PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 19, Fasc. 1 (1999), pp. 97–104

DECOMPOSITION OF CONVOLUTION SEMIGROUPS ON GROUPS AND THE 0-1 LAW

BY

H. BYCZKOWSKA AND T. BYCZKOWSKI* (WROCŁAW)

Abstract. Let $(X(t))_{t>0}$ be a stochastically continuous symmetric Lévy process with values in a complete separable group G. We denote by $(\mu_t)_{t>0}$ the semigroup of one-dimensional distributions of $X(t)$. Suppose that H is a Borel subgroup of G such that $\mu_t(H) > 0$ for all $t > 0$. We obtain a decomposition of the generator of the process $X(t)$ into a bounded part concentrated on H^c and the generator of a semigroup concentrated on H. This yields the 0-1 law for such processes. We also examine the differentiation of transition probability of the induced Markov process $\pi(X(t))$ on the homogeneous space G/H.

Introduction. The present paper is a continuation of [1]. For a given continuous symmetric convolution semigroup $(\mu_t)_{t>0}$ on a complete separable group G and a Borel subgroup H we decompose the generator of the above semigroup into a bounded part, concentrated on H^c, and the generator of a semigroup concentrated on H. This, in particular, yields the 0-1 law for such semigroups.

As in the above-mentioned paper, we apply a version of the so-called L^1 method and the Perturbation Formula, which establishes a link between the original semigroup, a bounded part of the generator, and the semigroup corresponding to an unbounded part. We do not use in our presentation the Trotter Approximation Theorem, as in the papers [2] and [4].

We adopt here the notation and terminology from [1].

Decomposition of semigroups. We first recall one result from [1]. By q we denote a fixed seminorm generating the topology of G.

PROPOSITION 1. *Let $(\mu_t)_{t>0}$ be a symmetric continuous convolution semigroup of probability measures on G acting on C_u, with generator N, and let $\mathscr{D}(N)$ be the domain of N. There exists a nonnegative measure ν, called the Lévy measure of $(\mu_t)_{t>0}$, such that for every $\eta > 0$ we have*

$$\nu\{q > \eta\} < \infty \quad \textit{and} \quad \lim_{t\downarrow 0}(1/t)\,\mu_t|_{q>\eta} = \nu_\eta \ \textit{weakly},$$

* H. Byczkowska and T. Byczkowski, Institute of Mathematics, Wrocław Technical University.

with $\nu_\eta = \nu|_{q>\eta}$, *whenever* $\nu\{q=\eta\} = 0$. *Moreover, for every* $\eta > 0$ *the following holds*:

$$Nf = (T_{\nu_\eta} - c_\eta I) f + N^\eta f, \quad f \in \mathscr{D}(N), \tag{1}$$

where $c_\eta = \nu\{q > \eta\}$ *and* N^η *is the generator of a convolution semigroup* $(\chi_t^\eta)_{t>0}$ *such that* $\mathscr{D}(N^\eta) = \mathscr{D}(N)$ *and*

$$\lim_{t\downarrow 0} (1/t)\, \chi_t^\eta \{q > \eta\} = 0 \quad \textit{if } \nu\{q=\eta\} = 0. \tag{2}$$

In this section we assume that $(\mu_t)_{t>0}$ is a fixed weakly continuous symmetric convolution semigroup of probability measures on G. We further assume that H is a Borel measurable subgroup of G. We state and prove here the main result, that is the decomposition of the generator of our given semigroup into a bounded part concentrated on H^c and the generator of a semigroup concentrated on H.

As mentioned before, we rely here on the so-called $L^1(\mu)$ method for μ defined by the formula

$$\mu = \int_0^\infty e^{-t} \mu_t \, dt.$$

We recall (cf. [2]) that $(\mu_t)_{t>0}$ acts as a strongly continuous semigroup on this space, with the norm $\|T_{\mu_s}\|_{L^1(\mu)} \leqslant e^s$. Observe that the symmetry of $(\mu_t)_{t>0}$ implies that if $\mu_{t_0}(H) > 0$ for a single t_0, then $\mu_t(H) > 0$ for all $t > 0$.

