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THEORETICAL PREDICTION OF PERIODICALLY CORRELATED SEQUENCES

BY

ANDRZEJ MAKAGON* (HAMPTON, VA, AND WROCŁAW)

Abstract. The paper deals with a spectral analysis and prediction of periodically correlated (PC) sequences. In particular, a moving average representation of a predictor is obtained and its coefficients are described in the language of outer factors of spectral line densities of the sequence. A comprehensive and self-contained overview of the spectral theory of PC sequences is included. The developed technique is used to compute the spectrum and an optimal moving average representation of a PC solution to a PARMA system of equations.

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1. INTRODUCTION

Given a sequence $(x(n))$ in a complex Hilbert space H, the theoretical prediction deals with the problem of finding the best approximation of a future element $x(n)$ in the past: $M_x(m) = \overline{\mathrm{sp}}\{x(k)\colon\ k \leqslant m\}$, $m < n$, assuming that the spectral characteristics of the sequence are known.

In this paper we study the prediction problem for periodically correlated (PC) sequences. PC sequences are directly related to T-dimensional stationary sequences; each PC sequence with period T partitioned into blocks of size T produces a T-dimensional stationary sequence, and vice versa. However, due to a rather complex relationship between the spectra of these two sequences, the mentioned partitioning a PC sequence is not a convenient tool in the prediction analysis. Theory of PC sequences has developed its own technique based on the fact that a PC sequence is a trajectory of a unitary group evalu-

* Department of Mathematics, Hampton University, Hampton, VA, USA, and The Hugo Steinhaus Center for Stochastic Methods, Wrocław, Poland.

ated at a periodic sequence. In the paper we will explore this approach and use it to study prediction problem for PC sequences.

The prediction problem is solved completely for one-dimensional stationary sequences, and is fairly advanced for multidimensional stationary sequences; however, in the latter case no explicit expression for a predictor in terms of the spectral density of a sequence is known yet. The available descriptions are in terms of *conjugate analytic* factors of a spectral density. Because of the mentioned correspondence between PC and T-dimensional stationary sequences, we cannot expect more explicit solutions in PC case either.

The paper is self-contained and is organized as follows. Section 2 summarizes basic facts about PC sequences. Section 3 contains a short review of prediction results for T-dimensional stationary sequences and precise statement of the problem. In Section 4 we study a certain T-dimensional stationary sequence, induced by a PC sequence, which will be the main tool used in the paper. In Section 5 we derive regularity conditions and give a spectral description of the coefficients in a moving average representation of a predictor of a PC sequence. Section 6 contains a short discussion of other stationary sequences associated with a PC sequence. Section 7 contains an example showing how the induced sequence technique can be applied to find spectrum and an optimal moving average representation of a PARMA sequence.

Notation. In the paper H and K will stand for complex Hilbert spaces, $(\cdot, \cdot)$ will denote the inner product, $(x \mid M)$ will denote the orthogonal projection of $x \in H$ onto a closed linear subspace M of H, and $H^T = H \oplus H \oplus \ldots \oplus H$ will denote the direct sum of T copies of a Hilbert space H. The letters $\mathscr{Z}$ and $\mathscr{C}$ will stand for the sets of integers and complex numbers, respectively. The dual of $\mathscr{Z}$ is identified with $[0, 2\pi)$, the operations of addition and multiplication in $[0, 2\pi)$ will always be *modulo* 2π, unless otherwise is stated. T will always be a positive integer. $\mathscr{Z}_T$ will denote the set of congruence classes *modulo* T, that is the set $\{0, \ldots, T-1\}$ with addition *modulo* T; any time we write $k, j \in \mathscr{Z}_T$, this will indicate that the addition, subtraction, multiplication, etc. of k and j are *modulo* T. If $n \in \mathscr{Z}$, then $[n]$ and $Q(n)$ will denote the remainder and the quotient in division of n by T, so that $n = Q(n)T + [n]$, $0 \leqslant [n] < T$. The mapping $\mathscr{Z} \ni n \to [n] \in \mathscr{Z}_T$ is a homomorphism of the group $\mathscr{Z}$ onto the group $\mathscr{Z}_T$.

