PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 19, Fasc. 1 (1999), pp. 23–41

LIMIT THEOREMS FOR THE HIERARCHY OF FREENESS

BY

UWE FRANZ (STRASBOURG) AND ROMUALD LENCZEWSKI* (WROCŁAW)

Abstract. The central limit theorem, the invariance principle and the Poisson limit theorem for the hierarchy of freeness are studied. We show that for given $m \in N$ the limit laws can be expressed in terms of non-crossing partitions of depth smaller than or equal to m. For $\mathscr{A} = C[x]$, we solve the associated moment problems and find explicitly the discrete limit measures.

1. Introduction. The notion of the *hierarchy of freeness* was introduced in [3] in the context of a unification of the main types of non-commutative independence (tensor, free, and Boolean, see the axiomatic approach in [7] and [8]). The main idea of the construction presented in [3] was to approximate the free product of states [11] through a sequence of products called *m-free products*, $m \in N$, using only tensor independence. In this way one obtains a hierarchy of products as well as a hierarchy of non-commutative probability spaces, the latter of which was called in [3] the *hierarchy of freeness.*

In the hierarchy of m-free products the two extremes are given by the Boolean product which corresponds to the first order approximation for $m = 1$ and the free product, obtained for $m = \infty$. Thus the hierarchy fills the "gap" between the Boolean product and the free product. Its another important feature is that it equips the combinatorics of non-crossing partitions with a hierarchic structure induced by their depths. Recall that the combinatorics of the Boolean product is based on the so-called *interval partitions,* and that of the free product — on all *non-crossing partitions.* By studying convolution-type limit theorems in this paper, we establish a connection between the combinatorics of the m-free product (or, rather, of the m-free convolution) and non-crossing partitions of depth $d(P) \leqslant m$. Thus the hierarchy also fills the "gap" between the combinatorics of interval partitions and that of all non-crossing partitions. Let us add that the hierarchy of freeness lends itself easily to certain generalizations, and in fact was introduced in [3] in the context of the conditionally free product of states (see [1]). Other generalizations were indicated in [2].

* Hugo Steinhaus Center for Stochastic Methods, Institute of Mathematics, Technical University of Wrocław. This work was supported by KBN (2PO3A 05415).

In this work we study the convolution-type central limit theorems, the invariance principles and Poisson's limit theorems for m-free products, calling those theorems *m-free limit theorems*. Let us only note that we do not use the m-free convolutions in our notation. Nevertheless, all theorems can be phrased using m-free convolutions introduced in [3]. It is well known that in the central limit theorem for free independence [11] only non-crossing pair partitions give rise to the limit Wigner semicircle law [9]. In our case we show that in the m-free central limit theorem only non-crossing pair partitions of depth less than or equal to m appear in the combinatorial form of the limit law for each $m \in \mathbf{N}$. For the special case of the algebra of polynomials in one variable $\mathbf{C}[x]$, we introduce a hierarchy of Cauchy transforms of the limit laws, which enables us to recover the corresponding hierarchy of discrete measures on the real line which approximate the Wigner measure. A similar approach is used for m-free Poisson's limit theorems.

Section 2 is of preliminary character and contains all needed facts on the hierarchy of freeness. In Section 3 we prove the central limit theorem for the hierarchy of freeness (Theorem 3.5). Note that our approach is based on the tensor product construction developed in [3] and as such gives a new (and probably the most explicit) proof of the free central limit theorem. In Section 4, the corresponding invariance principle is stated (Theorem 4.1) and a hierarchy of m-free Brownian motions is introduced. In Section 5, we restrict ourselves to $\mathbf{C}[x]$ and study the hierarchy of measures corresponding to the central limit laws. We show that they are discrete measures that approximate weakly the Wigner measure. Poisson's limit theorem for the hierarchy of freeness is proved in Section 6 and the associated moment problems are solved.

2. The hierarchy of freeness. This section is of preliminary character and contains all needed facts on the hierarchy of freeness. For more details, see [3] and [2].

Let $(\mathscr{A}_l)_{l\in I}$ be a family of unital *-algebras and let $(\phi_l)_{l\in I}$ be the corresponding family of states. We assume that $\mathscr{A}_l = \mathscr{A}_l^0 \oplus \mathbf{1}_l$, where $\mathscr{A}_l^0$ is a *-subalgebra of $\mathscr{A}_l$, and in the free product $*_{l\in I} A_l$ we identify units. Extend each $\mathscr{A}_l$ to $\tilde{\mathscr{A}}_l = \mathscr{A}_l * \mathbf{C}(t_l)$, where $\mathbf{C}(t_l)$ is the unital *-algebra generated by the projection t_l. Make $\tilde{\mathscr{A}}_l$ into a *-algebra in the canonical fashion. Finally, denote by $(\tilde{\phi}_l)_{l\in I}$ the Boolean extensions of $(\tilde{\phi}_l)_{l\in I}$, i.e. states on $(\tilde{\mathscr{A}}_l)_{l\in I}$ given by $\tilde{\phi}_l(\mathbf{1}_l) = 1$ and

$$\tilde{\phi}_l(t_l^r a^{(1)} t_l \ldots t_l a^{(p)} t_l^s) = \phi_l(a^{(1)}) \ldots \phi_l(a^{(p)})$$

for $a^{(1)}, \ldots, a^{(p)} \in \mathscr{A}_l^0$, $r, s \in \{0, 1\}$. For details, see [2].

Consider the quantum probability space $(\mathscr{B}, \tilde{\Phi})$, where

$$\mathscr{B} = \bigotimes_{l\in I} \tilde{\mathscr{A}}_l^{\otimes\infty}, \qquad \tilde{\Phi} = \bigotimes_{l\in I} \tilde{\phi}_l^{\otimes\infty},$$

and the tensor products are understood as in [2] with canonical involutions on $\bigotimes_{l\in I} \mathscr{A}_l$ and $\mathscr{B}$. This is the quantum probability space in which one can embed the hierarchy of freeness defined in [3] (see again [2]). Since we have two tensor products here (over I, and then over N for each $l \in I$), we will label tensor sites by (l, k), $l \in I$, $k \in N$, and we will refer to l and k as the outer and inner site, respectively.

In the definition of these embeddings the following notation will be used. For $l \in I$, $n \in N$, let

$$i_n^{(l)}: \tilde{\mathscr{A}}_l \to \tilde{\mathscr{A}}_l^{\otimes \infty}$$

be the linear mapping given by

$$i_n^{(l)}(a) = \mathbf{1}_l^{\otimes (n-1)} \otimes a \otimes \mathbf{1}_l^{\otimes \infty} \quad \text{for } a \in \tilde{\mathscr{A}}_l.$$

For the notational convenience we put $i_0^{(l)}(a) = 0$. Further, we denote by

$$t_{[k}^{(l)} = \mathbf{1}_l^{\otimes (k-1)} \otimes t_l^{\otimes \infty}$$

a projection in $\tilde{\mathscr{A}}_l^{\otimes \infty}$ which is built from projections t_l at all sites $\geqslant k$, $k \geqslant 1$, and we put $t_{[0}^{(l)} = 0$ for convenience.

We define the linear mappings

$$\gamma_k^{(l)}: \mathscr{A}_l^0 \to \mathscr{B}, \quad \gamma_k^{(l)}(a) = i_k^{(l)}(a) \otimes \bigotimes_{r \neq l} t_{[k}^{(r)},$$

$$\hat{\gamma}_k^{(l)}: \mathscr{A}_l^0 \to \mathscr{B}, \quad \hat{\gamma}_k^{(l)}(a) = i_k^{(l)}(a) \otimes \bigotimes_{r \neq l} t_{[k-1}^{(r)},$$

where $k \in N$, $l \in I$. Note that since $i_0^{(l)}(t) = 0$, we have $\hat{\gamma}_1^{(l)}(a) = 0$. In other words, $\gamma_k^{(l)}(a)$ puts $a \in \mathscr{A}_l^0$ at site (l, k) and projections t_r at sites (r, s) for all $r \neq l$ and $s \geqslant k$. In turn, $\hat{\gamma}_k^{(l)}(a)$ puts a at site (l, k) and projections t_r at sites (r, s) for all $r \neq l$ and $s \geqslant k-1$.

