PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 19, Fasc. 2 (1999), pp. 429–439

ABSOLUTE CONTINUITY, QUADRATIC FORMS, AND CAUSAL PERTURBATIONS OF PRODUCT MEASURES

BY

MAREK KANTER (BERKELEY, CALIFORNIA)

Abstract. Necessary and sufficient conditions are given for the absolute continuity of a causally perturbed product measure with respect to the non-perturbed measure. When the perturbation is linear, these conditions involve the convergence of a quadratic form of independent random variables. The convergence of this form is studied when the independent random variables are symmetric or strictly stable.

AMS 1991 Subject Classification: Primary: 60B10, 60B11.

Key words and phrases: Absolute continuity of measures, quadratic forms, stable random variables, symmetric random variables, product measures.

1. Introduction. This paper has two goals. Our first goal is to study the pairwise absolute continuity of the measures P_ψ on R^J generated by the discrete time stochastic sequence

$$(1.1) \qquad Y_j = \begin{cases} X_0 & \text{if } j = 0, \\ X_j + \psi_{j-1}(X_0, \ldots, X_{j-1}) & \text{if } j \geqslant 1,\ j \in J. \end{cases}$$

Here R stands for the real line, J stands for the non-negative integers, $X = (X_n\colon n \in J)$ is an infinite sequence of independent real random variables, and $\psi = (\psi_j\colon j \in J)$ is an infinite sequence of Borel measurable functions such that ψ_j maps R^{j+1} into R. (If $\psi_j \equiv 0$ for all $j \in J$, then the corresponding measure is denoted by P_0.) To accomplish our first goal we shall determine criteria for $P_0 \succ P_\psi$ and for $P_0 \approx P_\psi$, where $P_0 \succ P_\psi$ means that P_ψ is absolutely continuous with respect to P_0 while $P_0 \approx P_\psi$ means that $P_0 \succ P_\psi$ and $P_\psi \succ P_0$.

Given further technical conditions on the distribution of X, we shall prove that $P_0 \succ P_\psi$ if and only if

$$(1.2) \qquad \sum_{j \in J} \psi_j^2(X_0, \ldots, X_j) < \infty \quad \text{a.s.}$$

(An almost sure probability statement with no measure specified, such as (1.2), refers to a basic probability measure P on an underlying probability space Ω sufficiently rich to accommodate all random variables under consideration.) The extra conditions on X are that the random variables X_n have non-zero density functions f_n with uniformly bounded Fisher information and that there exist numbers $\theta > 0$, $p > 0$, $N > 0$ such that

$$P[|X_n| \leqslant \theta] \geqslant p \quad \text{for } n \geqslant N,\ n \in J. \tag{1.3}$$

The second goal of this paper is to establish criteria under which (1.2) holds when the functions ψ_j are linear. These criteria will provide a simple application of our results regarding absolute continuity. We shall also apply a dichotomy theorem in Kanter [7] which states that $P_0 \succ P_\psi$ implies $P_0 \approx P_\psi$ if the functions ψ_j are linear and the densities f_n are non-zero. The causal condition that ψ_j depends on $(X_0, \ldots, X_j)$ is not needed either in the dichotomy theorem or in any of the following results regarding (1.2) in the linear case.

Suppose $X = (X_n\colon n \in J)$ is a sequence of independent symmetrically distributed non-trivial random variables with values in R. Let $B = \{b_{jn}\colon j, n \in J, b_{jn} \in R\}$ be a matrix such that the limit

$$\langle b_j, X\rangle = \lim_{N\to\infty} \sum_{n=0}^{N} b_{jn} X_n \text{ a.s.} \tag{1.4}$$

exists for all j in J. (We identify $\psi_j(X)$ with $\langle b_j, X\rangle$.) We shall prove that

$$\sum_{j\in J} \langle b_j, X\rangle^2 < \infty \text{ a.s.} \tag{1.5}$$

if and only if

$$\sum_{n\in J} \|B_n\|^2 X_n^2 < \infty \text{ a.s.,} \tag{1.6}$$

where $B_n = (b_{jn}\colon j \in J)$ and $\|B_n\|$ is the norm of B_n in $l^2(J)$. An explicit criterion for the validity of (1.6) is

$$\sum_{n\in J} \varrho_n(\|B_n\|^2) < \infty, \tag{1.7}$$

where $\exp(-\varrho_n(t)) = E(\exp(-tX_n^2))$ for $t \geqslant 0$.