A *sequence* $(x(n))$ (or (x_n)) in H is a function from $\mathscr{Z}$ into H. A sequence $(x(n))$ is said to be *T-periodic* if $x(n+T) = x(n)$ for all $n \in \mathscr{Z}$. T-periodic sequences are in one-to-one correspondence with functions on $\mathscr{Z}_T$ via the mapping $(x(n)) \leftrightarrow x([n])$, and they will be often identified in the paper without mentioning. If $(s(n))$ is T-periodic, then ten sum $\sum_{k=r}^{r+T-1} s(k)$ does not depend on r and this also will be very often employed in the paper without warning.

Matrices will be denoted by $[X^{j,k}]$ or by bold-face letters. Multiplication of matrices is a standard matrix multiplication. We allow $X^{j,k}$ to be elements of a Hilbert space, and then the product of entries is an inner product in H, that is

$$[X^{j,k}][Y^{j,k}] = [\sum_p (X^{j,p}, Y^{p,k})].$$

If $A = [A^{j,k}]$ is a matrix, then A^* denotes its conjugate matrix, that is the transposed matrix (with conjugate entries, if $A^{j,k} \in \mathscr{C}$).

$L^2(\mu; H)$ will stand for a Hilbert space of H-valued functions on $[0, 2\pi)$ which are square-integrable with respect to a nonnegative measure μ. If μ is Lebesgue measure, then the letter μ will be dropped. If $f \in L^2(H)$, then its *Fourier transform* $\hat{f}$ is a square-summable sequence defined by

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-int} f(t)\,dt, \qquad n \in \mathscr{Z}. \tag{1}$$

With this normalization, $f(t) = (1/\sqrt{2\pi}) \sum_k \hat{f}(k) e^{ikt}$ in $L^2(H)$. A function $f \in L^2(H)$ is called *conjugate analytic* if $\hat{f}(n) = 0$ for all $n > 0$. The set of all H-valued square-integrable conjugate analytic functions is denoted by $L^2_-(H)$. A matrix-valued function is *conjugate analytic* if its coordinate functions are conjugate analytic.

If $(s(n))$ is a T-periodic sequence in H, then the *discrete Fourier transform* of $(s(n))$ is a T-periodic sequence defined by

$$\tilde{s}(n) = \frac{1}{T} \sum_{k=0}^{T-1} e^{-2\pi ink/T} s(k), \qquad n \in \mathscr{Z}. \tag{2}$$

The *inverse* discrete Fourier transform is given by

$$s(n) = \sum_{k=0}^{T-1} e^{2\pi ink/T} \tilde{s}(k), \qquad n \in \mathscr{Z}.$$

The paper deals with stochastic sequences of zero mean and finite variance complex random variables, which are represented here as sequences in a complex Hilbert space H. If $(x(n))$ is a stochastic sequence, then the function $K_x(n, m) = (x(n), x(m))$ is referred to as the *correlation function* of $(x(n))$. A sequence $(x(n))$ is called *harmonizable* if there is an H-valued finite Borel complex measure F on $[0, 2\pi) \times [0, 2\pi)$ such that

$$K_x(m, n) = \int_0^{2\pi} \int_0^{2\pi} e^{i(ms-nt)} F(ds, dt). \tag{3}$$

An H^T-valued sequence $[X^k(n)]$ will be called a *T-dimensional sequence in H*. T-dimensional sequences will be looked upon as column vectors ($T \times 1$ matrices). Coordinates of vectors and matrices are numbered from 0. Two T-dimensional sequences $[X^k(n)]$ and $[Y^k(n)]$ are said to be *equivalent* if there is an isometry that maps $X^k(n)$ onto $Y^k(n)$, that is if $(X^k(m), X^j(n)) = (Y^k(m), Y^j(n))$ for all $k, j = 0, \ldots, T-1$ and $n, m \in \mathscr{Z}$.