Now, suppose that H is a Borel subgroup of G such that $\mu(H) > 0$. Then, as a simple consequence of the fact that $T_{\mu_t} \mathbf{1}_H$ converges in $L^1(\mu)$ to $\mathbf{1}_H$ as $t\downarrow 0$, we obtain $\mu_t(H) \to 1$ at $t \downarrow 0$. In particular, $\mu_t(H) > 0$ for all $t > 0$.

As in the paper [1], we consider various L^1 spaces, steming from the Perturbation Formula.

Decomposition Theorem. *Let* $(\mu_t)_{t>0}$ *be a symmetric continuous convolution semigroup of probability measures on G acting on* C_u, *with the Lévy measure* ν *and the generator N. Let* $\mathscr{D}(N)$ *be the domain of N. Assume that H is a Borel subgroup of G such that* $\mu_t(H) > 0$ *for all* $t > 0$. *Then* $\nu(H^c) < \infty$ *and*

$$\lim_{t\downarrow 0} (1/t)\, \mu_{t|H^c} = \nu_{|H^c} \tag{3}$$

weakly. The following holds:

$$Nf = (T_{\nu|H^c} - cI) f + N^H f, \quad f \in \mathscr{D}(N), \tag{4}$$

where $c = \nu(H^c)$ *and* N^H *is the generator of a convolution semigroup* $(\chi_t)_{t>0}$ *concentrated on H with* $\mathscr{D}(N^H) = \mathscr{D}(N)$.

Proof. The proof is divided into three steps.

Step 1. Assume that $\lim_{t\downarrow 0} (1/t)\, T_{\mu_t|H^c} = \gamma$ weakly, where γ is a finite measure. We show that $\gamma(H) = 0$.

As in [1], we begin with the following decomposition:

$$(1/s)[T_{\mu_s} - I] = (\mu_s(H)/s)[T_{\mu_s^H} - I] + (\mu_s(H^c)/s)[T_{\mu_s^{H^c}} - I],$$

where μ_s^H and $\mu_s^{H^c}$ are conditional probabilities with respect to H and H^c, respectively. We rewrite the formula in the following form:

$$N_s = N_s^H + N_s^{H^c}, \tag{5}$$

regarding it as a decomposition of (bounded) generators of the corresponding convolution semigroups (acting on $C_u(G)$) denoted by $(\mu_t^s)_{t>0}$ and $(\chi_t^{H,s})_{t>0}$, $(\gamma_t^{H,s})_{t>0}$, respectively. Observe that

$$T_{\mu_t^s} \to T_{\mu_t} \tag{6}$$

strongly in C_u and also in $L^1(\mu)$. Indeed, for bounded Borel measurable functions f on G we have

$$\|T_{\mu_t} f\|_{L^1(\mu)} \leqslant e^t \|f\|_{L^1(\mu)},$$

so $(\mu_t)_{t>0}$ acts as a strongly continuous semigroup on $L^1(\mu)$. We also have the following estimate (see (4.1) in [1]):

$$\|T_{\mu_t^s} f\|_{L^1(\mu)} \leqslant \|f\|_{L^1(\mu*\mu_t^s)} \leqslant \exp\{t(e^s-1)/s\} \|f\|_{L^1(\mu)}, \tag{7}$$

and hence (6) also holds in $L^1(\mu)$.

We write the Perturbation Formula for the decomposition (5):

$$\mu_t^s = \exp\{-t\mu_s(H^c)/s\}\,\chi_t^{H,s} + \exp\{-t\mu_s(H^c)/s\} \int_0^t \chi_{t-u}^{H,s} * (1/s)\,\mu_{s|H^c} * \chi_u^{H,s}\,du \tag{8}$$

$$+\exp\{-t\mu_s(H^c)/s\} \sum_{n=2}^{\infty} \gamma_{t,n}^{H,s},$$

where

$$\gamma_{t,0}^{H,s} = \chi_t^{H,s}, \qquad \gamma_{t,n}^{H,s} = \int_0^t \chi_{t-u}^{H,s} * (1/s)\,\mu_{s|H^c} * \gamma_{u,n-1}^{H,s}\,du.$$

Define the measures m_s on $G \times G$ by the formula

$$m_s(dx, dz) = \exp\{-\mu_s(H^c)/s\} \int_0^1 (\mu * \chi_{1-u}^{H,s})(dx)(\chi_u^{H,s} * \mu)(dz)\,du.$$