We shall also show that (1.6) holds if and only if (1.4) and (1.5) both do, when the hypothesis of symmetry on X is replaced by strict stability of index $\alpha \in (0, 2]$. In that case (1.7) is rewritten as

$$\sum_{n\in J} \|B_n\|^\alpha < \infty. \tag{1.8}$$

Relation to previous work. Kadota and Shepp [3] considered the measures P_ψ in the case when $(X_n\colon n \in J)$ was a sequence of independent standard normal random variables and $(\psi_j\colon j \in J)$ was a general causal sequence of functions as in (1.1). They proved the equivalence of (1.2) with the condition that $P_0 \succ P_\psi$ and established various other related results in the Gaussian case. The equivalence of (1.8) to (1.4) and (1.5) when $(X_n\colon n \in J)$ is a sequence of independent, identically distributed, non-trivial random variables with symmetric stable distribution of index $\alpha \in (0, 2]$ can be gleaned from Linde [8]. We note that most papers concerning the convergence of quadratic functions of independent random variables do not stipulate non-negative definiteness as we do in (1.5). (For example, there is no such stipulation in Cambanis et al. [1].)

Organization. The rest of this paper is divided into three sections. In Section 2 we will establish the connection between (1.2) and the condition that $P_0 \succ P_\psi$. In Section 3 we will give criteria for (1.2) when the functions ψ_j are linear (not necessarily causal), while the random variables $(X_n\colon n \in J)$ are symmetric. In Section 4 we will extend the results in Section 3 to the case when the random variables $(X_n\colon n \in J)$ are strictly stable, not necessarily symmetric.

2. Causally perturbed product measures. In this section we study the pairwise absolute continuity of causally perturbed product measures. Given $\theta > 0$ and $x \in R$, let

$$h_\theta(x) = \begin{cases} x & \text{if } |x| \leqslant \theta, \\ \theta \operatorname{sign}(x) & \text{if } |x| > \theta. \end{cases}$$

THEOREM 2.1. *Let $X = (X_n\colon n \in J)$ be a sequence of independent real random variables and let (θ, p, N) be a triplet of positive numbers satisfying* (1.3). *Let ψ and P_ψ be defined as in the Introduction. Then $P_0 \succ P_\psi$ implies* (1.2).

Proof. For $j \in J$ let $Z_j = \psi_j(X_0, \ldots, X_j)$ and let $Z'_j = h_1(Z_j)$. Define $S_j = \operatorname{sign}(Z_j)$. (If $Z_j = 0$, let $S_j = 0$.) Write $h(x)$ to stand for $h_{\theta+1}(x)$. It is easy to verify that

$$h(x) S_j \leqslant h(x + Z'_j) S_j \leqslant h(x + Z_j) S_j \tag{2.1}$$

for $x \in R$ and $j \in J$, since h is non-decreasing. It follows from the first inequality in (2.1) and the definition of h that

$$|Z'_j| I_{A(j)} \leqslant \big(h(X_{j+1} + Z'_j) - h(X_{j+1})\big) S_j, \tag{2.2}$$

where $A(j) = [\max(|X_{j+1}|, |X_{j+1} + Z'_j|) \leqslant \theta + 1]$ and I_A is the indicator function of the set A.

Now let F_j stand for the σ-field generated by $(X_0, \ldots, X_j)$ and let $\mu_j = E\big(h(X_j)\big)$. It follows from (2.2) that for $j \geqslant N-1$

$$p\,|Z'_j| \leqslant S_j \big(E\big(h(X_{j+1} + Z'_j) \,|\, F_j\big) - \mu_{j+1}\big) \text{ a.s.} \tag{2.3}$$

since $[|X_{j+1}| \leqslant \theta] \subset A(j)$. Furthermore

$$S_j \big(E\big(h(X_{j+1} + Z'_j) \,|\, F_j\big) - \mu_{j+1}\big) \leqslant S_j \big(E\big(h(Y_{j+1}) \,|\, F_j\big) - \mu_{j+1}\big) \text{ a.s.} \tag{2.4}$$

by setting $x = X_{j+1}$ in the second inequality in (2.1) and remembering $Y_{j+1} = X_{j+1} + Z_j$.