A T-dimensional sequence $[X^k(n)]$ is called *stationary* if for every $k, j = 0, \ldots, T-1$ the cross-correlation function $(X^k(m), X^j(n))$ depends only on $m-n$. The correlation function of a T-dimensional stationary sequence is a $T \times T$-matrix valued sequence defined as $[K^{j,k}(n)] = [X^k(n)][X^k(0)]^*$, that is $K^{j,k}(n) = (X^j(n), X^k(0))$. A T-dimensional sequence $[X^k(n)]$ is stationary iff there is a unitary operator U in $M_X = \overline{\mathrm{sp}}\{X^k(m)\colon k = 0, \ldots, T-1, m \in \mathscr{Z}\}$ such that $X^k(n) = U^n X^k(0)$ for every $n \in Z$ and $k = 0, \ldots, T-1$. The operator U is referred to as the *shift operator* of $[X^k(n)]$. If $[X^k(n)]$ is a T-dimensional stationary sequence, then writing $U^n = \int_0^{2\pi} e^{inx} E(dx)$ we obtain

$$K^{j,k}(n) = \int_0^{2\pi} e^{inx} \Gamma^{j,k}(dx), \tag{4}$$

where $\Gamma^{j,k}(\Delta) = (E(\Delta) X^j(0), X^k(0))$, $k, j = 0, \ldots, T-1$, is a complex measure on $[0, 2\pi)$. The $T \times T$-matrix measure $\Gamma(\Delta) = [\Gamma^{j,k}(\Delta)]$ is called the *spectral measure* of a T-dimensional stationary sequence $[X^k(n)]$. If Γ is absolutely continuous with respect to a nonnegative σ-finite measure μ, then its Radon-Nikodym derivative $d\Gamma/d\mu(t)$ is μ-almost everywhere a nonnegative definite $T \times T$-matrix. Every $T \times d$-matrix valued function $A(t)$ with coordinates in $L^2(\mathscr{C})$ such that

$$\frac{d\Gamma}{d\mu}(t) = A(t) A(t)^*, \ \mu\text{-a.e.},$$

will be called a *square root* of the density $d\Gamma/d\mu(t)$.

If $A(\cdot)$ is a square root of $d\Gamma/d\mu$, then the rows $A^0(\cdot), \ldots, A^{T-1}(\cdot)$ of $A(\cdot)$ are elements of the Hilbert space $L^2(C^d)$ and

$$\int_0^{2\pi} e^{i(n-m)t} (A^j(t), A^k(t)) \mu(dt) = K^{j,k}(n-m), \quad j, k = 0, \ldots, T-1, \ n, m \in \mathscr{Z},$$

where the inner product $(A^j(t), A^k(t))$ is in $\mathscr{C}^d$. Therefore, any T-dimensional stationary sequence $[X^k(n)]$ is equivalent to an $L^2(C^d)$-valued T-dimensional stationary sequence $[Y^k(n)]$ defined by

$$Y^k(n)(\cdot) = e^{in\cdot} A^k(\cdot), \quad k = 0, \ldots, T-1, \ n \in \mathscr{Z}, \tag{5}$$

where $A(t)$ is a square root of the density $d\Gamma/d\mu$ of $[X^k(n)]$. This is known as *Kolmogorov's Isomorphism Theorem.*

2. PC SEQUENCES: BASIC FACTS

Theory of PC sequences was set by Gladyshev in [5]. In this section we review Gladyshev's results concerning the structure and the spectrum of PC sequences.

DEFINITION 2.1. A stochastic sequence $(x(n))$ is called *periodically correlated* (PC) with period T if $K_x(n, m) = K_x(n+T, m+T)$ for every $n, m \in \mathscr{Z}$.

If $(x(n))$ is PC with period T, then the function $K_x(n+p, n)$ is T-periodic in n for every $p \in \mathscr{Z}$, and hence

$$K_x(n+p, n) = \sum_{j=0}^{T-1} e^{2\pi ijn/T} a_j(p), \tag{6}$$

where

$$a_j(p) = (1/T) \sum_{n=0}^{T-1} e^{-2\pi ijn/T} K_x(n+p, n). \tag{7}$$

Let us put $M_x = \overline{\mathrm{sp}}\{x(n): n \in \mathscr{Z}\}$. If $(x(n))$ is PC with period T, then the mapping $V: x(n) \to x(n+T)$, $n \in \mathscr{Z}$, extends linearly to a unitary operator $V: M_x \xrightarrow{\text{onto}} M_x$. The operator V is called the *T-shift operator* of $(x(n))$. If U is a *unitary T-th root* of V, that is U is a unitary operator in a space $K \supseteq M_x$ such that $U^T = V$ on M_x, then $p(n) = U^{-n} x(n)$ is a T-periodic sequence in K and

$$x(n) = U^n p(n), \qquad n \in \mathscr{Z}. \tag{8}$$

Define $W^q(0) = (1/T)\sum_{n=0}^{T-1} e^{-2\pi inq/T} p(n)$ and $W^q(n) = U^n W^q(0)$. Then $[W^q(n)]$ is a T-dimensional stationary sequence in K and

$$x(n) = \sum_{q=0}^{T-1} e^{2\pi inq/T} W^q(n), \qquad n \in \mathscr{Z}. \tag{9}$$

