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A CENTRAL LIMIT THEOREM FOR STRICTLY STATIONARY SEQUENCES IN TERMS OF SLOW VARIATION IN THE LIMIT

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Abstract. For strictly stationary random sequences satisfying the "minimal" dependence condition, necessary and sufficient conditions for the weak convergence to the normal law in terms of slow variation in the limit are found.

1. Introduction and results. Let $\{X_k\}_{k\in Z}$ be a strictly stationary sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $S_n = \sum_{k=1}^{n} X_k$ and let $v_n \to +\infty$ be a sequence of positive numbers. Let $\mathscr{N}$ denote a standard normally distributed random variable.

Bernstein in [1] introduced a method for proving limit theorems for dependent variables known as "big blocks by small blocks separation". This method requires the following "dependence" Condition $B(v_n)$ (see [5]):

$$\max_{1\leqslant k+l\leqslant n}\left|E\left(\exp\left\{it\frac{S_{k+l}}{v_n}\right\}\right)-E\left(\exp\left\{it\frac{S_k}{v_n}\right\}\right)\cdot E\left(\exp\left\{it\frac{S_l}{v_n}\right\}\right)\right| \underset{n}{\to} 0 \tag{1.1}$$

for some sequence $v_n \to +\infty$ of nonnegative reals.

Following [2] we shall say that the sequence of measurable nonnegative functions f_n is $(-\gamma)$-*regularly varying in the limit* if there exists a "rate" sequence r_n, $r_n \to +\infty$, such that for any sequence x_n, dominated by the rate sequence (i.e., such that $x_n = o(r_n)$) and $x_n \to +\infty$, we have

$$x_n^{\gamma} f_n(x_n) \to c > 0.$$

In the case where $\gamma = 0$ we say that f_n is *slowly varying in the limit*. If L is slowly varying in the sense of Karamata, then the sequence of functions

$$\left\{f_n(x) = \frac{L(x\cdot n)}{L(n)}\right\}$$

is slowly varying in the limit.

A strictly stationary random sequence $\{X_k\}$ with symmetric partial sums S_n is in the domain of attraction of the symmetric strictly p-stable law, $p \in (0, 2)$, if and only if the sequence of functions

$$\{f_n(x) = x^p P(S_n > xv_n)\}$$

is slowly varying in the limit ([2], Theorem 1). For $p = 2$ the corresponding result is stated in the following theorem:

THEOREM 1. *Let $\{X_k\}$ be a strictly stationary sequence with symmetric sums S_n which satisfies* (1.1) *for some $v_n \to +\infty$. Then CLT for $\{X_k\}$ holds if and only if*

$$\left\{E\left(\frac{S_n^2}{v_n^2} \wedge x\right)\right\}$$

is slowly varying in the limit sequence of functions.

If $\{X_k\}$ is an i.i.d. sequence such that $EX_1 = 0$, $EX_1^2 = 1$, and the Cramér condition

$$E(\exp\{h|X_1|\}) < +\infty$$

holds for some $h > 0$, then the sequence of functions:

$$\left\{\frac{P(S_n > x\sqrt{n})}{P(\mathcal{N} > x)}\right\} \tag{1.2}$$

is slowly varying in the limit with the "rate" $r_n = n^{1/6}$ ([4], XVI, § 7, Theorem 1). On the other hand, Nagaev in [8] (Theorem 1) proved that if $x_n \geqslant \log n$, then for laws such that

$$x^{2+\varepsilon} P(X_1 > x) = L(x),$$

where $\varepsilon > 0$ and $L(x)$ is slowly varying in the sense of Karamata, the following relation holds:

$$P(S_n > x_n\sqrt{n}) \sim nP(X_1 > x_n\sqrt{n}), \qquad n \to +\infty.$$

Hence in the general case (such as the absence of variance) one cannot expect better than a logarithmic rate sequence in (1.2). However, the existence of any rate sequence is equivalent to CLT (for a similar result when $0 < p < 2$ see [6]).

THEOREM 2. *Let $\{X_k\}$ be a strictly stationary random sequence satisfying* (1.1) *for $v_n \to +\infty$. Then*

$$\mathscr{L}(v_n^{-1} S_n) \underset{w}{\to} \mathcal{N}(0, 1),$$

if and only if

$$\left\{\frac{P((-1)^m S_n > \sqrt{x}\, v_n)}{P((-1)^m \mathcal{N} > \sqrt{x})}\right\} \tag{1.3}$$

is slowly varying in the limit for $m = 1, 2$.