To prove the theorem, suppose by way of contradiction that $P_0 \succ P_\psi$ and (1.2) is false. Then $\sum (Z'_j)^2 = \infty$ with positive probability. Thus there exist $c = (c_j\colon j \in J)$ such that $c \in l^2(J)$, $c_j > 0$ for $j \in J$, and

$$P\left[\sum_{j\in J} c_j |Z'_j| = \infty\right] > 0. \tag{2.5}$$

Noting that

$$\mathrm{Var}\big(h(Y_{j+1})\big) \leqslant (\theta+1)^2 \quad \text{for } j \in J,$$

it follows that

$$\sum_{j\in J} c_j S_j\big(h(Y_{j+1}) - E\big(h(Y_{j+1} \mid F_j)\big)\big) = U$$

converges a.s. (The partial sums defining U are an L^2 bounded martingale with respect to the σ-fields F_j.)

On the other hand, the map from R^J into itself defined by (1.1) is invertible. Hence there exists a sequence $(\phi_j\colon j \in J)$ of Borel measurable functions such that ϕ_j maps R^{j+1} into R and

$$Z_j = \psi_j(X_0, \ldots, X_j) = \phi_j(Y_0, \ldots, Y_j) \text{ a.s.} \tag{2.6}$$

for $j \in J$. Arguing as was done to define U, note that

$$\sum_{j\in J} c_j \operatorname{sign}\big(\phi_j(X_0, \ldots, X_j)\big)\big(h(X_{j+1}) - \mu_{j+1}\big)$$

converges a.s. Use the hypothesis $P_0 \succ P_\psi$ to conclude that

$$\sum_{j\in J} c_j S_j\big(h(Y_{j+1}) - \mu_{j+1}\big) = V$$

converges a.s. since $S_j = \operatorname{sign}\big(\phi_j(Y_0, \ldots, Y_j)\big)$. Subtracting U from V it follows that

$$\sum_{j\in J} c_j S_j\big(E\big(h(Y_{j+1}) \mid F_j\big) - \mu_{j+1}\big)$$

converges a.s. But this is in contradiction to (2.3), (2.4), and (2.5). ∎

Remark 2.1. The key inequality (2.3) is essentially present in Kanter [6], which deals with the special case of Theorem 2.1 when ψ_j is a sequence of constants. (See also Kanter [5].)

We now present a partial converse of Theorem 2.1.

THEOREM 2.2. *Let $X = (X_n\colon n \in J)$ be a sequence of independent real random variables such that X_n has density $f_n(x) > 0$ for all $n \in J$, $x \in R$. Suppose also that there exists a finite number i_F such that the Fisher information $i_F(f_n)$ of f_n satisfies*

$$i_F(f_n) < i_F \quad \textit{for } n \in J. \tag{2.7}$$

Then (1.2) implies that $P_0 \succ P_\psi$.

Proof. Refer to (2.6) and note that (1.1) can be rewritten as

$$Y_j = \begin{cases} X_0 & \text{if } j = 0, \\ X_j + \phi_{j-1}(Y_0, \ldots, Y_{j-1}) & \text{if } j \geqslant 1, j \in J. \end{cases}$$

Let $p_j(z_0, \ldots, z_j)$ stand for the Radon–Nikodym derivative $dP_\psi^{(j)}/dP_0^{(j)}$ evaluated at the point $(z_0, \ldots, z_j) \in R^{j+1}$. (Here $P_\psi^{(j)}$ stands for P_ψ restricted to the σ-algebra generated by the functions $(e_0, \ldots, e_j)$, where $e_n(z) = z_n$ for $z = (z_0, \ldots, z_n, \ldots) \in R^J$.) For $j \in J$ and $(z_0, \ldots, z_{j+1}) \in R^{j+2}$, let

$$r_{j+1}(z_0, \ldots, z_{j+1}) = p_{j+1}(z_0, \ldots, z_{j+1})/p_j(z_0, \ldots, z_j).$$

It is clear that

$$r_{j+1}(z_0, \ldots, z_{j+1}) = f_{j+1}\big(z_{j+1} - \phi_j(z_0, \ldots, z_j)\big)/f_{j+1}(z_{j+1}). \tag{2.8}$$