Conversely, if $(x(n))$ has the form (9), then $x(n) = U^n p(n)$, where U is the shift of $[W^k(n)]$ and $(p(n))$ is a T-periodic function in K defined by

$$p(n) = \sum_{q=0}^{T-1} e^{2\pi inq/T} W^q(0). \tag{10}$$

This proves the following characterization of PC sequences:

PROPOSITION 2.2 (Gladyshev [5]). *A sequence $(x(n))$ is PC with period T iff there are a Hilbert space $K \supseteq M_x$, a unitary operator U in K, and a T-periodic sequence $(p(n))$ in K such that $x(n) = U^n p(n)$, $n \in \mathscr{Z}$, or, equivalently, iff there are a Hilbert space $K \supseteq M_x$ and a T-dimensional stationary sequence $[W^k(n)]$ in K such that $(x(n))$ has a representation* (9).

Any T-dimensional stationary sequence $[W^q(n)]$ in $K \supseteq M_x$ that satisfies (9) will be called a *generating sequence* of $(x(n))$.

If a T-dimensional stationary sequence $[W^k(n)]$ generates a PC sequence $(x(n))$, then

$$K_x(n+p, n) = \sum_{j=0}^{T-1} e^{2\pi ijn/T} \Big(\sum_{k=0}^{T-1} e^{2\pi ikp/T} \big(W^k(p), W^{k-j}(0)\big)\Big), \tag{11}$$

where $k, j \in \mathscr{Z}_T$ (that is, the subtraction is *modulo* T). Therefore the coefficients $a_j(p)$ in (6) can be written as

$$(12)\quad a_j(p) = \sum_{k=0}^{T-1} e^{2\pi i k p/T}\big(W^k(p), W^{k-j}(0)\big) = \int_0^{2\pi} e^{ips} \sum_{k=0}^{T-1} \Gamma^{k,k-j}(ds-2\pi k/T),$$

$j \in \mathscr{Z}_T$, $p \in \mathscr{Z}$, where $[\Gamma^{j,k}]$ is the spectral measure of $[W^q(n)]$ (algebraic operations in $[0, 2\pi)$ are *modulo* 2π). Writing $\gamma_j(ds) = \sum_{k=0}^{T-1} \Gamma^{k,k-j}(ds-2\pi k/T)$, we obtain

$$(13)\quad a_j(p) = \int_0^{2\pi} e^{ips}\gamma_j(ds), \qquad j \in \mathscr{Z}_T,\ p \in \mathscr{Z}.$$

Define a measure F on $[0, 2\pi)\times[0, 2\pi)$ as $F = \sum_{j=0}^{T-1} F_j$, where F_j is the image of γ_j through the mapping l_j: $s \to (s, s-2\pi j/T)$, which maps $[0, 2\pi)$ onto the line segment

$$L_j = \{(s, s-2\pi j/T)\colon s \in [0, 2\pi)\} \subset [0, 2\pi)\times[0, 2\pi).$$

Then

$$\int_0^{2\pi}\int_0^{2\pi} e^{i(ms-tn)} F(ds, dt) = \sum_{j=0}^{T-1} e^{2\pi i j n/T} \int_0^{2\pi} e^{i(m-n)s} F_j(ds, ds-2\pi j/T)$$
$$= \sum_{j=0}^{T-1} e^{2\pi i j n/T} \int_0^{2\pi} e^{i(m-n)s}\gamma_j(ds) = \sum_{j=0}^{T-1} e^{2\pi i j n/T} a_j(m-n) = K_x(m, n),$$

which shows that $(x(n))$ is harmonizable and its spectrum is equal to F.