2. Proofs. Condition (1.1) depends on a normalizing sequence v_n. Sometimes the information is required whether the sequence such that $\varsigma_n \geqslant v_n$ also satisfies (1.1). It turns out that this is the case where the sequence $\{v_n^{-1} S_n\}$ is stochastically compact [3], i.e., that every subsequence has a further subsequence which converges weakly to a nondegenerate limit.

LEMMA 1. *For a stochastically compact sequence* $\{v_n^{-1} S_n\}$ *the convergence in* (1.1) *is uniform on every* $[0, T]$, $T < +\infty$, *and*

$$\lim_n \max_{1 \leqslant l \leqslant n} \frac{v_l}{v_n} < +\infty. \tag{2.1}$$

Proof of Lemma 1. Let $\{\hat{X}_k\}$ be an independent copy of $\{X_k\}$ and $\hat{S}_n = \sum_1^n \hat{X}_k$. Assume that for some $l_n \leqslant n$ the sequence $\{v_n^{-1}(S_{l_n} - \hat{S}_{l_n})\}$ is not tight. For any symmetric and independent random variables we have

$$P(X+Y > x) \geqslant P(X > x) \cdot P(Y \geqslant 0) = \tfrac{1}{2} P(X > x).$$

Hence

$$\mathscr{L}\left(\frac{S_{l_n} - \hat{S}_{l_n}}{v_n}\right) * \mathscr{L}\left(\frac{S_{n-l_n} - \hat{S}_{n-l_n}}{v_n}\right)$$

is not tight, which together with (1.1) contradicts that $\{v_n^{-1} S_n\}$ is tight. Now by the tightness of $\{v_n^{-1}(S_{l_n} - \hat{S}_{l_n})\}$ we have

$$\lim_n \max_{1 \leqslant l \leqslant n} \frac{v_l}{v_n} < +\infty.$$

Assume that this is not the case. Then there exists a subsequence n' such that

$$\lim_{n'} v_{n'}^{-1} v_{l_{n'}} = +\infty$$

and

$$\mathscr{L}\left(\frac{S_{l_{n'}} - \hat{S}_{l_{n'}}}{v_n}\right) = \mathscr{L}\left(\sqrt{\frac{v_{l_{n'}}}{v_{n'}}} \cdot \frac{S_{l_{n'}}}{v_{l_{n'}}}\right) * \mathscr{L}\left(-\sqrt{\frac{v_{l_{n'}}}{v_{n'}}} \cdot \frac{S_{l_{n'}}}{v_{l_{n'}}}\right),$$

which is not possible since any weak limit $\{v_{l_{n'}}^{-1} S_{l_{n'}}\}$ is nondegerate and the left-hand side is tight.

Now, let us assume that there exists $T > 0$ such that (1.1) does not hold uniformly on $[0, T]$. Hence there exists a subsequence n' such that $t_{n'} \to t_0 \leqslant T$ and

$$\lim_{n'} \left| E\left(\exp\left\{it_{n'} \frac{S_{k_{n'}+l_{n'}}}{v_{n'}}\right\}\right) - E\left(\exp\left\{it_{n'} \frac{S_{k_{n'}}}{v_{n'}}\right\}\right) \cdot E\left(\exp\left\{it_{n'} \frac{S_{l_{n'}}}{v_{n'}}\right\}\right)\right| > 0,$$

while by (2.1) and the tightness there exist random variables Z, Z_1, Z_2 such that

$$\mathscr{L}(v_{n'}^{-1} S_{k_{n'}+l_{n'}}) \underset{w}{\to} \mathscr{L}(Z), \qquad \mathscr{L}(v_{n'}^{-1} S_{k_{n'}}) \underset{w}{\to} \mathscr{L}(Z_1), \qquad \mathscr{L}(v_{n'}^{-1} S_{l_{n'}}) \underset{w}{\to} \mathscr{L}(Z_2).$$

Hence

$$\lim_{n'}\left|E\left(\exp\left\{iut_{n'}\frac{S_{k_{n'}+l_{n'}}}{v_{n'}}\right\}\right)-E\left(\exp\left\{iut_{n'}\frac{S_{k_{n'}}}{v_{n'}}\right\}\right)\cdot E\left(\exp\left\{iut_{n'}\frac{S_{l_{n'}}}{v_{n'}}\right\}\right)\right|$$
$$= |E(\exp\{iut_0 Z\})-E(\exp\{iut_0 Z_1\})E(\exp\{iut_0 Z_2\})|,$$

but, by (1.1), $\mathscr{L}(Z)=\mathscr{L}(Z_1)*\mathscr{L}(Z_2)$. Thus the right-hand side equals 0, which is not possible. This completes the proof. ∎