We define

$$h_j(z_0, \ldots, z_j) = 1 - \int \big(f_{j+1}\big(z - \phi_j(z_0, \ldots, z_j)\big) f_{j+1}(z)\big)^{1/2}\, dz \qquad \text{for } j \in J.$$

We can write

$$h_j(z_0, \ldots, z_j) = \int |\hat{q}_{j+1}(s)|^2 \big(1 - \exp\big(is\, \phi_j(z_0, \ldots, z_j)\big)\big)\, ds,$$

where $q_j(x) = \big(f_j(x)\big)^{1/2}$ and

$$\hat{q}_j(s) = (2\pi)^{-1/2} \int e^{isx} q_j(x)\, dx$$

is the Fourier transform of q_j. Since $|\hat{q}_j(s)|^2$ is an even function of s, it follows that

$$0 \leqslant h_j(z_0, \ldots, z_j) \leqslant 2^{-1} \big(\int |s q_{j+1}(s)|^2\, ds\big) \big(\phi_j(z_0, \ldots, z_j)\big)^2.$$

As shown in Shepp [9], we may identify $i_F(f_n)$ with $4 \int |s\hat{q}_n(s)|^2\, ds$ for $n \in J$. Thus

$$h_j(z_0, \ldots, z_j) \leqslant (i_F/8)\big(\phi_j(z_0, \ldots, z_j)\big)^2. \tag{2.9}$$

We can use (2.8) to write

$$h_j(z_0, \ldots, z_j) = 1 - E_0\big((r_{j+1})^{1/2} \mid e_n = z_n,\ 0 \leqslant n \leqslant j\big) \tag{2.10}$$

for P_0 almost all z in R^J, where $E_0(U \mid A)$ denotes the conditional expectation of U given A with respect to P_0. Now (2.10) involves only $j+1$ coordinates and $P_0^{(j)} \succ P_\psi^{(j)}$ for all $j \in J$. An easy argument shows that the distribution of the random variables

$$\big(1 - E_0\big((r_{j+1})^{1/2} \mid e_n = z_n,\ 0 \leqslant n \leqslant j\big) \colon j \in J\big)$$

under P_ψ is the same as the distribution of the random variables $\big(h_j(Y_0, \ldots, Y_j) \colon j \in J\big)$ on (Ω, P). Recalling (2.6), note that

$$\sum_{j \in J} \big(\psi_j(X_0, \ldots, X_j)\big)^2 = \sum_{j \in J} \big(\phi_j(Y_0, \ldots, Y_j)\big)^2 \text{ a.s.}$$

We conclude from (1.2) and (2.9) that the series

$$\sum_{j\in J} 1-E_0\left((r_{j+1})^{1/2} \mid e_n = z_n,\ 0 \leqslant n \leqslant j\right)$$

converges for P_ψ almost all $z \in R^J$. Applying Theorem 4 in Shiryayev [10], p. 496, it follows that $P_0 \succ P_\psi$, since $P_0^{(j)} \succ P_\psi^{(j)}$ for all $j \in J$. ∎

Remark 2.2. It is natural to ask if Theorems 2.1 and 2.2 hold in the "two-sided" case when J is replaced by L, the set of all integers. In that case (1.1) is replaced by the equation

$$Y_j = X_j + \psi_{j-1}(\ldots, X_{j-1}) \quad \text{for } j \in L,$$

where $X = (X_j\colon j \in L)$ is a sequence of independent random variables indexed by L. The proofs of Theorems 2.1 and 2.2 no longer go through, because the analogue to (2.6) may not hold in the two-sided case. More explicitly, it is not known if there exists a sequence $(\phi_j\colon j \in J)$ of Borel functions such that

$$\psi_j(\ldots, X_j) = \phi_j(\ldots, Y_j) \text{ a.s.} \quad \text{for } j \in L.$$

(Note that the domains of ϕ_j and ψ_j in the two-sided case are the sets $D_j = \{x = (x_n)\colon x_n \in R,\ n \in L,\ n \leqslant j\}$.)

We can combine Theorems 2.1 and 2.2, thereby summarizing our results in this section.