It was shown in [2] that the mappings

$$j_l^{(m)}: \mathscr{A}_l^0 \to \mathscr{B}, \quad j_l^{(m)} = \sum_{k=1}^{m} j_{l,k} \equiv \sum_{k=1}^{m} (\gamma_k^{(l)} - \hat{\gamma}_k^{(l)}),$$

where $l \in I$, $m \in N$, are *-homomorphisms. Using them, we can define for each $m \in N$ the *-homomorphism

$$j^{(m)}: \mathop{*}_{l \in I} \mathscr{A}_l \to \mathscr{B}$$

as the linear extension of $j^{(m)}(\mathbf{1}) = \bigotimes_{l \in I} \mathbf{1}_l^{\otimes \infty}$ and

$$j^{(m)}(a_1 \ldots a_n) = j_{l_1}^{(m)}(a_1) \ldots j_{l_n}^{(m)}(a_n),$$

where $a_i \in \mathscr{A}_{l_i}^0$, $l_i \in I$, $i = 1, \ldots, n$.

DEFINITION 2.1. The sequence of quantum probability spaces $(\mathscr{A}^{(m)}, \Phi^{(m)})_{m \in N}$, where $\mathscr{A}^{(m)} = j^{(m)}(*_{l \in I} \mathscr{A}_l)$ and $\Phi^{(m)}$ is the restriction of $\tilde{\Phi}$ to $\mathscr{A}^{(m)}$, is called the *hierarchy of freeness*. The state $\tilde{\Phi}^{(m)}$ is called the *m-free product state* and $j^{(m)}(a)$, $a \in \mathscr{A}_l^0$, are called the *m-free random variables*.

Remark. Note that $\tilde{\Phi} \circ j^{(m)}$ defines a state on $*_{l \in I} \mathscr{A}_l$.

The GNS representation of the hierarchy of freeness [2] will also be useful here. Thus, let $(\mathscr{H}_l, \pi_l, \Omega_l)$ be the GNS triple associated with the pair $(\mathscr{A}_l, \phi_l)$, i.e. $\mathscr{H}_l$ is a pre-Hilbert space, π_l is a *-representation of $\mathscr{A}_l$, and Ω_l is a cyclic vector such that $\phi_l(x) = \langle \Omega_l, \pi_l(x)\Omega_l\rangle$ for any $x \in \mathscr{A}_l$. We start with the infinite tensor product pre-Hilbert space

$$\mathscr{H}^{\otimes} = \bigotimes_{l\in I} \mathscr{H}_l^{\otimes\infty}$$

with respect to the vector $\Omega = \bigotimes_{l\in I}\Omega_l^{\otimes\infty}$ and denote by

$$\Gamma_k^{(l)}: \mathscr{A}_l^0 \to \mathscr{L}(\mathscr{H}^{\otimes}), \qquad \hat{\Gamma}_k^{(l)}: \mathscr{A}_l^0 \to \mathscr{L}(\mathscr{H}^{\otimes})$$

the *-homomorphisms corresponding to $\gamma_k^{(l)}$, $\hat{\gamma}_k^{(l)}$, i.e.

$$\Gamma_k^{(l)}(a) = i_k^{(l)}(\pi_l(a)) \otimes \bigotimes_{j\neq l} P_{[k}^{(j)}, \qquad \hat{\Gamma}_k^{(l)}(a) = i_k^{(l)}(\pi_l(a)) \otimes \bigotimes_{j\neq l} P_{[k-1}^{(j)}$$

for $a \in \mathscr{A}_l^0$, where $P_{[k}^{(j)} = \mathrm{Id}^{\otimes(k-1)} \otimes (P^{(j)})^{\otimes\infty}$, $P^{(j)}$ is the projection onto the vacuum Ω_j in $\mathscr{H}_j$, and $P_{[0} = 0$. Then the GNS representation $\pi^{\otimes m}$ of $(*_{l\in I}\mathscr{A}_l, \Phi\circ j^{(m)})$ is given by

$$\pi^{\otimes m}(\mathbb{1}) = \bigotimes_{l\in I} \mathrm{Id}_l^{\otimes\infty} \quad \text{and} \quad \pi^{\otimes m} = \mathop{*}_{l\in I} \pi_l^{\otimes m} \text{ on } \mathop{*}_{l\in I} \mathscr{A}_l^0,$$

where

$$\pi_l^{\otimes m}(a) = \sum_{k=1}^{m} \left(\Gamma_k^{(l)}(a) - \hat{\Gamma}_k^{(l)}(a)\right) \quad \text{for } a \in \mathscr{A}_l^0.$$

For each $m \in N$ the cyclic vector is Ω and the carrier space of $\pi^{\otimes m}$ is

$$\mathscr{H}^{\otimes m} = \pi^{\otimes m}\big(\mathop{*}_{l\in I} \mathscr{A}_l\big)\Omega.$$

We need to take a closer look at the correlations

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \dots j_{l_n}^{(m)}(a_n)\big) = \sum_{1\leqslant m_1,\dots,m_n\leqslant m} \tilde{\Phi}\big(j_{l_1,m_1}(a_1) \dots j_{l_n,m_n}(a_n)\big)$$

for any tuple $(l_1, \dots, l_n)$, $a_i \in \mathscr{A}_{l_i}^0$, $i = 1, \dots, n$. Equivalently, we can write

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \dots j_{l_n}^{(m)}(a_n)\big) = \tilde{\Phi}\circ j^{(m)}(a_1, \dots, a_n).$$

Before we derive some results which are specific to the central limit theorem and use the assumption on the zero mean, we prove a "pyramid formula" (slightly more general than the one in [3]), which always allows us to reduce the summation in the above sum to a "pyramid". We also give a new proof, using the GNS representation.

PROPOSITION 2.2. *The following formula holds*:

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \dots j_{l_n}^{(m)}(a_n)\big) = \sum_{(m_1,\dots,m_n)\in\Upsilon_n^m} \tilde{\Phi}\big(j_{l_1,m_1}(a_1) \dots j_{l_n,m_n}(a_n)\big),$$

where $\Upsilon_n^m = \{(p_1, \dots, p_n) \mid 1 \leqslant p_k, p_{n-k} \leqslant k \wedge m,\ 1 \leqslant k \leqslant n/2\}$ *and* $k \wedge m = \min\{k, m\}$.

Proof. Using the GNS representation, we obtain

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1)\dots j_{l_n}^{(m)}(a_n)\big) = \langle \Omega, \pi^{\otimes m}(a_1), \dots, \pi^{\otimes m}(a_n)\,\Omega\rangle,$$

and thus, in order to prove the proposition, it is enough to show that if $(m_1, \dots, m_n) \notin \Upsilon_n^m$, then

$$\langle \Omega, \big(\Gamma_{m_1}^{(l_1)}(a_1) - \hat{\Gamma}_{m_1}^{(l_1)}(a_1)\big) \dots \big(\Gamma_{m_n}^{(l_n)}(a_n) - \hat{\Gamma}_{m_n}^{(l_n)}(a_n)\big)\,\Omega\rangle = 0. \tag{1}$$

Introduce the filtration

$$\mathscr{H}_{0]}^{\otimes} \subset \mathscr{H}_{1]}^{\otimes} \subset \dots \subset \mathscr{H}_{k]}^{\otimes} \subset \dots$$

of subspaces of $\mathscr{H}^{\otimes}$ given by $\mathscr{H}_{0]} = \mathbf{C}\Omega$ and

$$\mathscr{H}_{k]}^{\otimes} = \mathrm{Lin}\,\{\bigotimes_{l\in I} (x_{l,1} \otimes \dots \otimes x_{l,k} \otimes \Omega_l^{\otimes\infty})\}.$$

Note that if $k > 1$, then $\Gamma_k^{(l)}(a)$ agrees with $\hat{\Gamma}_k^{(l)}(a)$ on $\mathscr{H}_{k-2]}$. Moreover,

$$\big(\Gamma_k^{(l)}(a) - \hat{\Gamma}_k^{(l)}(a)\big)\,\mathscr{H}_{k-1]}^{\otimes} \subset \mathscr{H}_{k]}^{\otimes}$$

for any $k \geqslant 1$. These two facts imply that

$$\big(\Gamma_{m_1}^{(l_1)}(a_1) - \hat{\Gamma}_{m_1}^{(l_1)}(a_1)\big) \dots \big(\Gamma_{m_n}^{(l_n)}(a_n) - \hat{\Gamma}_{m_n}^{(l_n)}(a_n)\big)\,\Omega = 0$$

if $(m_1, \dots, m_n) \notin \Theta_n^m$, where

$$\Theta_n^m = \{(p_1, \dots, p_n) \mid 1 \leqslant p_i \leqslant (n-i+1) \wedge m\}.$$