Remark 1. The Lévy metric satisfies the following inequality ([9], Theorem 1.5.2):

$$d_L(X,Y)\leqslant\frac{1}{\pi}\int_0^T|E(\exp\{itX\})-E(\exp\{itY\})|\frac{dt}{t}+\left(4\sqrt{2}+\frac{1}{80\pi}\right)\frac{\ln(1+T)}{T}.$$

Hence, if $\{v_n^{-1}S_n\}$ is stochastically compact, then condition (1.1) is equivalent to

$$\max_{1\leqslant k+l\leqslant n} d_L\big(\mathscr{L}(S_{k+l}/v_n),\ \mathscr{L}(S_k/v_n)*\mathscr{L}(S_l/v_n)\big)\xrightarrow[n]{}0. \tag{2.2}$$

Proof of Theorem 1. Assume that

$$2\int_0^{\sqrt{x_n}} yP(|S_n|>yv_n)\,dy = E\left(\frac{S_n^2}{v_n^2}\wedge x_n\right)\to 1.$$

Let $y_n=o(x_n)$, $y_n\to\infty$. Then

$$2\int_{\sqrt{y_n}}^{\sqrt{x_n}} yP(|S_n|>yv_n)\,dy=\left\{E\left(\frac{S_n^2}{v_n^2}\wedge x_n\right)-E\left(\frac{S_n^2}{v_n^2}\wedge y_n\right)\right\}\xrightarrow[n]{}0.$$

Hence

$$x_n(1-y_n/x_n)P(|S_n|>\sqrt{x_n}\,v_n)$$
$$=P(|S_n|>\sqrt{x_n}\,v_n)\cdot 2\int_{\sqrt{y_n}}^{\sqrt{x_n}} y\,dy\leqslant 2\int_{\sqrt{y_n}}^{\sqrt{x_n}} yP(|S_n|>yv_n)\,dy\to 0.$$

Thus

$$x_nP(|S_n|>\sqrt{x_n}\,v_n)\to 0.$$

On the other hand,

$$E\left(\frac{S_n^2}{v_n^2}I\left(\left|\frac{S_n}{v_n}\right|\leqslant\sqrt{x_n}\right)\right)=-x_nP(|S_n|>\sqrt{x_n}\,v_n)+2\int_0^{\sqrt{x_n}} yP(|S_n|>yv_n)\,dy.$$

Taking $x_n=o(\sqrt{s_n})$, $x_n\to\infty$ in the above, we get

$$x_n^2P(|S_n|>x_nv_n)\to 0,\qquad E\left(\frac{S_n^2}{v_n^2}I\left(\left|\frac{S_n}{v_n}\right|\leqslant x_n\right)\right)\to 1.$$

Since $\{S_k\}$ are symmetric, Theorem 1 follows by Theorem 1 in [7] or by Theorem 9.5 in [5]. ∎

Proof of Theorem 2. It is enough to show that (1.3) implies CLT. Observe that if for any $K \geqslant 1$ we have

$$\lim_k P(S_{n_k} > K v_{n_k}) = 0,$$

then for any $l \geqslant 1$

$$\frac{P(S_{n_k} > \sqrt{l}\, K v_{n_k})}{P(\mathcal{N} > \sqrt{l}\, K)} \underset{k}{\to} 0$$

holds. Hence, for $\sqrt{x_k} = \sqrt{y_k}\, K$, $y_k = o(s_k' \wedge s_{n_k})$ and $y_k \to +\infty$, we obtain

$$\lim_k \frac{P(S_{n_k} > \sqrt{x_k}\, v_{n_k})}{P(\mathcal{N} > \sqrt{x_k})} = 0,$$

which contradicts (1.3). Thus further we may assume that

$$\liminf_n P(S_n > K v_n) > 0, \qquad \liminf_n P(S_n < -K v_n) > 0 \tag{2.3}$$

hold for any $K \geqslant 1$.