COROLLARY 2.1. *Let $X = (X_n\colon n \in J)$ be a sequence of independent random variables satisfying* (1.3) *for some triplet (θ, p, N) of strictly positive numbers. Suppose also that X_n has density $f_n(x) > 0$ satisfying* (2.7). *Then $P_0 \succ P_\psi$ if and only if* (1.2) *holds.*

3. The symmetric linear case. For statistical investigations it is more convenient to know that $P_0 \approx P_\psi$ than $P_0 \succ P_\psi$. As an application of the results in Section 2 we will start this section by giving conditions for $P_0 \approx P_\psi$ based on the assumption that Y_j in (1.1) are linearly dependent on $X = (X_n\colon n \in J)$. We will then use these conditions as motivation for studying (1.5).

LEMMA 3.1. *Suppose $X = (X_n\colon n \in J)$ is a sequence of independent random variables all with non-zero density functions. Suppose further that there exists a matrix $B = (b_{jn}\colon j, n \in J,\ b_{jn} \in R)$ such that for $j \in J$, $\psi_j(x) = \langle b_j, x\rangle$ for P_0 almost all x, where $\langle b_j, x\rangle$ is defined as in* (1.4). *Then $P_0 \succ P_\psi$ implies $P_0 \approx P_\psi$.*

Proof. Given a subgroup $D \subset R^J$, refer to the definition of D-ergodic and D-smooth measures on R^J in Kanter [7]. Upon examining the example considered in the first section of that paper, it can be deduced that there exists a collection of countable subgroups D_ψ of R^J, indexed by $\psi = (\psi_j\colon j \in J)$, such that P_ψ is D_ψ-ergodic and D_ψ-smooth. Now apply Theorem 2.1 in the same paper. ∎

Suppose the functions ψ_j are causal as in Theorem 2.1 and linear as in Lemma 3.1. Suppose further that the technical conditions (1.3) and (2.7) hold. It follows from Corollary 2.1 and Lemma 3.1 that $P_0 \approx P_\psi$ if and only if (1.5) holds. It is therefore of interest to study when (1.4) and (1.5) hold in order to present a specific application of our previous results. The causal condition that ψ_j depends on $(x_0, \ldots, x_j)$ plays no role in this application and is therefore dropped.

We shall consider this question in the remainder of this section under the hypothesis that X is symmetric. This hypothesis allows us to make use of a classical theorem due to Kahane [4], pp. 24–25, which involves a sequence $(U_n\colon n \in J)$ of independent, symmetric, ± 1 valued, random variables and a sequence $(h_n\colon n \in J)$ of vectors in a Hilbert space H. Kahane's theorem states that $\sum U_n h_n$ converges a.s. in H if and only if $\sum \|h_n\|^2$ is finite.

THEOREM 3.1. *Let $X = (X_n\colon n \in J)$ be a sequence of independent, symmetrically distributed, non-trivial, real random variables. Let $B = (b_{jn}\colon j, n \in J, b_{jn} \in R)$. Then* (1.4) *and* (1.5) *hold if and only if* (1.6) *does.*

Proof. Given any finite subsets S and T of J, write

$$\sum_{j \in S} \langle b_j(T), X\rangle^2 = \Big\|\sum_{n \in T} X_n B_n(S)\Big\|^2 \text{ a.s.,} \tag{3.1}$$

where $B_n(S) = \big(b_{jn} I_S(j)\colon j \in J\big)$ and $b_j(T) = \big(b_{jn} I_T(n)\colon n \in J\big)$. (Note that $B_n(S) \in l^2(J)$ for $n \in J$.)

Suppose first that (1.6) holds. It follows by conditioning on $(|X_n|\colon n \in J)$ and using Kahane's theorem that for any integer $N \in J$ the random vector

$$W_N = \sum_{n \geqslant N} X_n B_n$$

converges a.s. in $l^2(J)$. Furthermore, $\big((W_N, G_N)\colon N \in J\big)$ is a reverse martingale in $l^2(J)$, where G_N is the σ-field generated by $(X_n\colon n \geqslant N)$. Standard arguments show that W_N converges a.s. to a constant vector $W_\infty \in l^2(J)$. On the other hand, W_N is symmetrically distributed, hence $W_\infty = 0$. Thus