We can repeat this argument for the adjoints and obtain a mirror reflection of this condition ($(m_n, \dots, m_1) \notin \Theta_n^m$), which finally leads to (1) if $(m_1, \dots, m_n) \notin \Upsilon_n^m$. ∎

PROPOSITION 2.3. *If $\mathscr{A}_l = \mathscr{A}$, $\phi_l = \phi$, $l \in I$, then the correlations of m-free random variables are invariant under permutations π of N, i.e.*

$$\tilde{\Phi}\big(j_{\pi(l_1)}^{(m)}(a_1)\dots j_{\pi(l_n)}^{(m)}(a_n)\big) = \tilde{\Phi}\big(j_{l_1}^{(m)}(a_1)\dots j_{l_n}^{(m)}(a_n)\big).$$

Moreover, if $\{l_1, \dots, l_r\} \cap \{l_{r+1}, \dots, l_n\} = \emptyset$, then

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1)\dots j_{l_n}^{(m)}(a_n)\big) = \tilde{\Phi}\big(j_{l_1}^{(m)}(a_1)\dots j_{l_r}^{(m)}(a_r)\big)\,\tilde{\Phi}\big(j_{l_{r+1}}^{(m)}(a_{r+1})\dots j_{l_n}^{(m)}(a_n)\big).$$

Proof. From the properties of the tensor product and the fact that $\phi_l = \phi$ for all $l \in I$, we obtain

$$\tilde{\Phi}\big(\gamma_{k_1}^{\sigma(\pi(l_1))}(a_1)\dots \gamma_{k_n}^{\sigma(\pi(l_n))}(a_n)\big) = \tilde{\Phi}\big(\gamma_{k_1}^{\sigma(l_1)}(a_1)\dots \gamma_{k_n}^{\sigma(l_n)}(a_n)\big)$$

for any $1 \leqslant k_1, \dots, k_n \leqslant m$, where $\gamma_k^{\sigma(l)}(a) = \gamma_k^{(l)}(a), \hat{\gamma}_k^{(l)}(a)$. From this we get the first part of the proposition. The second part is obvious. ∎

3. A central limit theorem. In this section we prove the central limit theorem for the sums of m-free independent random variables. We show that in the limit only the non-crossing pair partitions P of depth $d(P) \leqslant m$ give a non-vanishing contribution.

DEFINITION 3.1. A pair partition $P = \{P_1, \ldots, P_k\}$, where $P_j = \{\alpha(j), \beta(j)\}$, $j = 1, \ldots, k$, of the set $\{1, \ldots, 2k\}$ is *crossing* if there exist $1 \leqslant p, q \leqslant k$ such that $\alpha(p) < \alpha(q) < \beta(p) < \beta(q)$. If P is not a crossing partition, then it is called *non-crossing*. If P is non-crossing, then by $d(P)$ we denote its *depth*, i.e. the maximum of all integers d for which there exist $1 \leqslant s_1, \ldots, s_d \leqslant k$ such that $\alpha(s_1) < \ldots < \alpha(s_d)$ and $\beta(s_1) > \ldots > \beta(s_d)$. We will denote the set of all non-crossing pair partitions P of depth $d(P) \leqslant m$ of the set $\{1, \ldots, n\}$ by $NC_n^{\mathrm{pair}}(m)$.

Remark. If we link each $\alpha(l)$ with $\beta(l)$ in a pair-partition P by "bridges", then a pair-partition is non-crossing if and only if it is possible to draw these bridges without intersections. The depth $d(P)$ of P is then the maximal number of bridges that pass over the same "gap".

Note that with each tuple $(l_1, \ldots, l_n)$, $l_1, \ldots, l_n \in I$, we can associate a partition P of $\{1, \ldots, n\}$. This can be done as follows. Let $K = \{k_1, \ldots, k_r\} = \{l_1, \ldots, l_n\}$ with $k_1 < k_2 < \ldots < k_r$ and put

$$P_i = \{p \mid k_p = i\}.$$

Then we will say that the partition P is *associated with the tuple* $(l_1, \ldots, l_n)$.

LEMMA 3.2. *Assume that the partition P associated with the tuple $(l_1, \ldots, l_n)$, where $n = 2k$, is a non-crossing pair-partition of depth $d(P) > m$. If $\phi(a_i) = 0$ for $i = 1, \ldots, n$, then*

$$\tilde{\Phi}\left(j_{l_1}^{(m)}(a_1) \ldots j_{l_n}^{(m)}(a_n)\right) = 0.$$

Proof. First of all note that each site can be occupied by at most two elements since P is a pair-partition. Assume that $d(P) > m$. Each $j_{l_r}^{(m)}(a_r)$, $1 \leqslant r \leqslant n$, is a sum of m terms in which a_r appears at m different sites, namely (l_r, u), $1 \leqslant u \leqslant m$. Since P is a pair-partition, and thus a given a_r has only one "partner", say a_s at site (l_s, w) with $l_s = l_r = l$, the only way to avoid "singletons" (first-order moments) is for each pair to occupy the same inner site, i.e. $u = w$. Now, we have at least $d(P)$ pairs to occupy at most m different inner sites. Since $d(P) > m$, at least one inner site, say u, must be occupied by two pairs, say (a_r, a_s) and (a_p, a_q), $l_r = l_s = l$, $l_p = l_q = l'$. Now, since P is non-crossing, we must have $r < p < q < s$ or $p < r < s < q$. In the first case, at site (l, u) we obtain

$$\ldots a_r t \ldots t a_s \ldots$$

since $j_{l',u}(a_p)$ and $j_{l',u}(a_q)$ put a projection t at all sites (b, c), $b \neq l'$ and $c \geqslant u$. Thus a_r and a_s are separated by t which produces first moments, therefore gives zero by our zero mean assumption. The second case is analogous. ∎

LEMMA 3.3. *Assume that the partition P associated with the tuple $(l_1, \ldots, l_n)$, where $n = 2k$, is a non-crossing pair-partition of depth $d(P) \leqslant m$. If*

$\phi(a_i) = 0$ *for* $i = 1, \ldots, n$, *then*

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_n}^{(m)}(a_n)\big) = \prod_{i=1}^{k} \phi(a_{P_i}),$$

where $a_J = \prod_{l \in J} a_l$ *for any* $J \subset \{1, \ldots, n\}$, *with the product taken in the natural order.*

Proof. The proof will proceed by induction. Clearly, the case $m = 1$ boils down to considering interval pair-partitions (only they can be of depth $d(P) \leqslant 1$), i.e. take

$$P = \{\{i_1, i_2\}, \ldots, \{i_{2k-1}, i_{2k}\}\}.$$

Then

$$\tilde{\Phi}\big(j_{l_1}^{(1)}(a_1) \ldots j_{l_{2k}}^{(1)}(a_{2k})\big) = \phi(a_1 a_2) \ldots \phi(a_{2k-1} a_{2k}).$$

Assume now that

$$\tilde{\Phi}\big(j_{l_1}^{(m-1)}(a_1) \ldots j_{l_n}^{(m-1)}(a_{2k})\big) = \prod_{i=1}^{k} \phi(a_{P_i})$$

for $d(P) \leqslant m-1$ and any k. We will show that the same property holds for $j^{(m)}$ and non-crossing partitions of depth $d(P) \leqslant m$.

