha=m^{-1}$, $\delta=m^{-1}$, by the Markov property we obtain

$$X^x(2t/m)\in L^2(\Omega,\,\mathscr{F}_t,\,\boldsymbol{P},\,V_\varrho^{2m^{-1},n}).$$

Proceeding in this way, after m steps we obtain the claim. ■

4.3. Strict positivity. In this brief subsection we show that any stationary solution of equation (4.1) with values in the set of non-negative continuous functions on $\boldsymbol{R}^d$ has, in reality, values on the space of strictly positive functions on $\boldsymbol{R}^d$. This is done by using the following *strict positivity* result proved in [15]. Remark that, by [13], if x belongs to the space L^2, then equation (4.1) has a unique solution X^x in $\mathscr{C}_{\mathscr{P}}(0,\,T,\,L^2(\Omega,\,L^2))$.

THEOREM 4.6. *Assume that Hypothesis* 2.1 (i) *holds and let* x *belong to* $L^2(\mathbf{R}^d)$. *If* $x \geqslant 0$ *and, for some* $\varepsilon > 0$, *the set* $\{\xi \in \mathbf{R}^d\colon x(\xi) \geqslant \varepsilon\}$ *has the non-empty interior, then* $X^x(t) > 0$ *almost surely in* $\Omega \times \mathbf{R}^d$ *for all* $t > 0$.

The result can be easily extended to weighted spaces by using a standard comparison result (see Section 4 in [15], or [12], or [8]).

COROLLARY 4.7. *Let* ϱ *be an admissible weight and assume that Hypothesis* 2.1 (i) *holds and let* $x \in L^2_\varrho$. *If* $x \geqslant 0$ *and, for some* $\varepsilon > 0$, *the set* $\{\xi \in \mathbf{R}^d\colon x(\xi) \geqslant \varepsilon\}$ *has the non-empty interior, then* $X^x(t) > 0$ *almost surely in* $\Omega \times \mathbf{R}^d$ *for all* $t > 0$.

Proof. We can assume that $x \geqslant \varepsilon I_{B(\xi,r)}$ for some $\varepsilon > 0$, $r > 0$, $\xi \in \mathbf{R}^d$, where by $B(\xi, r)$ we denote the open ball in $\mathbf{R}^d$ of center ξ and radius r. Let $x_r = x I_{B(\xi,r)}$ and $x_R = x I_{B(\xi,R)}$ for $R > r$. Then $x_r \in L^2(\mathbf{R}^d)$, $x_R \in L^2(\mathbf{R}^d)$; moreover, $\varepsilon I_{B(\xi,r)} \leqslant x_r \leqslant x_R$ and, finally, $x_R \to x$ in L^2_ϱ as $R \nearrow +\infty$. By [15], Proposition 4.1, we know that, for fixed t, $X^{x_r}(t) \leqslant X^{x_R}(t)$ almost surely in $\mathbf{P} \otimes \mu$. Moreover, as can be easily proved by a parameter-depending contraction principle (see Section 2), $X^{x_R}(t) \to X^x(t)$ in L^2_ϱ as $R \nearrow +\infty$. The claim follows since, by Theorem 4.6, $X^{x_r}(t) > 0$ almost surely in $\Omega \times \mathbf{R}^d$. ∎

We are now in a position to prove the main result of this subsection. We denote by $\mathbf{B}_\varrho$ the set $\{x \in L^2_\varrho\colon x(\zeta) > 0$ for almost every $\zeta \in \mathbf{R}^d\}$. Notice that

$$\bar{\mathbf{B}}_\varrho = \{x \in L^2_\varrho\colon x(\zeta) \geqslant 0 \text{ for almost every } \zeta \in \mathbf{R}^d\}.$$

THEOREM 4.8. *Assume that Hypothesis* 2.1 (i) *holds, let* ϱ *be an admissible weight, and* μ *an invariant measure for equation* (4.1) *in* L^2_ϱ. *Suppose that* $\mu(\bar{\mathbf{B}}_\varrho) = 1$, $\mu(\{\mathbf{0}\}) = 0$, *and the set of elements of* L^2_ϱ *which have no continuous modification is of measure* 0 *relatively to* μ. *Then* μ *is concentrated on* $\mathbf{B}_\varrho$, *that is:* $\mu(\mathbf{B}_\varrho) = 1$.

Proof. Our assumptions imply that μ-almost every $x \in L^2_\varrho$ belongs to $\bar{\mathbf{B}}_\varrho - \{\mathbf{0}\}$ and has a continuous modification. Moreover, if $x \neq \mathbf{0}$ lies in $\bar{\mathbf{B}}_\varrho$ and has a continuous modification, then for a suitable $\varepsilon > 0$ the set $\{\xi \in \mathbf{R}^d\colon x(\xi) \geqslant \varepsilon\}$ has a non-empty interior. Therefore, by Corollary 4.7, for fixed $t > 0$, $X^x(t) \in \mathbf{B}_\varrho$ $\mathbf{P}$-a.s. The claim follows since, by definition,

$$\mu(\mathbf{B}_\varrho) = \int_{L^2_\varrho} I_{\mathbf{B}_\varrho}(X^x(t))\,\mu(dx) = 1. \ \blacksquare$$

4.4. Applicability of the Cole–Hopf transform. The previous parts of Section 4 were devoted to the study of specific properties of the support of invariant measures to the stochastic linear heat equation (1.3). One of our main motivations in doing so was to find conditions under which there exists a stationary solution of equation (1.3) to which the Cole–Hopf transform can be applied.