Observe that, by (7) and (8), for a bounded Borel function h we get

$$\begin{aligned} &(1/s)\Big|\int_{G\times G} T_{\mu_{s|H^c}}(h_z)(x)\,m_s(dx, dz)\Big| \\ &\leqslant \int_{G^3} |h(xyz)|\,(1/s)\,\mu_{s|H^c}(dy)\,m_s(dx, dz) \\ &= \int_G \int_0^1 |h(z)| \exp\{-\mu_s(H^c)/s\}\big(\mu * \chi_{1-u}^{H,s} * (1/s)\,\mu_{s|H^c} * \chi_u^{H,s} * \mu\big)(dz)\,du \\ &\leqslant \int_G |h(z)|\,\mu * \mu_1^s * \mu(dz) \leqslant \exp\{(e^s-1)/s\} \int_G |h(z)|\,\mu * \mu(dz) \\ &= \exp\{(e^s-1)/s\}\,\|h\|_{L^1(\mu*\mu)}. \end{aligned} \tag{9}$$

By (7) and (8) we also obtain

(10) $m_s(dx, dz)$

$$= \int_0^1 (\mu * \exp\{-(1-u)\mu_s(H^c)/s\}\chi_{1-u}^{H,s})(dx)(\exp\{-u\mu_s(H^c)/s\}\chi_u^{H,s} * \mu)(dz)\,du$$

$$\leqslant \int_0^1 \mu * \mu_{1-u}^s(dx)\,\mu * \mu_u^s(dz)\,du$$

$$\leqslant \int_0^1 \exp\{(1-u)(e^s-1)/s\}\,\mu(dx)\exp\{u(e^s-1)/s\}\,\mu(dz)\,du$$

$$= \exp\{(e^s-1)/s\}\,\mu(dx)\,\mu(dz).$$

This yields the following estimate:

(11) $$\Big|\int_{G\times G} T_\gamma(h_z)(x)\,m_s(dx, dz)\Big| \leqslant \int_{G^3} |h(xyz)|\,\gamma(dy)\,m_s(dx, dz)$$

$$\leqslant \exp\{(e^s-1)/s\} \int_{G^3} |h(xyz)|\,\mu(dx)\,\gamma(dy)\,\mu(dz) = \exp\{(e^s-1)/s\}\,\|h\|_{L^1(\mu*\gamma*\mu)}.$$

We also obtain for $f \in C_u(G)$

(12) $$\int_{G\times G} |(1/s)\,T_{\mu_s|H^c}(f_z) - T_\gamma(f_z)|\,m_s(dx, dz)$$

$$\leqslant \exp\{(e^s-1)/s\} \int_{G\times G} |(1/s)\,T_{\mu_s|H^c}(f_z) - T_\gamma(f_z)|\,\mu(dx)\,\mu(dz).$$

We now estimate the following expression, where $f \in C_u$ and g is a bounded Borel measurable function on G:

$$|(1/s)\,T_{\mu_s|H^c}(g_z)(x) - T_\gamma(g_z)(x)|$$

$$\leqslant |(1/s)\,T_{\mu_s|H^c}[g_z - f_z](x)| + |(1/s)\,T_{\mu_s|H^c}(f_z)(x) - T_\gamma(f_z)(x)| + |T_\gamma[g_z - f_z](x)|.$$

Integrating the above expression with respect to m_s and taking into account (9) applied for $h = g - f$ we estimate the integral of the first term on the right-hand side by $\exp\{(e^s-1)/s\}\,\|g-f\|_{L^1(\mu*\mu)}$. From (11), applied for $h = g-f$, we estimate the integral of the last term on the right by $\exp\{(e^s-1)/s\}\,\|g-f\|_{L^1(\mu*\gamma*\mu)}$.

Note that, by the assumption and (12), the second term on the right-hand side of the estimated expression tends to zero as $s\downarrow 0$.