The proof of that fact will be carried out by induction with respect to k. If $k = 1$, then we clearly have

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) j_{l_2}^{(m)}(a_2)\big) = \phi(a_1 a_2).$$

Assume that

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_{2k-2}}^{(m)}(a_{2k-2})\big) = \prod_{i=1}^{k-1} \phi(a_{S_i})$$

for any tuple $(l_1, \ldots, l_{2k-2})$, where S is the partition associated with it and $d(S) \leqslant m$. Now, when considering $\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_{2k}}^{(m)}(a_{2k})\big)$, it is enough to consider the case when $l_1 = l_{2k}$ since otherwise P would separate into subpartitions and the correlation would factorize by Proposition 2.3, thus we could apply the inductive assumption with respect to k. By Proposition 2.2,

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_{2k}}^{(m)}(a_{2k})\big) = \sum_{(m_1, \ldots, m_{2k}) \in \Upsilon_{2k}^{m}} \tilde{\Phi}\big(j_{l_1, m_1}(a_1) \ldots j_{l_{2k}, m_{2k}}(a_{2k})\big).$$

Keeping in mind that $j_{l_i, m_i}(a_i) = \gamma_{m_i}^{(l_i)}(a) - \hat{\gamma}_{m_i}^{(l_i)}(a)$, $1 \leqslant i \leqslant 2k$, we can see that the only way to avoid a separation of a_1 from a_{2k} (which would produce two singletons and thus give zero contribution) is to take into account in the above sum only those tuples $(m_1, \ldots, m_{2k}) \in \Upsilon_{2k}^{m}$ for which $m_2, \ldots, m_{2k-1} \neq 1$ (i.e., in particular, $m_2 = m_{2k-1} = 2$), and, moreover, assume that the products start with $\gamma_2^{(l_2)}(a_2)$ and end with $\gamma_2^{(l_{2k-1})}(a_{2k-1})$. Then, at site $(l_1, 1)$ we get $a_1 a_{2k}$, and at $(l_p, 1)$, $p \in \{2, \ldots, k\}$, we get either the projection t or the unit $\mathbb{1}$, and $\tilde{\phi}$ sends them to 1. Therefore, we obtain

$$\begin{aligned}\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_{2k}}^{(m)}(a_{2k})\big) &= \phi(a_1 a_{2k})\, \tilde{\Phi}\big(j^{(m-1)}(a_2) \ldots j^{(m-1)}(a_{2k-1})\big) \\ &= \phi(a_1 a_{2k}) \prod_{i=1}^{k-1} \phi(a_{P_i'}) = \prod_{i=1}^{k} \phi(a_{P_i})\end{aligned}$$

by the inductive assumption with respect to m, where

$$P = \{P_1, \ldots, P_k\}, \quad P' = \{P_2, \ldots, P_k\} \quad \text{and} \quad P_1 = \{1, 2k\}. \blacksquare$$

LEMMA 3.4. *Assume that the partition P associated with the tuple $(l_1, \ldots, l_n)$, where $n = 2k$, is a crossing pair-partition. If $\phi(a_i) = 0$ for $i = 1, \ldots, n$, then*

$$\tilde{\Phi}\big(j_{l_1}^{(m)}(a_1) \ldots j_{l_{2k}}^{(m)}(a_n)\big) = 0.$$

Proof. We will show that the correlation which corresponds to a crossing pair-partition P of $\{1, \ldots, 2k\}$ produces a singleton, and thus vanishes by the mean zero assumption.

There exist $1 \leqslant p < q < r < s \leqslant 2k$ such that $l_p = l_q = l$, $l_r = l_s = l'$. It is enough to consider those terms from the "pyramid" in which $m_p = m_q = u$ and $m_r = m_s = w$ since otherwise we obtain at least one singleton which makes the contribution vanish. Suppose now that $u \leqslant w$. Then $j_{l,u}(a_p)$ and $j_{l,u}(a_q)$ put a projection t at site (l', w) since they put a t at all sites (b, c), where $b \neq l$ and $c \geqslant u$. Thus, at site (l', w) we obtain

$$\ldots t \ldots a_r \ldots t \ldots a_s \ldots,$$

and thus t separates a_r and a_s. If $u > w$, then a similar thing happens to a_p and a_q at site (l, u). This makes the contribution of all terms vanish. $\blacksquare$

Assume now that $\mathscr{A}_l = \mathscr{A}$, $l \in N$. We will derive the central limit theorem for the sums of m-free "independent" variables (in other words, the central limit theorem for m-free convolutions)

$$S_N^{(m)}(a) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} j_k^{(m)}(a), \quad \text{where } a \in \mathscr{A}^0.$$

THEOREM 3.5. *Let $m \in N$, $a_1, \ldots, a_n \in \mathscr{A}$, and let ϕ be a state on $\mathscr{A}$ for which $\phi(a_i) = 0$, $i = 1, \ldots, n$. Then*

$$\lim_{N \to \infty} \tilde{\Phi}\big(S_N^{(m)}(a_1) \ldots S_N^{(m)}(a_n)\big) = \sum_{\{P_1, \ldots, P_k\} \in NC_n^{\text{pair}}(m)} \phi(a_{P_1}) \ldots \phi(a_{P_k})$$

if $n = 2k$. If n is odd, then the above limit vanishes.

Proof. Using Proposition 2.2 and typical central limit arguments (see, for instance, the limit theorem for correlations which are invariant under order-preserving injections in [4] or [10]) we know that only pair-partitions may give a non-vanishing contribution as $N \to \infty$. Now use Lemmas 3.2–3.4 to see that out of these only the non-crossing pair-partitions of depth $\leqslant m$ really give a non-vanishing contribution. The second part of the theorem is again standard and follows from the assumption on the zero mean. $\blacksquare$

COROLLARY 3.6. *In particular, if $\mathscr{A} = C[x]$, $x^* = x$, and $\phi(x^2) = 1$, then*

$$M_n^{(m)} \equiv \lim_{N \to \infty} \tilde{\Phi}\big((S_N^{(m)}(x))^n\big) = |NC_n^{\text{pair}}(m)| \quad \textit{for } n \textit{ even}.$$

The odd limit moments vanish.

The corollary follows immediately from Theorem 3.5.

Remark. Knowing that m-freeness approximates freeness, we automatically obtain the central limit theorem for free random variables (as well as conditionally free random variables or their possible generalizations as discussed in [2]). For that purpose and for given $n = 2k$ it is enough to take the k-free product state.

In Section 5 we will solve the moment problem for the limit moments given by Corollary 3.6 for each m.

4. An invariance principle and m-free Brownian motions. In this section we state an invariance principle for the hierarchy of freeness. We also define a corresponding *hierarchy of Brownian motions* and show that, under some additional assumptions on the state ϕ, the limit distributions obtained from the invariance principle are the distributions of the hierarchy of Brownian motions.

Let us begin with the invariance principle. Let $a \in \mathscr{A}^0$ and instead of the sums $S_N^{(m)}(a)$ consider now sample sums

$$S_{N,f}^{(m)}(a) = \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} j_k^{(m)}(a) \int_{k-1}^{k} f\left(\frac{t}{N}\right) dt,$$

indexed not only by N and m, but also by $f \in L_c^2(\boldsymbol{R}_+)$, where $L_c^2(\boldsymbol{R}_+)$ stands for the square integrable real-valued functions with compact support on $\boldsymbol{R}$.

THEOREM 4.1. *Let $f_1, \ldots, f_n \in L_c^2(\boldsymbol{R}_+)$, $a_1, \ldots, a_n \in \mathscr{A}^0$, $m, N \in \boldsymbol{N}$. Then*

$$\lim_{N \to \infty} \tilde{\Phi}(S_{N,f_1}^{(m)}(a_1) \ldots S_{N,f_n}^{(m)}(a_n))$$
$$= \sum_{\{P_1, \ldots, P_k\} \in NC_n^{\text{pair}}(m)} \phi(a_{P_1}) \ldots \phi(a_{P_k}) \prod_{r=1}^{k} \int_0^{\infty} f_{\alpha(r)}(t) f_{\beta(r)}(t)\, dt$$

if $n = 2k$, where $P_i = \{\alpha(i), \beta(i)\}$, $i = 1, \ldots, k$. If n is odd, then the above limit vanishes.

Proof. This is a special case of the invariance principle for correlations invariant under order-preserving injections proved in [10]. ∎

Under certain additional assumptions one can realize the limit distribution in terms of creation and annihilation operators on a suitable Fock space. Note that the only difference between our invariance principle and the invariance principle for free independence is that in the case of m-freeness only non-crossing partitions of depth $\leqslant m$ survive in the limit.

To take that into account it is enough to define the *m-free Fock space*

$$\mathscr{F}^{(m)} \equiv \mathscr{F}^{(m)}(L^2(\boldsymbol{R}_+)) = \boldsymbol{C} \oplus \bigoplus_{k=1}^{m} L^2(\boldsymbol{R}_+)^{\otimes k}$$

with the vacuum vector $\Omega_m = 1 \oplus 0 \oplus \ldots \oplus 0$ and the canonical scalar product $\langle \cdot, \cdot \rangle_{\mathscr{F}^{(m)}}$.