$$\lim_{N \to \infty} W_N = 0 \text{ a.s.} \tag{3.2}$$

We now note that, for fixed T, the left-hand side of (3.1) is increasing with S. Letting $S \uparrow J$, we get

$$\sum_{j \in J} \langle b_j(T), X\rangle^2 = \Big\|\sum_{n \in T} X_n B_n\Big\|^2 \text{ a.s.} \tag{3.3}$$

for any finite subset T of J. It follows from (3.2) and (3.3) that (for fixed j) the sequence of random variables $\langle b_j(T), X\rangle$ is Cauchy in probability as $T \uparrow J$. This proves (1.4). Furthermore,

$$\sum_j \langle b_j, X\rangle^2 \leqslant \liminf_{T \uparrow J} \sum_j \langle b_j(T), X\rangle^2$$

by Fatou's lemma, and

$$\lim_{T\uparrow J} \sum_{j\in J} \langle b_j(T), X\rangle^2 = \|W_0\|^2 \text{ a.s.},$$

using (3.2) and (3.3). This proves (1.5).

Suppose now that (1.4) and (1.5) hold. Let $T\uparrow J$ in (3.1) and apply (1.4) to get

$$\sum_{j\in S} \langle b_j, X\rangle^2 = \Big\|\sum_{n\in J} X_n B_n(S)\Big\|^2 \text{ a.s.} \tag{3.4}$$

Write $S'-S$ to stand for the complement of S in S' and use (1.5) to obtain

$$\begin{aligned}&\lim_{S\uparrow J} \sup_{S'\supset S} \Big\|\sum_{m\in J} X_m B_m(S) - \sum_{n\in J} X_n B_n(S')\Big\|^2 \\ &\qquad = \lim_{S\uparrow J} \sup_{S'\supset S} \Big\|\sum_{n\in J} X_n B_n(S'-S)\Big\|^2 = \lim_{S\uparrow J} \sup_{S'\supset S} \sum_{j\in S'-S} \langle b_j, X\rangle^2 = 0 \text{ a.s.}\end{aligned}$$

It follows that $\sum X_n B_n(S)$ is a.s. Cauchy in $l^2(J)$ as $S\uparrow J$. Thus

$$\lim_{S\uparrow J} \sum_{n\in J} X_n B_n(S) = \sum_{n\in J} X_n B_n \text{ a.s.}$$

with $B_n \in l^2(J)$ for $n\in J$ as a consequence of Proposition 4.1 in Cambanis et al. [1]. We now apply Kahane's theorem to deduce (1.6). ∎

The convergence of (1.6) can be given more explicitly.

THEOREM 3.2. *Let $X=(X_n\colon n\in J)$ be a sequence of independent random variables. Then* (1.6) *and* (1.7) *are equivalent conditions on X.*

Proof. First note that for any sequence c_n of non-negative real numbers

$$E\Big(\exp\big(-\sum_{n\in J} c_n X_n^2\big)\Big) = \exp\big(-\sum_{n\in J} \varrho_n(c_n)\big). \tag{3.5}$$

Letting $c_n = \|B_n\|^2$, it follows immediately that (1.6) implies (1.7). Conversely, if (1.7) holds, then clearly the sum in (1.6) converges with positive probability as a consequence of (3.5). The Hewitt–Savage 0-1 law now guarantees that a.s. convergence holds in (1.6). ∎

4. The strictly stable linear case. In the previous section we established (1.5) as a criterion for $P_0 \approx P_\psi$ in the linear case, and we analyzed this criterion in the symmetric case. In this section we drop the hypothesis of symmetry and study the same criterion when X is strictly stable. (See Feller [2] for background on stable random variables.) We start by stating a known result for symmetric stable random variables.

THEOREM 4.1. *Suppose $X=(X_n\colon n\in J)$ is a sequence of symmetric, independent, identically distributed, non-trivial real random variables, stable of index $\alpha\in(0,2]$. Let B be given as in Theorem* 3.1. *Then* (1.4) *and* (1.5) *hold if and only if* (1.8) *does.*

There are several ways to verify Theorem 4.1. One way is to deduce it from Corollary 7.8.3 in Linde [8], p. 154. Another way is to apply Theorem 3.1 in conjunction with Exercise 1 in Kahane [4], p. 29.