Let us write $Z_n = S_n - \hat{S}_n$, $u_n = s_n v_n^2$ and $\varsigma_n^2 = E(Z_n^2 \wedge u_n)$, where $\hat{S}_n$ is an independent copy of S_n. Now

$$P(Z_n > 2v_n) = P(S_n - \hat{S}_n > 2v_n) \geqslant P(S_n > v_n) \cdot P(S_n < -v_n).$$

Hence, by (2.3),

$$\liminf_n P(Z_n > 2v_n) > 0,$$

and

$$0 < \liminf_n P(Z_n > 2v_n) \leqslant \liminf_n E\left(\frac{Z_n^2}{v_n^2} \wedge s_n\right).$$

Consequently,

$$\liminf_n \frac{\varsigma_n^2}{v_n^2} = C > 0, \qquad \varsigma_n^2 \to +\infty.$$

Since

$$P(Z_n^2 > y u_n) \leqslant 2P(S_n^2 > 4^{-1} y u_n) \underset{n}{\to} 0,$$

so by the Lebesgue theorem we have

$$\frac{\varsigma_n^2}{u_n} = \int_0^1 P(Z_n^2 > y u_n)\, dy \underset{n}{\to} 0.$$

Let $x_n \to +\infty$ be a sequence such that $x_n = o(u_n/\varsigma_n^2)$. Then

$$\frac{E(Z_n^2 \wedge u_n) - E(Z_n^2 \wedge x_n \varsigma_n^2)}{\varsigma_n^2} = \varsigma_n^{-2} \int_{x_n\varsigma_n^2}^{u_n} P(Z_n^2 > y)\,dy$$

$$= \frac{u_n}{\varsigma_n^2} \int_{(x_n\varsigma_n^2)/u_n}^{1} P(\mathcal{N}^2 > ys_n) \frac{P(Z_n^2 > ys_n v_n^2)}{P(\mathcal{N}^2 > ys_n)}\,dy$$

$$\leqslant \frac{u_n}{\varsigma_n^2}\Big(\int_{(x_n\varsigma_n^2)/u_n}^{1} P(\mathcal{N}^2 > ys_n)\,dy\Big)\cdot \sup_{\left(\frac{x_n\varsigma_n^2}{v_n^2 s_n}\right) \leqslant y \leqslant 1} \frac{P(Z_n^2 > ys_n v_n^2)}{P(\mathcal{N}^2 > ys_n)}$$

$$\leqslant \frac{u_n}{s_n\varsigma_n^2}\Big(\int_{(x_n\varsigma_n^2 s_n)/u_n}^{s_n} P(\mathcal{N}^2 > y)\,dy\Big)\cdot \sup_{\left(\frac{x_n\varsigma_n^2}{v_n^2 s_n}\right) \leqslant y \leqslant 1} \frac{2P(S_n^2 > 4^{-1} ys_n v_n^2)}{P(\mathcal{N}^2 > ys_n)}$$

$$\leqslant \frac{v_n^2}{\varsigma_n^2}\left(E(\mathcal{N}^2 \wedge s_n) - E\left(\mathcal{N}^2 \wedge \frac{x_n\varsigma_n^2}{v_n^2}\right)\right)\cdot O(1)$$

$$\leqslant O(1)\cdot\frac{1}{C}\cdot\left(E(\mathcal{N}^2 \wedge s_n) - E\left(\mathcal{N}^2 \wedge \frac{x_n\varsigma_n^2}{v_n^2}\right)\right) \underset{n}{\to} 0.$$

Hence

$$\frac{E(Z_n^2 \wedge x_n\varsigma_n^2)}{\varsigma_n^2} = E\left(\frac{Z_n^2}{\varsigma_n^2} \wedge x_n\right) \underset{n}{\to} 1$$

if $x_n \to \infty$ and $x_n = o(u_n/\varsigma_n^2)$. By (1.3) and (2.3) we observe that $\{v_n^{-1} Z_n\}$ is a stochastically compact sequence. It is easy to see that $\{Z_n\}$ satisfies Condition $B(v_n)$. Hence, by Lemma 1 and by the relation $\limsup_n \varsigma_n^{-2} v_n^2 = C^{-1} < +\infty$ it follows that $\{Z_n\}$ satisfies Condition $B(\varsigma_n)$. Now, by Theorem 1, for random variables $\{Z_n\}$ we have

$$\mathcal{L}\left(\frac{S_n - \hat{S}_n}{\varsigma_n}\right) = \mathcal{L}\left(\frac{Z_n}{\varsigma_n}\right) \underset{w}{\to} \mathcal{N}(0, 1).$$