Next, we define the *m-free creation operators*

$$a^{(m)*}(f)\colon \mathscr{F}^{(m)} \to \mathscr{F}^{(m)},$$

$$a^{(m)*}(f)\, f_1 \otimes \ldots \otimes f_n = \begin{cases} f \otimes f_1 \otimes \ldots \otimes f_n & \text{if } 1 \leqslant n < m, \\ 0 & \text{if } n = m \end{cases}$$

with $a^{(m)*}(f)\,\Omega_m = f$ and the *m-free annihilation operators*

$$a^{(m)}(f)\colon \mathscr{F}^{(m)} \to \mathscr{F}^{(m)},$$

$$a^{(m)}(f)\, f_1 \otimes \ldots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n$$

if $1 \leqslant n \leqslant m$ and $a^{(m)}(f)\,\Omega_m = 0$. Note that $a^{(m)*}(f),\ a^{(m)}(f) \in B(\mathscr{F}^{(m)})$.

We are ready to find a realization of the invariance principle limit in terms of the m-free creation and annihilation operators under standard assumptions. For simplicity we assume that $\mathscr{A}$ is the *-algebra generated by one element a and we write $\mathscr{A} = \boldsymbol{C}\langle a, a^* \rangle$.

THEOREM 4.2. *Let ϕ be a state on $\boldsymbol{C}\langle a, a^*\rangle$ such that $\phi(a) = \phi(a^*) = \phi(aa) = \phi(a^* a) = \phi(a^* a^*) = 0$, $\phi(aa^*) = 1$. Then*

$$\lim_{N\to\infty} \tilde{\Phi}\big(S_{N,f_1}^{(m)}(a^{\varepsilon_1}) \ldots S_{N,f_n}^{(m)}(a^{\varepsilon_n})\big) = \langle \Omega_m, a^{(m)\varepsilon_1}(f_1) \ldots a^{(m)\varepsilon_n}(f_n)\,\Omega_m \rangle_{\mathscr{F}^{(m)}}$$

for all $n \in \boldsymbol{N}$, $a^{\varepsilon_1}, \ldots, a^{\varepsilon_n} \in \{a, a^*\}, f_1, \ldots, f_n \in L_c^2(\boldsymbol{R}_+)$.

Proof. It is enough to notice that the m-truncated creation and annihilation operators are defined in such a way that there can be no contribution from pair-partitions of depth greater than m since the latter would require a tensor product of order greater than m. ∎

For each $m \in \boldsymbol{N}$ denote by $\mathscr{C}^{(m)}$ the C^*-algebra generated by $a^{(m)*}(f)$, $a^{(m)}(f)$, $f \in L^2(\boldsymbol{R}_+)$, and let φ_m be the vacuum expectation state in the m-free Fock space. Then the pair $(\mathscr{C}^{(m)}, \varphi_m)$ can be viewed as the *m-free Brownian motion* and the collection $(\mathscr{C}^{(m)}, \varphi_m)_{m\in \boldsymbol{N}}$ as the *hierarchy of m-free Brownian motions.*

5. The hierarchy of limit measures. In this section we solve the moment problem for the *m-free central limit laws* obtained in Section 3 in the case when $\mathscr{A} = \boldsymbol{C}[x]$, where $x = x^*$. We obtain a sequence $(\mu_m)_{m\in\boldsymbol{N}}$ of discrete measures that approximate the Wigner measure.

For that purpose, let us introduce the *hierarchy of Cauchy transforms* $(G_m(z))_{m\in\boldsymbol{N}}$ for the sequence of limit laws given by Corollary 3.6:

$$G_m(z) = \sum_{n=0}^{\infty} M_n^{(m)} z^{-n-1},$$

where $M_n^{(m)} = |NC_n^{\mathrm{pair}}(m)|$, $m, n \in \boldsymbol{N}$, and, in addition, $M_0^{(m)} = 1$, $m \in \boldsymbol{N}$. We also adopt the convention that $M_n^{(0)} = \delta_{n,0}$, which gives $G_0 = 1/z$. For the use of

Cauchy transforms in the case of freeness (conditional freeness), see [12] and [5] (cf. also [1]).

The moments $M_n^{(m)}$ grow less rapidly as $N \to \infty$ than the moments M_n of the Wigner measure. Therefore it is clear that for each m there exists a unique measure $\mu^{(m)}$ of which G_m is the Cauchy transform. In particular, $\mu^{(0)} = \delta_0$. We will find the explicit form of $\mu^{(m)}$ for each $m \in N$.

LEMMA 5.1. *The hierarchy of Cauchy transforms satisfies the recurrence relation*

$$G_m(z) = \frac{1}{z - G_{m-1}(z)},$$

where $m \in N^*$ *with* $G_0(z) = 1/z$ *if* $|z| > 2$.

Proof. Let us assume that we know the number of non-crossing pair-partitions of depth less than or equal to m of the set $\{1, \ldots, 2k\}$ for any $k \leqslant n$. To get a non-crossing pair-partition of depth less than or equal to m of the set $\{1, \ldots, 2n+2\}$, we have to choose a number $k \in \{2, \ldots, 2n+2\}$ that will form a pair with 1, then choose a non-crossing pair-partition of depth less than or equal to $m-1$ for the numbers between 1 and k, i.e. of the set $\{2, \ldots, k-1\}$, and a non-crossing pair-partition of depth less than or equal to m for the numbers from $k+1$ to $2n+2$, i.e. of the set $\{k+1, \ldots, 2n+2\}$.

Therefore, there are exactly $|NC_{k-2}^{\mathrm{pair}}(m-1)|\,|NC_{2n-k+2}^{\mathrm{pair}}(m)|$ such pair-partitions in which 1 is paired with k. For the total number of non-crossing pair-partitions of depth less than or equal to m of the set $\{1, \ldots, 2n+2\}$ we get

$$|NC_{2n+2}^{\mathrm{pair}}(m)| = \sum_{k=2}^{2n+2} |NC_{k-2}^{\mathrm{pair}}(m-1)|\,|NC_{2n-k+2}^{\mathrm{pair}}(m)|.$$

The terms with odd k give zero since there can be no pair-partition of a set with an odd number of elements. Hence

$$M_{2n+2}^{(m)} = \sum_{k=2}^{2n+2} M_{k-2}^{(m-1)} M_{2n-k+2}^{(m)} = \sum_{l=1}^{n+1} M_{2l-2}^{(m-1)} M_{2n-2l+2}^{(m)}.$$

The recurrence relation for the moments leads easily to the desired recurrence relation for the Cauchy transforms if $|z| > 2$ since

$$\begin{aligned} G_m(z) &= \sum_{n=0}^{\infty} M_{2n}^{(m)} z^{-2n-1} = \frac{1}{z} + \sum_{n=0}^{\infty} M_{2n+2}^{(m)} z^{-2n-3} \\ &= \frac{1}{z} + \frac{1}{z} \sum_{n=0}^{\infty} \sum_{l=1}^{n+1} M_{2l-2}^{(m-1)} z^{-2l+1} M_{2n-2l+2}^{(m)} z^{-2n+2l-3} = \frac{1}{z} + \frac{G_m(z)\,G_{m-1}(z)}{z}, \end{aligned}$$

and therefore

$$G_m(z) = \frac{1/z}{1 - G_{m-1}(z)/z} = \frac{1}{z - G_{m-1}(z)},$$

which completes the proof. ■

Remark 1. Note that the series given by $G_m(z)$ converges absolutely for $|z| > 2$ and all $m \in N$ since

$$|NC_{2k}^{\text{pair}}(m)| \leqslant |NC_{2k}^{\text{pair}}|,$$

where

$$|NC_{2k}^{\text{pair}}| = \frac{1}{k+1}\binom{2k}{k}$$

denotes the number of *all* non-crossing partitions of the set $\{1, \ldots, 2k\}$. Clearly,

$$|NC_n^{\text{pair}}| = |NC_n^{\text{pair}}(m)| = 0 \quad \text{if } n \text{ is odd.}$$

Remark 2. The Cauchy transforms $G_m(z)$ are rational functions of the complex variable z. In particular,

$$G_0(z) = \frac{1}{z}, \quad G_1(z) = \frac{1}{z - 1/z}, \quad G_2(z) = \frac{1}{z - \dfrac{1}{z - 1/z}}, \ldots$$

We will show below that G_m has $m+1$ simple poles in the interval $(-2, 2)$ (and none anywhere else). For that purpose we use the Chebyshev polynomials of the second kind

$$U_m(x) = \frac{\sin[(m+1)\arccos(x)]}{\sin[\arccos(x)]}$$

for $x \in (-1, 1)$, $m \in N \cup \{0\}$. They satisfy the recurrence relation

$$U_{m+1}(x) = 2x\, U_m(x) - U_{m-1}(x)$$

with $U_0(x) = 1$. Denote by $U_m(z)$ the analytic extension of $U_m(x)$. Note that $U_m(z)$ has exactly m simple zeros

$$u_{m,k} = \cos\left(\frac{k\pi}{m+1}\right), \quad k = 1, \ldots, m,$$

and that the zeros of $U_m(z)$ differ from those of $U_{m+1}(z)$. This enables us to define the meromorphic function

$$W_m(z) = \frac{U_m(z/2)}{U_{m+1}(z/2)}, \quad m \in N \cup \{0\},$$

with $m+1$ simple poles on the real line given by

$$z_{m,k} = 2\cos\left(\frac{k\pi}{m+2}\right), \quad k = 1, \ldots, m+1.$$

We show below that $W_m(z)$ coincides with $G_m(z)$.