The most elegant way to verify Theorem 4.1 is to deduce the case $\alpha \in (0, 2)$ from the case $\alpha = 2$. First note that when $\alpha = 2$, Theorem 4.1 is well known and easy to verify. On the other hand, if $\alpha \in (0, 2)$, then write $X_n = V_n^{1/2} Z_n$ for $n \in J$, where

(i) $V = (V_n\colon n \in J)$ and $Z = (Z_n\colon n \in J)$ are independent of each other,

(ii) Z is an i.i.d. sequence of standard normal random variables,

(iii) V is an i.i.d. sequence of positive random variables, stable of index $\alpha/2$.

(See Feller [2], p. 596.) If we condition by V and apply Theorem 4.1 in the case $\alpha = 2$, then we see that (1.4) and (1.5) hold if and only if

$$\sum_j \sum_n b_{jn}^2 V_n < \infty \text{ a.s.} \tag{4.1}$$

Now use the strict stability of V to see that (4.1) is equivalent to (1.8).

THEOREM 4.2. *Let $(X_n\colon n \in J)$ be a sequence of independent, identically distributed, real, non-trivial random variables, strictly stable of index $\alpha \in (0, 2]$. Let B be given as in Theorem* 3.1. *Then* (1.4) *and* (1.5) *hold if and only if* (1.8) *does.*

Proof. Suppose first that (1.4) and (1.5) hold. It follows that the functions

$$B_{jN}(x) = \sum_{n=0}^{N} b_{jn} x_n \quad \text{for } j \in J$$

converge for P_0 almost all x in R^J as $N \to \infty$. We let A_j stand for the linear set of x in R^J such that $\lim_{N\to\infty} B_{jN}(x)$ exists. Given $x \in R^J$, define

$$\langle b_j, x\rangle = \begin{cases} \lim_{N\to\infty} B_{jN}(x) & \text{for } x \in A_j, \\ +\infty & \text{for } x \notin A_j. \end{cases}$$

Let $B^{(2)}(x) = (\sum \langle b_j, x\rangle^2)^{1/2}$. It is easy to see that $B^{(2)}$ is a convex function from R^J into the extended non-negative real numbers $[0, \infty]$. Furthermore, $B^{(2)}(x) < \infty$ for P_0 almost all x in R^J as a consequence of (1.4) and (1.5).

It is clear that $B^{(2)}$ satisfies the triangle inequality

$$B^{(2)}(x \pm y) \leqslant B^{(2)}(x) + B^{(2)}(y) \quad \text{for } x, y \in R^J. \tag{4.2}$$

(Note $B^{(2)}$ is homogeneous of order 1 and satisfies $B^{(2)}(-x) = B^{(2)}(x)$ for $x \in R^J$.) Now let $\tilde{X}_n = X_n - X'_n$ for $n \in J$, where $X' = (X'_n\colon n \in J)$ is an independent copy of X. Let $\tilde{P}_0$ stand for the probability measure induced on R^J by the process $\tilde{X} = (\tilde{X}_n\colon n \in J)$. It follows from (4.2) that $B^{(2)}(\tilde{x}) < \infty$ for $\tilde{P}_0$ almost all $\tilde{x}$ in R^J. We may thus conclude that (1.8) holds as a consequence of Theorem 4.1 applied to the symmetric process $\tilde{X}$.

Conversely, if (1.8) holds, then so do (1.4) and (1.5) with $\tilde{X}$ substituted for X, by virtue of Theorem 4.1. It follows that $B^{(2)}(\tilde{x}) < \infty$ for $\tilde{P}_0$ almost all $\tilde{x}$

in R^J. Fubini's theorem establishes the existence of $x' \in R^J$ such that $B^{(2)}(X-x') < \infty$ a.s. Furthermore $B^{(2)}(X'-x') < \infty$ a.s., since X' is distributed like X. Apply (4.2) to write

$$B^{(2)}(X+X'-2x') \leqslant B^{(2)}(X-x')+B^{(2)}(X'-x').$$

Using the homogeneity of $B^{(2)}$, it follows that

$$B^{(2)}\big(2^{-1/\alpha}(X+X'-2x')\big) < \infty \ \text{ a.s.}$$

Now let $\delta = 1-\alpha^{-1}$. Note that $2^{-1/\alpha}(X+X')$ is distributed like X by hypothesis, hence $B^{(2)}(X-2^{\delta}x') < \infty$ a.s. Use (4.2) again to write

$$|1-2^{\delta}|\, B^{(2)}(x') \leqslant B^{(2)}(X-x')+B^{(2)}(-X+2^{\delta}x').$$