Since we can approximate $\mathbf{1}_H$ in measures $\mu*\mu$ and $\mu*\gamma*\mu$ by functions $f \in C_u$, we thus have proved that

$$\lim_{s\downarrow 0} \int_{G^2} \big|(1/s)\,T_{\mu_s|H^c}((\mathbf{1}_H)_z)(x) - T_\gamma((\mathbf{1}_H)_z)(x)\big|\,m_s(dx, dz) = 0.$$

However, for $x, z \in H$ we have $(1/s)\,T_{\mu_s|H^c}((\mathbf{1}_H)_z)(x) = 0$, while the integration of $T_\gamma((\mathbf{1}_H)_z)(x)$ with respect to m_s over $H\times H$ gives $\gamma(H)\exp\{(-\mu_s(H^c)/s\}\,\mu(H)^2$, which converges to $\gamma(H)e^{-c}\mu(H)^2$ as $s\downarrow 0$. Since $\mu(H) > 0$, this implies that $\gamma(H) = 0$, which completes the proof of this step.

Step 2. By a method similar to that in the proof of Proposition 1 in [1] we prove that (3) and (4) hold for a finite measure γ instead of $\nu_{|H^c}$. To do this we replace the set $\{q > \eta\}$ by H^c and $\{q \leqslant \eta\}$ by H and proceed as in this proof. By Step 1 we obtain $\gamma(H) = 0$. This and the same arguments as those used in the proof of Proposition 1 in [1] to show (2) yield

$$\lim_{t \downarrow 0} (1/t)\, \chi_t(H^c) = 0. \tag{13}$$

Observe that the application of Proposition 2 from [1] (more precisely: the Differentiation Lemma) implies, in view of (13), that the semigroup $(\chi_t)_{t>0}$ is concentrated on H. However, we provide here an alternative argument, based again on the Perturbation Formula.

First, we observe that $\chi_t(H) > 0$ for all $t > 0$. Indeed, assume the contrary. By symmetry, this means that $\chi_t(Hx) = 0$ for all $x \in G$ and all $t > 0$. But then we have

$$\gamma_{t,k}(H) = \int_0^t \chi_{t-u} * \gamma * \gamma_{u,k-1}(H)\, du = \int_0^t \int_G \chi_{t-u}(Hy^{-1})(\gamma * \gamma_{u,k-1})(dy)\, du = 0$$

for $k = 1, \ldots$, where $\gamma_{t,k}$ are the corresponding measures from the Perturbation Formula written for the decomposition (4) with γ instead of $\nu_{|H^c}$. This, however, gives $\mu_t(H) = 0$ for all $t > 0$, which contradicts our assumption and justifies our claim.

As a consequence, writing (8) for χ_t instead of μ_t, we get

$$\chi_t^s(H) = \exp\{-t\chi_s(H^c)/s\}\,[\tilde{\chi}_t^{H,s}(H) + \sum_{k=1}^{\infty} \tilde{\gamma}_t^{H,s,k}(H)] \geqslant \exp\{-t\chi_s(H^c)/s\}, \tag{14}$$

where χ_t^s, $\tilde{\chi}_t^{H,s}$ and $\tilde{\gamma}_t^{H,s,k}$ are the corresponding measures in the considered Perturbation Formula. The last inequality follows from the fact that the measures $\tilde{\chi}_t^{H,s}$ are concentrated on H. Applying (6) for χ_t^s and χ_t we infer that, as $s \to 0$, the left-hand side of (14) converges to $\chi_t(H)$, while the right-hand side converges to 1 because of (13). This completes the proof of this step.

Step 3. Assume that $\mu_t(H) > 0$ for all $t > 0$. By the proof of Theorem 1 in [1], for all $\eta > 0$ we obtain

$$\lim_{t \downarrow 0} (1/t)\,[\mu_{t|H^c} - \chi^\eta_{t|H^c}] = \nu_{|q>\eta \cap H^c}$$

weakly, where χ_t^η are as in Proposition 1. If $\nu\{q = \eta\} = 0$, we further obtain by Proposition 1:

$$\lim_{t \downarrow 0} (1/t)\, \mu_{t|q>\eta\cap H^c} = \lim_{t \downarrow 0} (1/t)\,[\mu_{t|q>\eta\cap H^c} - \chi^\eta_{t|q>\eta\cap H^c}] = \nu_{|q>\eta\cap H^c}.$$

By Step 2, the left-hand side converges to $\gamma_{|q>\eta}$, whenever $\gamma\{q = \eta\} = 0$. Thus, we have

$$\gamma_{|q>\eta} = \nu_{|q>\eta\cap H^c}$$

if only $\nu\{q=\eta\} = \gamma\{q=\eta\} = 0$. Letting $\eta \downarrow 0$, we obtain

$$\gamma = \nu_{|H^c}.$$

This completes the proof of the theorem.