LEMMA 5.2. *Let* $m \in N \cup \{0\}$. *The Cauchy transform* $G_m(z)$ *agrees with* $W_m(z)$ *for* $z \notin \{z_{m,k} \mid 1 \leqslant k \leqslant m+1\}$.

Proof. Clearly, $W_0(z) = G_0(z) = 1/z$ since $U_0(z) = 1$ and $U_1(z) = 2z$. Let us show that the functions $W_m(z)$ satisfy the recurrence relation given by Lemma 5.1. If $m \geqslant 1$, then the recurrence relation for the Chebyshev polynomials of the second kind gives

$$W_{m+1}(z) = \frac{U_{m+1}(z/2)}{U_{m+2}(z/2)} = \frac{U_{m+1}(z/2)}{zU_{m+1}(z/2) - U_m(z/2)}$$
$$= \frac{1}{z - U_m(z/2)/U_{m+1}(z/2)} = \frac{1}{z - W_m(z)}$$

for all $z \notin \{z_{m,k} \mid 1 \leqslant k \leqslant m+1\}$. Therefore, $G_m(z)$ must agree with $W_m(z)$ also for $m \geqslant 1$ on the intersection of their domains, which completes the proof. ∎

THEOREM 5.3. *The measures* $\mu^{(m)}$ *take the form*

$$\mu^{(m)} = \sum_{k=1}^{m+1} b_{m,k}\, \delta_{z_{m,k}},$$

where

$$b_{m,k} = \frac{2\sin^2[k\pi/(m+2)]}{m+2}$$

for $m \in N \cup \{0\}$ *and* $k = 1, \ldots, m+1$.

Proof. We have to invert the Cauchy transforms. By Lemma 5.2, $G_m(z)$ is a rational function with the degree of the denominator exceeding that of the numerator and with simple poles at $z_{m,k}$, $1 \leqslant k \leqslant m+1$. Thus its decomposition into partial fractions takes the form

$$G_m(z) = \sum_{k=1}^{m+1} \frac{b_{m,k}}{z - z_{m,k}}.$$

This shows that $G_m(z)$ is the Cauchy transform of a discrete measure with point masses at $z_{m,k}$, $1 \leqslant k \leqslant m+1$. The calculation of the residues gives the masses

$$b_{m,k} = \lim_{z \to z_{m,k}} \frac{\sin[(m+1)\arccos(z/2)]}{(d/dz)\sin[(m+2)\arccos(z/2)]}$$
$$= \frac{2\sin^2[k\pi/(m+2)]}{m+2},$$

which completes the proof. ∎

EXAMPLE. The measures $\mu^{(0)}, \mu^{(1)}, \mu^{(2)}$ are given by

$$\mu^{(0)} = \delta_0, \quad \mu^{(1)} = \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1, \quad \mu^{(2)} = \tfrac{1}{4}\delta_{-\sqrt{2}} + \tfrac{1}{2}\delta_0 + \tfrac{1}{4}\delta_{\sqrt{2}}.$$

Since the moment problems are determined for all $m \in N$, i.e. the measures $\mu^{(m)}$ are uniquely determined, $\mu^{(m)}$ converges weakly to the Wigner measure μ_W.

6. Poisson's limit theorem. In this section we study Poisson's limit theorem for the hierarchy of freeness and solve the moment problems for the associated limit laws called *m-free Poisson laws.* By $|NC_n(b, m)|$ we denote the number of non-crossing partitions of $\{1, \ldots, n\}$ with b blocks and depth less than or equal to m.

THEOREM 6.1. *Let* $\mathscr{A}_l = \mathscr{A} = C[x]$, $l \in N$, $x^* = x$, *and assume that* $N\phi^N(x^k) \to \lambda$, $k \in N$, $\lambda > 0$. *Let* $S_{m,N} = \sum_{k=1}^{N} j_k^{(m)}(x)$ *and denote by* $\tilde{\Phi}^{(m,N)}$ *the m-free product state corresponding to* ϕ^N. *Then*

$$\lim_{N \to \infty} \tilde{\Phi}^{(m,N)}(S_{m,N}^n) = \sum_{q=1}^{n} \lambda^q |NC_n(q, m)| \equiv M_n^{(m)}(\lambda).$$

Proof. We have

$$\tilde{\Phi}^{(m,N)}(S_{m,N}^n) = \sum_{1 \leqslant k_1, \ldots, k_n \leqslant N} \tilde{\Phi}^{(m,N)}\big(j_{k_1}(x) \ldots j_{k_n}(a)\big) = \sum_{P \in \mathscr{P}_n} (N)_{b(P)} m(P),$$

where P_n denotes partitions of $\{1, \ldots, n\}$, $m(P) = \tilde{\Phi}^{(m,N)}\big(j_{k_1}(x) \ldots j_{k_n}(x)\big)$ for any tuple $(k_1, \ldots, k_n)$ associated with the partition P, $b(P)$ denotes the number of blocks of P and $(N)_r = N(N-1) \ldots (N-r+1)$.

Now we apply the usual Poisson's limit arguments. The only partitions P which survive in the limit $N \to \infty$ are those for which the expression for $m(P)$ contains a term of type $\lambda^{b(P)}$ (i.e. the number of blocks of P is equal to the number of moments in the given term). If P is a crossing partition, then $m(P)$ "factorizes" into more than b moments, and thus gives no contribution to the limit. If P is non-crossing, then we have two cases: (i) $d(P) > m$ and (ii) $d(P) \leqslant m$. In the case (i) the contribution is zero even before taking the limit by the GNS representation. In the case (ii) the contribution is $\lambda^{b(P)}$, which completes the proof. ∎

In order to solve the associated moment problem, we want to find the generating functions for $|NC_n(b, m)|$. Thus, let

$$H^{(m)}(\lambda, z) = \sum_{n,b=0}^{\infty} |NC_n(b, m)| \lambda^b z^{-n-1}$$

for $m \geqslant 1$ and $H^{(0)}(\lambda, z) = 1/z$, where we adopt the conventions that

$$|NC_n(b, 0)| = \delta_{n0}\delta_{b0} \quad \text{and} \quad |NC_n(0, m)| = \delta_{n0}.$$

Clearly, $|NC_n(b, m)| = 0$ for $b > n > 0$, so the summation over b is finite for fixed n.

Note that $H^{(m)}(z)$, $m \geqslant 0$, converge absolutely for $|z|$ sufficiently large, say $|z| > R(\lambda) = (\sqrt{\lambda}+1)^2$. Moreover, they go to zero as $|z|$ goes to infinity (since there is no constant term in the series). Thus $|H^{(m)}(\lambda, z)| < 1$ for $|z| > R'(\lambda)$ for some sufficiently large $R'(\lambda)$ (it depends on λ but not on m by comparison with the free Poisson law, i.e. $|NC_n(b, m)| \leqslant |NC_n(b)|$, and therefore

$|H^{(m)}(\lambda, z)| \leqslant H(|\lambda|, |z|)$, where $|NC_n(b)|$ denotes the number of non-crossing partitions of $\{1, \ldots, n\}$ of b blocks and $H(\lambda, z)$ is the generating function for the free Poisson law).