Now $1-2^{\delta} \neq 0$ if $\alpha \neq 1$; hence $B^{(2)}(X) < \infty$ (a.s.) in that case. (Note that

$$B^{(2)}(X) \leqslant B^{(2)}(X-x')+B^{(2)}(x'),$$

and both terms on the right-hand side have already been shown to be a.s. finite.) This proves (1.4) and (1.5) for $\alpha \neq 1$.

If $\alpha = 1$, then by strict stability there exists $c \in R$ such that $X_n = Y_n + c$ for all $n \in J$, where $Y = (Y_n\colon n \in J)$ is a sequence of independent, identically distributed, symmetric Cauchy random variables. (See Feller [2], equation (3.19), p. 570). Let $\hat{c}$ in R^J be defined by $\hat{c}_n = c$ for all n in J and note that

$$B^{(2)}(Y+\hat{c}-x') = B^{(2)}(X-x') < \infty \ \text{ a.s.}$$

It follows that $B^{(2)}(-Y+\hat{c}-x') < \infty$ a.s. since Y is symmetric. Furthermore,

$$2B^{(2)}(\hat{c}-x') \leqslant B^{(2)}(Y+\hat{c}-x')+B^{(2)}(-Y+\hat{c}-x').$$

Hence $B^{(2)}(Y) < \infty$ a.s. since

$$B^{(2)}(Y) \leqslant B^{(2)}(Y+\hat{c}-x')+B^{(2)}(x'-\hat{c}).$$

Now, note that if (1.8) holds with $\alpha = 1$, then

$$\sum_j \Big(\sum_n |b_{jn}|\Big)^2 < \infty$$

by the triangle inequality for $\|\cdot\|$. It follows immediately that $B^{(2)}(\hat{c}) < \infty$. Thus

$$B^{(2)}(X) = B^{(2)}(Y+\hat{c}) \leqslant B^{(2)}(Y)+B^{(2)}(\hat{c}) < \infty \ \text{ a.s.}$$

This proves (1.4) and (1.5) for $\alpha = 1$. ∎

Remark 4.1. If X and B are as in Theorem 4.2, then (1.4) holds if and only if $b_j \in l^{\alpha}(J)$ for all $j \in J$.

Acknowledgments. My thanks to Jack Feldman for a helpful conversation regarding ergodic theory, which led to Remark 2.2.

REFERENCES

[1] S. Cambanis, J. Rosiński and W. A. Woyczyński, *Convergence of quadratic forms in p-stable random variables and Θ_p-radonifying operators*, Ann. Probab. 20 (1985), pp. 885–897.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications* 2, 2nd edition, Wiley, New York 1971.

[3] T. T. Kadota and L. A. Shepp, *Conditions for absolute continuity between a certain pair of probability measures*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 16 (1970), pp. 250–260.

[4] J.-P. Kahane, *Some Random Series of Functions*, D. C. Heath & Co., Lexington, Mass., 1968.

[5] M. Kanter, *On distinguishing translates of measures*, Ann. Math. Statist. 40 (1969), pp. 1773–1777.

[6] — *Correction to "On distinguishing translates of measures"*, Ann. Probab. 3 (1975), pp. 189–190.

[7] — *Equivalence singularity dichotomies for a class of ergodic measures*, Math. Proc. Cambridge Philos. Soc. 81 (1977), pp. 249–252.

[8] W. Linde, *Probability in Banach Spaces. Stable and Infinitely Divisible Distributions*, Wiley, New York 1986.

[9] L. A. Shepp, *Distinguishing a sequence of random variables from a translate of itself*, Ann. Math. Statist. 36 (1965), pp. 1107–1112.

[10] A. N. Shiryayev, *Probability*, Springer, New York 1984.

1216 Monterey Avenue
Berkeley, California 94707
E-mail: MXK8@PGE.COM

Received on 27.5.1999