PROPOSITION 2.3 (Gladyshev [5]). *A sequence $(x(n))$ is PC with period T iff*

$$K_x(m, n) = \int_0^{2\pi}\int_0^{2\pi} e^{i(ms-tn)} F_x(ds, dt),$$

where $F_x(ds, dt) = \sum_{j=0}^{T-1} F_j(ds, dt)$ and support of $F_j \subseteq L_j$, $j = 0, \ldots, T-1$.

Note that although a generating sequence $[W^k(n)]$ of $(x(n))$ is not unique, the measure F_x is unique. In consequence, the measures γ_j, $j \in \mathscr{Z}_T$, are unique in the sense that two PC sequences with period T have the same sets of γ_j's iff they are equivalent. Customarily, not F_x, but the family (γ_j), $j \in \mathscr{Z}_T$, is named the *spectrum* of a PC sequence.

DEFINITION 2.4. Let $(x(n))$ be a PC sequence with period T. The family of measures (γ_j), $j \in \mathscr{Z}_T$, defined by (13) is called the *spectrum* of $(x(n))$.

From the definition of (γ_j) it follows that if $[W^k(n)]$ generates $(x(n))$ and $[\Gamma^{j,k}]$ is the spectral measure of $[W^k(n)]$, then

$$(14)\quad \gamma_j(ds) = \sum_{k=0}^{T-1} \Gamma^{k,k-j}(ds-2\pi k/T) = \sum_{k=0}^{T-1} \big(E_U(ds-2\pi k/T)\, W^k(0), W^{k-j}(0)\big),$$

where E_U is the spectral resolution of the shift U of $[W^k(n)]$. From the Schwarz inequality we infer therefore that $|\gamma_j(ds)|^2 \leqslant \gamma_0(ds)\gamma_0(ds-2\pi j/T)$, $j \in \mathscr{Z}_T$, so all the measures γ_j are absolutely continuous with respect to γ_0.

The weakness of Gladyshev's construction of a generating sequence is that neither a triple $(U^n, p(n), K)$ in Proposition 2.2 nor a generating sequence $[W^q(n)]$ in (9) nor a measure $[\Gamma^{j,k}]$ in (14) are uniquely determined by $(x(n))$ or (γ_j). Although the T-shift operator V is uniquely defined, there are many different unitary T-th roots U of V and each of them leads to a different generating sequence. If $U = \int_0^{2\pi} e^{it} E_U(dt)$ is such that $U^T = V$ on M_x, then

$$V = \int_0^{2\pi} e^{itT} E_U(dt) = \int_0^{2\pi} e^{is} E_V(ds),$$

where the spectral resolution E_V of V is obtained by "stretching" the measure E_U by a factor T to a measure over the interval $[0, 2\pi T)$, and then "wrapping" it T times around the unit circle $[0, 2\pi)$, that is

$$E_V(ds) = \sum_{r=0}^{T-1} E_U\left(\frac{ds}{T}+\frac{2\pi r}{T}\right). \tag{15}$$

If E_V is given, then the construction of a unitary T-th root U of V means an opposite operation, that is "splitting" E_V into T mutually orthogonal projection valued measures E_V^r, $r = 0, \ldots, T-1$, placing them on intervals $[2\pi r, 2\pi(r+1))$, $r = 0, \ldots, T-1$, respectively, and "squeezing" the sum to $[0, 2\pi)$. If $T > 1$ and the dimension of M_x is more than one, then this clearly can be done in many different ways.

The standard example of a generating stationary sequence is the sequence created by the principal T-th root of V defined as $U = \int_0^{2\pi} e^{it/T} E_V(dt)$, where E_V is the spectral resolution of V (see [5]). In terms of "splitting" E_V, the spectral resolution of U is obtained when $E_V^0 = E_V$ and the other E_V^r's are zero. The generating sequence produced in such a way will be called the *principal root sequence.* The principal root sequence is given by

$$W^q(n) = \int_0^{2\pi/T} e^{itn} E_V(T\,ds)\, W^q(0), \qquad n \in \mathscr{Z},\ q = 0, \ldots, T-1, \tag{16}$$

where $W^q(0) = (1/T)\sum_{k=0}^{T-1} e^{-2\pi i k q/T}\big(U^{-k}x(k)\big)$. Note that since the spectral measure of $[W^q(n)]$ is supported on $[0, 2\pi/T)$, the principal root sequence is always deterministic (Proposition 3.1), and hence it fails to reflect regularity properties of $(x(n))$. This observation prompts a problem of constructing a generating sequence which shares the prediction properties of a PC sequence it generates and the spectrum of which can be uniquely expressed in terms of (γ_j). Such a construction will be presented in Section 4.