LEMMA 6.2. *The hierarchy of generating functions $(H^{(m)})_{m \geqslant 0}$ satisfies the recurrence relation*

$$H^{(m)}(\lambda, z) = \frac{1-H^{(m-1)}(\lambda, z)}{z-zH^{(m-1)}(\lambda, z)-\lambda}$$

for $m = 1, 2, \ldots$ and $|z| > R'(\lambda)$.

Proof. To get a non-crossing partition of $\{1, \ldots, n\}$ $(n \geqslant 1)$ we pick the elements that will be put in the same block as the first element; denote this block by $\{1, 1+k_1, 1+k_1+k_2, \ldots, 1+k_1+ \ldots +k_{r-1}\}$, and then choose non-crossing partitions for the remaining intervals $\{2, \ldots, k_1\}$, $\{k_1+2, \ldots, k_1+k_2\}, \ldots, \{k_1+ \ldots +k_{r-2}+2, \ldots, k_1+ \ldots +k_{r-1}\}$, $\{k_1+ \ldots +k_{r-1}+2, \ldots, n\}$. We will denote the number of elements of the last interval by k_r. If we want the resulting partition to have depth $\leqslant m$, then the partitions chosen for $\{2, \ldots, k_1\}, \ldots, \{k_1+ \ldots +k_{r-2}+2, \ldots, k_1+ \ldots +k_{r-1}\}$ must have depth $\leqslant m-1$, and those chosen for $\{k_1+ \ldots +k_{r-1}+2, \ldots, n\}$ must have depth $\leqslant m$. Let b_k be the number of blocks of the partition of the k-th interval; then the number of blocks of the whole partition is $b_1+ \ldots +b_r+1$. Therefore the number of non-crossing partitions of $\{1, \ldots, n\}$ with b blocks and depth $\leqslant m$ can be calculated recursively by the formula

$$|NC_n(b, m)| = \sum_{r=1}^{n} \sum_{\substack{k_1,\ldots,k_{r-1} \geqslant 1; k_r \geqslant 0 \\ k_1+\ldots+k_r = n-1}} \sum_{\substack{b_1,\ldots,b_r \geqslant 0 \\ b_1+\ldots+b_r = b-1}} |NC_{k_1-1}(b_1, m-1)| \times \ldots |NC_{k_{r-1}-1}(b_{r-1}, m-1)| \, |NC_{k_r}(b_r, m)|$$

for $n \geqslant 1$, if we use the conventions $|NC_n(b, 0)| = \delta_{n0}\delta_{b0}$ and $|NC_n(0, m)| = \delta_{n0}$. By these conventions we have $H^{(0)}(\lambda, z) = 1/z$.

Let now $m \geqslant 1$ and $|z| \geqslant R'$. Then we have

$$\begin{aligned}
H^{(m)}(\lambda, z) &= \sum_{n,b=0}^{\infty} |NC_n(b, m)| \lambda^b z^{-n-1} \\
&= \frac{1}{z}+\frac{\lambda}{z} \sum_{n,b=1}^{\infty} \sum_{r=1}^{n} \sum_{\substack{k_1,\ldots,k_{r-1} \geqslant 1; k_r \geqslant 0 \\ k_1+\ldots+k_r = n-1}} \sum_{\substack{b_1,\ldots,b_r \geqslant 0 \\ b_1+\ldots+b_r = b-1}} |NC_{k_1-1}(b_1, m-1)| \lambda^{b_1} z^{-k_1} \\
&\quad \times \ldots |NC_{k_{r-1}-1}(b_{r-1}, m-1)| \lambda^{b_{r-1}} z^{-k_{r-1}} |NC_{k_r}(b_r, m)| \lambda^{b_r} z^{-k_r-1} \\
&= \frac{1}{z}+\frac{\lambda}{z} \sum_{r=1}^{\infty} \left(\sum_{\beta,\nu=0}^{\infty} |NC_\nu(\beta, m-1)| \lambda^\beta z^{-\nu-1} \right)^{r-1} \sum_{\mu,\alpha=0}^{\infty} |NC_\mu(\alpha, m)| \lambda^\alpha z^{-\mu-1} \\
&= \frac{1}{z}+\frac{\lambda H^{(m)}(\lambda, z)}{z(1-H^{(m-1)}(\lambda, z))},
\end{aligned}$$

where the summations can be interchanged since all sums converge absolutely (remember that $|H^{(m-1)}(\lambda, z)| < 1$ for $|z| > R'(\lambda)$), and therefore

$$H^{(m)}(\lambda, z) = \frac{1-H^{(m-1)}(\lambda, z)}{z - zH^{(m-1)}(\lambda, z) - \lambda}. \quad \blacksquare$$

We will now give an explicit expression for the solution of this recurrence relation. To this end we will again use the Chebyshev polynomials of the second kind.

PROPOSITION 6.3. *Let $\lambda > 0$, $m \in N \cup \{0\}$. The meromorphic functions*

$$F_\lambda^{(m)}(z) = \frac{(z-\lambda)\, U_m\big((z-\lambda-1)/(2\sqrt{\lambda})\big) - \sqrt{\lambda}\, U_{m+1}\big((z-\lambda-1)/(2\sqrt{\lambda})\big)}{zU_m\big((z-\lambda-1)/(2\sqrt{\lambda})\big)}$$

solve the recurrence relation

$$F_\lambda^{(m)}(z) = \frac{1-F_\lambda^{(m-1)}(z)}{z - zF_\lambda^{(m-1)}(z) - \lambda} \quad \text{for } m \geq 1, \; F_\lambda^{(0)}(z) = \frac{1}{z},$$

and therefore we have $H^{(m)}(\lambda, z) = F_\lambda^{(m)}(z)$ for $|z| > R'(\lambda)$.

Furthermore, $F_\lambda^{(m)}(z)$ has the partial fraction decomposition

$$F_\lambda^{(m)}(z) = \sum_{k=0}^{m} \frac{a_{m,k}(\lambda)}{z - y_{m,k}(\lambda)},$$

where

$$y_{m,0}(\lambda) = 0,$$

$$y_{m,k}(\lambda) = 2\sqrt{\lambda}\cos\left(\frac{k\pi}{m+1}\right) + \lambda + 1, \qquad k = 1, \ldots, m,$$

$$a_{m,0}(\lambda) = \sqrt{\lambda}\, \frac{U_{m+1}\big((\lambda+1)/(2\sqrt{\lambda})\big)}{U_m\big((\lambda+1)/(2\sqrt{\lambda})\big)} - \lambda,$$

$$a_{m,k}(\lambda) = \frac{2\lambda \sin^2[k\pi/(m+1)]}{(m+1)\big[2\sqrt{\lambda}\cos[k\pi/(m+1)] + \lambda + 1\big]}, \qquad k = 1, \ldots, m,$$

for $m \in N$.

Proof. Fix λ and let $F_\lambda^{(m)}(z) = P_\lambda^{(m)}(z)/Q_\lambda^{(m)}(z)$, where

$$P_\lambda^{(m)}(z) = \lambda^{m/2}(z-\lambda)\, U_m\left(\frac{z-\lambda-1}{2\sqrt{\lambda}}\right) - \lambda^{(m+1)/2}\, U_{m+1}\left(\frac{z-\lambda-1}{2\sqrt{\lambda}}\right),$$

$$Q_\lambda^{(m)}(z) = \lambda^{m/2} z U_m\left(\frac{z-\lambda-1}{2\sqrt{\lambda}}\right).$$

From the recurrence relation for the Chebyshev polynomials of the second kind it follows that $P_\lambda^{(m)}(z)$ and $Q_\lambda^{(m)}(z)$ satisfy the coupled recurrence relations

$$P_\lambda^{(m)}(z) = Q_\lambda^{(m-1)}(z) - P_\lambda^{(m-1)}(z),$$
$$Q_\lambda^{(m)}(z) = (z-\lambda)\, Q_\lambda^{(m-1)}(z) - zP_\lambda^{(m-1)}(z)$$

for $m \geqslant 1$, and $P_\lambda^{(0)}(z) = 1$, $Q_\lambda^{(0)}(z) = z$.