Let us recall (see also [1] and [7]) that the Cole–Hopf transform associates with a real function $\varphi\colon \mathbf{R}^d \to \mathbf{R}$ the vector field $\mathscr{H}\varphi\colon \mathbf{R}^d \to \mathbf{R}^d$ defined by

$$\mathscr{H}\varphi(\xi) = \left(-\frac{\partial}{\partial \xi_1}\log\varphi(\xi), \ldots, -\frac{\partial}{\partial \xi_d}\log\varphi(\xi)\right).$$

It is evident that the definition of $\mathscr{H}$ requires that φ belongs to the set

$$H = \{\varphi \in H^1_{\mathrm{loc}}(\boldsymbol{R}^d)\colon\ \varphi(\xi) > 0 \text{ for almost every } \xi \in \boldsymbol{R}^d\}.$$

The Cole–Hopf transform was introduced in the deterministic framework to construct a solution to the Burgers equation and has been recently used (see [7]) in the case of the stochastic Burgers equation:

$$\partial_t u(t, \xi) = \tfrac{1}{2}\Delta_\xi u(t, \xi) - \langle u(t, \xi), \nabla_\xi\rangle u(t, \xi) + \nabla\dot{\mathscr{W}}(t, \xi) \tag{4.4}$$

(see [1] for the one-dimensional case). Note that if $X(t)$, $t \geqslant 0$, is a solution to equation (1.3) and one defines $u(t) = \mathscr{H}(X(t))$, $t \geqslant 0$, then differentiating $u(t)$ formally, by the Itô rule, one gets exactly (4.4).

The following theorem gives conditions under which the Cole–Hopf transform can be applied to a stationary solution of (4.1).

THEOREM 4.9. *Assume that* $d > 2$,

$$\frac{\Gamma(d/2-1)}{4\pi^{d/2}} \int_{\boldsymbol{R}^d} \tilde{\Gamma}(\zeta)|\zeta|^{2-d}\, d\zeta < 1$$

with $\tilde{\Gamma}$ *defined as in Theorem* 3.3, *and that*

$$\int_{\boldsymbol{R}^d} (1+|\xi|^{2n})^p \gamma^p(\xi)\, d\xi < \infty$$

with $p > 1$, $(1-1/p)(d/2) < 1$, $n \in \boldsymbol{N}$, $n > d/2$. *Let moreover* ϱ *be an admissible weight such that*

$$\int_{\boldsymbol{R}^d} \frac{\varrho(\xi)}{\hat{\varrho}(\xi)}\, d\xi < +\infty$$

for some admissible weight $\hat{\varrho}$. *Then there exists a measure* $\bar{\mu}$ *on* L^2_ϱ *invariant for equation* (4.1) *and satisfying* $\bar{\mu}(\boldsymbol{H}) = 1$. *In particular, if* $X(t)$ *is a stationary solution to equation* (4.1) *with* $\mathscr{L}(X(t)) = \bar{\mu}$, *then the Cole–Hopf transform is applicable to* $X(t)$.

Proof. Theorems 3.1 and 3.3 imply that there exists a sequence $T_j \nearrow +\infty$ such that the sequence

$$\frac{1}{T_j} \int_1^{T_j+1} \mathscr{L}(X^1(t))\, dt, \qquad j = 1, 2, \ldots,$$

weakly converges to a measure μ on L^2_ϱ invariant for equation (4.1). Theorem 3.3 also yields that the assumptions of Theorem 4.1 are satisfied. Therefore we have $\mu(\{\boldsymbol{0}\}) < 1$. Moreover, by Theorem 4.4 it follows that $\mu(H^n_\varrho) = 1$. Finally, Corollary 4.7 implies that $X^1(t) \in \boldsymbol{B}_\varrho$ $\boldsymbol{P}$-almost surely; therefore $\mu(\bar{\boldsymbol{B}}_\varrho) = 1$ (for the definition of $\boldsymbol{B}_\varrho$ see Subsection 4.3).

Now let

$$\bar{\mu} = \frac{1}{1-\mu(\boldsymbol{0})}\, \mu - \frac{\mu\{\boldsymbol{0}\}}{1-\mu\{\boldsymbol{0}\}}\, \delta_{\{\boldsymbol{0}\}}.$$

Since $\delta_{\boldsymbol{0}}$ is an invariant measure for equation (4.1), $\bar{\mu}$ is an invariant measure as well and

$$\bar{\mu}(\bar{B}_\varrho - \{0\}) = 1 \quad \text{and} \quad \bar{\mu}(H^n_\varrho) = 1.$$

Moreover, by the Sobolev embedding theorem, we know that H^n_ϱ, with $n > d/2$, is included in the set of functions of L^2_ϱ having a continuous modification. So we can conclude that the measure $\bar{\mu}$ satisfies the conditions of Theorem 4.8. Therefore we have $\bar{\mu}(B \cap H^n_\varrho) = 1$ and the claim follows being $H^n_\varrho \subset H^1_{\mathrm{loc}}(\boldsymbol{R}^d)$ and $\boldsymbol{B} \cap H^n_\varrho \subset \boldsymbol{H}$. ∎

Remark 4.10. In reality, what we obtain, under the assumptions of Theorem 4.9, is that $\bar{\mu}(\boldsymbol{B} \cap H^n_\varrho) = 1$, which is a stronger property than what we were looking for. This additional regularity was needed to ensure that the measure μ was concentrated on continuous functions, and thus the applicability of Theorem 4.8.

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