Remark. By the Perturbation Formula it is not difficult to see that if the semigroup $(\mu_t)_{t>0}$ has the generator as in (4) with $\nu(H^c) < \infty$ and with $\exp(tN^H)$ concentrated on H, then $\mu_t(H) > 0$ for all $t > 0$.

As an application of the Decomposition Theorem we obtain

COROLLARY. *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup of probability measures on groups with the Lévy measure ν and let H be a Borel subgroup of G. Then $\nu(H^c) = \infty$ yields $\mu_t(H) = 0$ for all $t > 0$, while $\nu(H^c) = 0$ gives $\mu_t(H) = 0$ for all $t > 0$ or $\mu_t(H) = 1$ for all $t > 0$. If $\mu_t(H) > 0$ and $0 < \nu(H^c) < \infty$, then $\mu_t(H^c) > 0$ for all $t > 0$.*

Proof. By the Decomposition Theorem we infer that $\nu(H^c) = \infty$ yields $\mu_t(H) = 0$ for all $t > 0$. Assume now that $\mu_t(H) > 0$ for all $t > 0$. If $\nu(H^c) = 0$, then $\mu_t = \chi_t$ for all $t > 0$. Since the measures χ_t are concentrated on H, this concludes the proof of the first part.

Let now $\mu_t(H) > 0$ for all $t > 0$ and assume that $0 < \nu(H^c) < \infty$. Let χ_t be as in the Decomposition Theorem. Since χ_t are concentrated on H, by the Perturbation Formula applied to (4) we obtain

$$\mu_t(H^c) \geqslant \int_0^t \chi_{t-u} * \nu_{|H^c} * \chi_u(H^c)\, du = t\nu(H^c) > 0,$$

which shows that the 0-1 law does not hold in this case.

Differentiation of transition probabilities. In this section we obtain some results on differentiability of transition probabilities induced by $(\mu_t)_{t>0}$ on the homogeneous space G/H. We indicate some basic results, using semigroups technique. Results of this kind for general Markov chains were obtained by different methods by Doblin (see [3]).

Assume that $(\mu_t)_{t>0}$ is a fixed weakly continuous symmetric convolution semigroup of probability measures on G. By $(X(t))_{t>0}$ we denote a homogeneous process on G with right independent increments and one-dimensional distributions μ_t, i.e. such that for every $0 \leqslant t_1 < t_2 < \ldots < t_k$ the increments

$$X(t_1),\, X(t_1)^{-1}X(t_2),\, \ldots,\, X(t_{k-1})^{-1}X(t_k)$$

are independent G-valued random variables with distributions given by $\mu_{t_1}, \mu_{t_2-t_1}, \ldots, \mu_{t_k-t_{k-1}}$. Assume that H is a Borel subgroup of G. Let $\pi\colon G \to G/H$ be the canonical mapping onto the right cosets space with the σ-field generated by π from the Borel σ-algebra on G. It is not difficult to see that $\pi(X(t))_{t>0}$ is a Markov process on G/H. For a measurable subset $D \subseteq G/H$ and a coset Hx we obtain

$$P_u(Hx, D) = P\big(HX(t+u)\in D \mid HX(t) = Hx\big)$$
$$= P\big(HX(t)\big(X(t)^{-1}X(t+u)\big)\in D \mid HX(t) = Hx\big)$$
$$= P\big(Hx\,X(u)\in D\big) = \mu_u\{y;\ Hxy\in D\}.$$

When $D = Hz$ we obtain

$$P_u(Hx, Hz) = \int \mathbf{1}_{Hz}(xy)\mu_u(dy)$$
$$= T_{\mu_u}\mathbf{1}_{Hz}(x) = T_{\mu_u}\mathbf{1}_{Hz}(Hx) = \mu_u(x^{-1}Hz).$$

Observe that $P_u(Hx, Hz) > 0$ for some $u > 0$ yields, by symmetry, $\mu_t(z^{-1}Hz) > 0$ and $\mu_t(x^{-1}Hx) > 0$ for all $t > 0$. Now, we state and prove one result concerning differentiability of the above transition probabilities on the homogeneous space G/H.