Now we shall establish that symmetricity can be dropped. Let us write

$$U_n = S_n I(|S_n| \leqslant \sqrt{x_n}\, v_n), \qquad \hat{U}_n = \hat{S}_n I(|\hat{S}_n| \leqslant \sqrt{x_n}\, v_n)$$

for some fixed $x_n \to \infty$, $x_n = o(s_n)$. By (1.3) we have

$$P(|S_n I(|S_n| > \sqrt{x_n}\, v_n)| > \varepsilon\varsigma_n) \leqslant P(|S_n| > \sqrt{x_n}\, v_n) \underset{n}{\to} 0,$$

and hence

$$\mathcal{L}\left(\frac{(U_n - EU_n) - (\hat{U}_n - E\hat{U}_n)}{\varsigma_n}\right) \sim \mathcal{L}\left(\frac{S_n - \hat{S}_n}{\varsigma_n}\right). \tag{2.4}$$

We shall prove that

$$\limsup_n E\left(\frac{U_n-EU_n}{\varsigma_n}\right)^2 < +\infty.$$

For this, note that

$$\begin{aligned}
E(U_n-\hat{U}_n)^2 &= 2\int_0^{2\sqrt{x_n}v_n} yP(|U_n-\hat{U}_n|>y)\,dy\\
&= 4\int_0^{2\sqrt{x_n}v_n} yP(U_n-\hat{U}_n>y)\,dy \leqslant 4\int_0^{2\sqrt{x_n}v_n} yP(S_n-\hat{S}_n>2^{-1}y)\,dy\\
&\quad +4\int_0^{2\sqrt{x_n}v_n} y\big(P\big(S_n I(|S_n|>\sqrt{x_n}\,v_n)-\hat{S}_n I(|\hat{S}_n|>\sqrt{x_n}\,v_n)>2^{-1}y\big)\big)\,dy\\
&\leqslant 8\int_0^{\sqrt{x_n}v_n} yP(S_n-\hat{S}_n>y)\,dy\\
&\quad +8\int_0^{2\sqrt{x_n}v_n} yP\big(|S_n|\,I(|S_n|>\sqrt{x_n}\,v_n)>4^{-1}y\big)\,dy\\
&\leqslant 4E(Z_n^2\wedge x_n v_n^2)+8P(|S_n|>\sqrt{x_n}\,v_n)\cdot\int_0^{2\sqrt{x_n}v_n} y\,dy\\
&\leqslant 4E\left(Z_n^2\wedge\left(\frac{x_n v_n^2}{\varsigma_n^2}\right)\varsigma_n^2\right)+16x_n v_n^2\,P(|S_n|>\sqrt{x_n}\,v_n).
\end{aligned}$$

Since $\limsup_n \varsigma_n^{-2} v_n^2 = C^{-1} < +\infty$, so by the relations

$$\limsup_n \varsigma_n^{-2} E(U_n-\hat{U}_n)^2 \leqslant 4+\lim_n \varsigma_n^{-2} v_n^2\cdot 16x_n P(|S_n|>\sqrt{x_n}\,v_n) = 4,$$

we have

$$\begin{aligned}
\limsup_n \varsigma_n^{-2} E\big(U_n-E(U_n)\big)^2 &= 2^{-1}\limsup_n \varsigma^{-2} E\big((U_n-EU_n)-(\hat{U}_n-E\hat{U}_n)\big)^2\\
&= 2^{-1}\limsup_n \varsigma_n^{-2} E(U_n-\hat{U}_n)^2 \leqslant 2^{-1}\cdot 4 = 2.
\end{aligned}$$

Now, by (2.4) and the Cramér theorem, we know that any weak limit of $\{\varsigma_n^{-1}(U_n-EU_n)\}_n$ is of the form $\mathcal{N}(a, 2^{-1})$. The sequence $\{\varsigma_n^{-2}E(U_n-EU_n)^2\}_n$ is bounded, so the only possibility is $a=0$. On the other hand, $\{\varsigma_n(U_n-EU_n)\}$ is a tight sequence, and hence

$$\mathcal{L}\left(\frac{U_n-EU_n}{\varsigma_n}\right)\underset{w}{\to}\mathcal{N}\left(0,\frac{1}{2}\right).$$

By (1.3) we obtain

$$P\big(\big|S_n I(|S_n|>\sqrt{x_n}\,v_n)\big|>\varepsilon\varsigma_n\big)\underset{n}{\to}0,$$

whence

$$\mathscr{L}\left(\frac{S_n - ES_n I(|S_n| \leqslant \sqrt{x_n}\, v_n)}{\sqrt{2}\,\varsigma_n}\right) \underset{n}{\to} \mathscr{N}(0, 1).$$