For $m = 0$ we have $F_\lambda^0(z) = P_\lambda^{(0)}(z)/Q_\lambda^{(0)}(z) = 1/z$, and for $m \geqslant 1$,

$$F_\lambda^{(m)}(z) = \frac{P_\lambda^{(m)}(z)}{Q_\lambda^{(m)}(z)} = \frac{Q_\lambda^{(m-1)}(z) - P_\lambda^{(m-1)}(z)}{(z-\lambda)\, Q_\lambda^{(m-1)}(z) - zP_\lambda^{(m-1)}(z)}$$
$$= \frac{1 - P_\lambda^{(m-1)}(z)/Q_\lambda^{(m-1)}(z)}{z-\lambda - zP_\lambda^{(m-1)}(z)/Q_\lambda^{(m-1)}(z)} = \frac{1 - F_\lambda^{(m-1)}(z)}{z - zF_\lambda^{(m-1)}(z) - \lambda}.$$

It is easy to deduce from the recurrence relation that $P_\lambda^{(m)}(z)$ has degree $\leqslant m$. From the definition of $Q_\lambda^{(m)}(z)$ we immediately see that it has $m+1$ distinct simple roots, $y_{m,0}(\lambda) = 0$, and

$$y_{m,k}(\lambda) = 2\sqrt{\lambda}\cos\left(\frac{k\pi}{m+1}\right) + \lambda + 1, \qquad k = 1, \ldots, m.$$

Therefore $F_\lambda^{(m)}(z)$ has the form stated in the proposition. The calculation of the residues gives

$$a_{m,0}(\lambda) = \lim_{z\to 0} zF_\lambda^{(m)}(z) = \sqrt{\lambda}\,\frac{U_{m+1}\big((\lambda+1)/(2\sqrt{\lambda})\big)}{U_m\big((\lambda+1)/(2\sqrt{\lambda})\big)} - \lambda,$$

$$a_{m,k}(\lambda) = \lim_{z\to y_{m,k}} (z - y_{m,k})\, F_\lambda^{(m)}(z)$$
$$= -\frac{2\lambda}{2\sqrt{\lambda}\cos[k\pi/(m+1)] + \lambda + 1} \lim_{x\to x_{m,k}} \frac{\sin[(m+2)\arccos(x)]}{(d/dx)\sin[(m+1)\arccos(x)]}$$
$$= \frac{2\lambda\sin^2(k\pi/(m+1))}{(m+1)\big[2\sqrt{\lambda}\cos(k\pi/(m+1)) + \lambda + 1\big]} \qquad \text{for } m \geqslant k \geqslant 1,$$

where $x_{m,k} = \cos(k\pi/(m+1))$. ∎

THEOREM 6.4. *Let $m \in N$, $\lambda > 0$. The moments $(M_n^{(m)}(\lambda))_{n\in N}$ determine a unique measure on the real line of the form*

$$\mu_\lambda^{(m)} = \sum_{k=0}^{m} a_{m,k}(\lambda)\, \delta_{y_{m,k}(\lambda)}.$$

Proof. The moments $(M_n^{(m)}(\lambda))_{n\in N}$ grow less rapidly as $n \to \infty$ than the moments of the free Poisson limit measure, therefore it is clear that the moment problem has a unique solution $\mu_\lambda^{(m)}$. Denote its Cauchy transform by

$$G_\lambda^{(m)}(z) = \int_{R} \frac{1}{z-x}\, d\mu_\lambda^{(m)}(x).$$

By Lemma 6.2 we know that $H^{(m)}(\lambda, z) = \sum_{n=0}^{\infty} M_n^{(m)}(\lambda) z^{-n-1}$ converges absolutely for $|z| \geqslant R(\lambda)$, therefore it coincides with the Cauchy transform of $\mu_\lambda^{(m)}$ for $|z| \geqslant R(\lambda)$. By Proposition 6.3 we now have $G_\lambda^{(m)}(z) = F_\lambda^{(m)}(z)$ for $|z| > R'(\lambda)$, and then also for all $z \in \mathbf{C} \setminus \mathbf{R}$, since both functions are analytic on $\mathbf{C} \setminus \mathbf{R}$.

It now follows immediately from the partial fraction decomposition of Proposition 6.3 that $\mu_\lambda^{(m)}$ has the form stated in the theorem. ■

EXAMPLE. We get

$$\mu_\lambda^{(0)} = \delta_0, \qquad \mu_\lambda^{(1)} = \frac{1}{1+\lambda}\delta_0 + \frac{\lambda}{1+\lambda}\delta_{1+\lambda},$$

$$\mu_\lambda^{(2)} = \frac{1}{1+\lambda+\lambda^2}\delta_0 + \frac{\lambda}{2(1+\sqrt{\lambda}+\lambda)}\delta_{1+\sqrt{\lambda}+\lambda} + \frac{\lambda}{2(1-\sqrt{\lambda}+\lambda)}\delta_{1-\sqrt{\lambda}+\lambda}.$$

Remark 1. A realization of a quantum random variable with the m-free Poisson law (corresponding to the measure $\mu_\lambda^{(m)}$) on the quantum probability space $(M_{m+1}(\mathbf{C}), \varphi)$, where $\varphi(\cdot) = \langle \Omega, \cdot\, \Omega \rangle$, is given by the $(m+1)\times(m+1)$-matrix

$$N_\lambda^{(m)} = \begin{pmatrix} 1 & \sqrt{\lambda} & 0 & \dots & \dots & \dots & 0 \\ \sqrt{\lambda} & 1+\lambda & \sqrt{\lambda} & 0 & \dots & \dots & 0 \\ 0 & \sqrt{\lambda} & 1+\lambda & \sqrt{\lambda} & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & \sqrt{\lambda} & 1+\lambda & \sqrt{\lambda} & 0 \\ 0 & \dots & \dots & 0 & \sqrt{\lambda} & 1+\lambda & \sqrt{\lambda} \\ 0 & \dots & \dots & \dots & 0 & \sqrt{\lambda} & \lambda \end{pmatrix}$$

if $\Omega = (0, \dots, 0, 1)^T$.

Remark 2. Let X be a quantum random variable (in some quantum probability space) whose distribution is the $2m$-free central limit law (corresponding to the measure $\mu^{(2m)}$). Then X^2 has the m-free Poisson limit law corresponding to the measure $\mu_1^{(m)}$, as can easily be seen from the respective Cauchy transform. This is the m-free version of a more general observation in the free case, namely that the product XPX, where X and P are free, X has the Wigner law, and P is a projection, has the free Poisson law (with parameter $\varphi(P)$); cf. [6].

REFERENCES

[1] M. Bożejko, M. Leinert and R. Speicher, *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math. 175, No. 2 (1996), pp. 357–388.

[2] U. Franz, R. Lenczewski and M. Schürmann, *The GNS construction for the hierarchy of freeness*, Preprint No. 9/98, Technical University of Wrocław, 1998.

[3] R. Lenczewski, *Unification of independence in quantum probability*, Inf. Dim. Anal., Quant. Probab. & Rel. Top., Vol. 1, No. 3 (1998), pp. 383–405.

[4] — *A noncommutative limit theorem for homogeneous correlations*, Studia Math. 129 (1998), pp. 225–252.

[5] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. 106 (1992), pp. 409–438.

[6] A. Nica and R. Speicher (with an appendix by D. Voiculescu), *On the multiplication of free N-tuples of noncommutative random variables*, Amer. J. Math. 118 (1996), pp. 799–837.

[7] M. Schürmann, *Non-commutative probability on algebraic structures*, in: *Probability Measures on Groups and Related Structures*, Vol. XI (Oberwolfach, 1994), World Scientific, River Edge, NJ, 1995, pp. 332–356.

[8] — *Direct sums of tensor products and non-commutative independence*, J. Funct. Anal. 133 (1995), pp. 1–9.

[9] R. Speicher, *A new example of "independence" and "white noise"*, Probab. Theory Related Fields 84 (1990), pp. 141–159.

[10] — and W. von Waldenfels, *A general central limit theorem and invariance principle*, in: *Quantum Probability and Related Topics*, Vol. IX, World Scientific, 1994, pp. 371–387.

[11] D. V. Voiculescu, *Symmetries of some reduced free product $\mathscr{C}^*$-algebras*, in: *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Math. 1132, Springer, Berlin 1985, pp. 556–588.

[12] — *Addition of certain non-commuting random variables*, J. Funct. Anal. 66 (1986), pp. 323–346.

Institut de Recherche Mathématique Avancée
Université Louis Pasteur
Rue René Descartes
F-67084 Strasbourg Cedex, France
E-mail: franz@math.u-strasbg.fr

Hugo Steinhaus Center for Stochastic Methods
Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: lenczew@im.pwr.wroc.pl

Received on 31.7.1998