PROPOSITION 2. *Let $(\mu_t)_{t>0}$ and H be as above. Define as above transition probabilities $P_u(Hx, Hz)$ on the space of right cosets G/H. There exists a subset $G_0 \subseteq G$ such that $HG_0 = G_0$ and the transition probabilities $P_u(Hx, Hz)$ are continuous on G_0, while for x or z from G_0^c we obtain $P_u(Hx, Hz) = 0$ for all $u > 0$. Then for every $x, z\in G$, $x \neq z$, there exist*

$$\lim_{t\downarrow 0}(1/t)[1-P_t(Hx, Hx)] = q(x), \qquad \lim_{t\downarrow 0}(1/t)P_t(Hx, Hz) = q(x, z) < \infty,$$

and $q(x) = \infty$, whenever $x\in G_0^c$, while $q(x) < \infty$ for $x\in G_0$; if either x or z are in G_0^c, then $q(x, z) = 0$.

Proof. Define

$$G_0 = \{x\in G;\ \mu_t(x^{-1}Hx) > 0 \text{ for some } t > 0\}.$$

Let us note that if $z\in G_0$, then $Hz \subseteq G_0$. Moreover, if $P_u(Hx, Hz) = \mu_u(x^{-1}Hz) > 0$, then we have $\mu_{2u}(x^{-1}Hx) > 0$ and $\mu_{2u}(z^{-1}Hz) > 0$, so $x, z\in G_0$. Thus, we have obtained $P_u(Hx, Hz) = 0$ if either z or x is in G_0^c.

On the other hand, if $x_0\in G_0$, then by the $L^1(\mu)$ method, applied for $x_0^{-1}Hx_0$ instead of H, we obtain $\mu_t(x_0^{-1}Hx_0) \to 0$ as $t\downarrow 0$. This yields that μ_t is uniformly continuous on bounded measurable functions on G_0/H. Indeed, for a bounded measurable function g which is constant on right cosets of H and $x_0\in G_0$ we obtain:

$$|T_{\mu_t}g(Hx_0)-g(Hx_0)| = \Big|\int_G g(Hx_0y)\mu_t(dy)-g(Hx_0)\Big|$$
$$\leqslant \Big|\int_{x_0^{-1}Hx_0} g(Hx_0y)\mu_t(dy)-g(Hx_0)\Big| + \int_{(x_0^{-1}Hx_0)^c} g(Hx_0y)\mu_t(dy)$$
$$\leqslant g(Hx_0)\mu_t\big((x_0^{-1}Hx_0)^c\big) + \int_{(x_0^{-1}Hx_0)^c} g(Hx_0y)\mu_t(dy)$$
$$\leqslant 2\|g\|_\infty \mu_t\big((x_0^{-1}Hx_0)^c\big) \to 0.$$

This means that the semigroup T_{μ_t} is uniformly continuous on bounded measurable functions on G_0/H. Hence the generator of the semigroup is bounded and defined on the whole space. This completes the proof of the proposition.

As an application we obtain the following result, playing an essential role in [1] and [4] (with different proofs therein):

DIFFERENTIATION LEMMA. *Let* $(\mu_t)_{t>0}$ *be a symmetric convolution semigroup on* G *and* H *a Borel subgroup of* G. *If* $\mu_t(H) > 0$ *for all* $t > 0$, *then* $\lim_{t\downarrow 0}(1/t)\,\mu_t(H^c)$ *exists and is finite.*

REFERENCES

[1] H. Byczkowska and T. Byczkowski, *Zero-one law for symmetric convolution semigroups of measures on groups*, J. Theoret. Probab. 289 (1998), pp. 633–643.

[2] T. Byczkowski and A. Hulanicki, *Gaussian measure of normal subgroups*, Ann. Probab. 11 (1983), pp. 685–691.

[3] M. Loève, *Probability Theory*, Part II, Springer, New York 1978.

[4] E. Siebert, *Decomposition of convolution semigroups on Polish groups and zero-one law*, Hokkaido Math. J. 16 (1987), pp. 235–255.

Institute of Mathematics
Wrocław Technical University
50-370 Wrocław, Poland

Received on 11.11.1998