Taking $A_n = ES_n I(|S_n| < \sqrt{x_n}\, v_n)$ and $B_n = \sqrt{2}\,\varsigma_n$ in Theorem 10.3 in [5], we see that the limit $\lim_n n^{-1} A_n = A$ exists and

$$\mathscr{L}\left(\frac{S_n - nA}{B_n}\right) \underset{w}{\to} \mathscr{N}(0, 1)$$

holds. Also for the sequence $\{A_n\}$ we have

$$\left|\frac{A_n}{n}\right| \leqslant \frac{E|S_n| I(|S_n| \leqslant \sqrt{x_n}\, v_n)}{n} \leqslant \frac{\sqrt{x_n}\, v_n}{n} \leqslant \frac{\sqrt{x_n}(1/\sqrt{C}+\varepsilon)\varsigma_n}{n},$$

and hence $|n^{-1} A_n| \to 0$ by the slow variation of the sequence $n^{-1}\varsigma_n^2$ ([5], Theorem 3.1). Consequently, we get

$$\mathscr{L}\left(\frac{S_n}{B_n}\right) = \mathscr{L}\left(\frac{S_n}{\sqrt{2}\,\varsigma_n}\right) \underset{w}{\to} \mathscr{N}(0, 1).$$

The proof will be completed if we show that $B_n^{-1} v_n \to 1$. We know that

$$\frac{x_n B_n^2}{C v_n^2} = \frac{x_n \sqrt{2}\,\varsigma_n^2}{C v_n^2} \to +\infty \qquad \text{for } x_n = o(\varsigma_n^{-2} u_n).$$

Hence by (1.3) we obtain

$$\frac{P\left(S_n^2 > ((x_n B_n^2)/(C v_n^2))\, v_n^2\right)}{P\left(\mathscr{N}^2 > (x_n B_n^2)/(C v_n^2)\right)} \to 1.$$

Now, since $C^{-1} x_n = o(\varsigma_n^{-2} u_n)$, by what has been proved we have

$$\frac{P(S_n^2 > C^{-1} x_n B_n^2)}{P(\mathscr{N}^2 > C^{-1} x_n)} \to 1$$

and, consequently,

$$\frac{P\left(\mathscr{N}^2 > (x_n B_n^2)/(C v_n^2)\right)}{P(\mathscr{N}^2 > x_n/C)} \to 1.$$

Observe that

$$P(\mathscr{N}^2 > x) \sim \frac{1}{\sqrt{2\pi}} x^{-1/2} \exp\left\{-\frac{x}{2}\right\},$$

whence

$$\frac{1}{\sqrt{(x_n B_n^2)/(C v_n^2)}} \exp\left\{-\frac{1}{2}\cdot\frac{x_n B_n^2}{C v_n^2}\right\} \sqrt{\frac{x_n}{C}} \exp\left\{\frac{x_n}{2C}\right\} \sim 1.$$

Thus

$$\frac{v_n}{B_n}\exp\left\{\frac{x_n}{2C}\left(1-\frac{B_n^2}{v_n^2}\right)\right\}\to 1, \tag{2.5}$$

$$\frac{B_n}{v_n}\exp\left\{\frac{x_n}{2C}\left(\frac{B_n^2}{v_n^2}-1\right)\right\}\to 1. \tag{2.6}$$

Since

$$\liminf_n \frac{B_n^2}{v_n^2} = \liminf_n \frac{(\sqrt{2}\,\varsigma_n)^2}{v_n^2} = 2C > 0,$$

by (2.6) we obtain

$$\limsup_n \frac{B_n^2}{v_n^2} \leqslant 1 \quad \text{and} \quad \liminf_n \frac{v_n^2}{B_n^2} \geqslant 1.$$

By (2.5) we have

$$\liminf_n \frac{B_n^2}{v_n^2} \geqslant 1$$

(if this is not true, then we have along the subsequence n_k:

$$\frac{v_{n_k}}{B_{n_k}}\exp\left\{\frac{x_{n_k}}{2C}\left(1-\frac{B_{n_k}^2}{v_{n_k}^2}\right)\right\}\to 0,$$

which contradicts (2.5)). Finally, $B_n \sim v_n$, and hence

$$\mathscr{L}\left(\frac{S_n}{v_n}\right) \underset{w}{\to} \mathscr{N}(0, 1).$$

This completes the proof of Theorem 2. ∎

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