We finish this section with a restatement of Proposition 2.2 in the spirit of Kolmogorov's Isomorphism Theorem. In what follows, a nonnegative measure μ on $[0, 2\pi)$ is called *$(2\pi/T)$-invariant* if $\mu(\Delta) = \mu(\Delta + 2\pi/T)$ for every Borel set $\Delta \subseteq [0, 2\pi)$.

Proposition 2.5. *A sequence $(x(n))$ is PC with period T iff there are a $(2\pi/T)$-invariant measure μ, an integer $0 < d \leqslant T$, and a T-periodic sequence $(C(n))$ of functions in $L^2(\mu; C^d)$ such that $(x(n))$ is equivalent to an $L^2(\mu; C^d)$-valued sequence*

$$y(n)(\cdot) = e^{in\cdot}\, C(n)(\cdot), \qquad n \in \mathscr{Z}. \tag{17}$$

If this is the case, then the spectral measures (γ_j) of $(x(n))$ are absolutely continuous with respect to μ and

$$\frac{d\gamma_j}{d\mu}(ds) = \sum_{k=0}^{T-1} \big(A^k(s-2\pi k/T),\, A^{k-j}(s-2\pi k/T)\big),$$

where $A^q(\cdot) = (1/T)\sum_{n=0}^{T-1} e^{-2\pi iqn/T}\, C(n)(\cdot)$, $q = 0, \ldots, T-1$.

The proposition above is merely a restatement of Proposition 2.2, and hence inherits its nonuniqueness. The proof follows immediately from Proposition 2.2 and (5).

3. PREDICTION PROBLEM

For any T-dimensional sequence $[X^k(n)]$ in a Hilbert space H let us write

$$M_X(n) = \overline{\mathrm{sp}}\,\{X^k(m)\colon k = 0, \ldots, T-1, m \leqslant n\}, \qquad M_X(-\infty) = \bigcap_n M_X(n),$$

$$M_X = M_X(\infty), \qquad \text{and} \qquad N_X(n) = M_X(n) \ominus M_X(n-1).$$

If $M_X(n) = M_X$ for all $n \in \mathscr{Z}$, then the sequence is called *deterministic*; if $M_X(-\infty) = \{0\}$, then the sequence is called *regular*. If $[X^k(n)]$ is stationary, then the dimension of $N_X(0)$ is called the *rank* of the sequence $[X^k(n)]$.

The objectives of prediction theory is to describe the *predictor* $(X^p(n) \,|\, M_X(m))$, $m < n$, in terms of the spectrum of the sequence, which is assumed to be known. Intermediate questions are concerned with a decomposition of a sequence into deterministic and regular parts, spectral characterizations of regularity, computation of the prediction error, etc. The prediction problem is almost completely solved for stationary sequences, however still no explicit analytic expressions for regularity or predictor of a T-dimensional stationary sequence in terms of its density are available for $T > 1$ (for more information and references see [18] or [16]). The accessible characterizations are in the language of *conjugate analytic* square roots of the spectral density, and the prediction formula is in the equivalent language of *moving average* series. The next two propositions summarize known results in the stationary case.

If the spectral measure $[\Gamma^{j,k}]$ of a T-dimensional stationary sequence is absolutely continuous with respect to the Lebesgue measure, then its Radon-Nikodym derivative $G(s) = [d\Gamma^{j,k}/ds(s)]$ will be referred to as the *spectral* *density* of the sequence. Any d-dimensional stationary sequence $[\xi_n^k]$, $n \in \mathscr{Z}$, such that $[\xi_n^k][\xi_m^k]^* = I_d \delta_{n-m}$, $n, m \in \mathscr{Z}$, where I_d is the $d \times d$ identity matrix, will be called a *d-dimensional innovation* (or *innovation*, if $d = 1$).

PROPOSITION 3.1. *Let $[X^k(n)]$ be a T-dimensional stationary sequence. The following conditions are equivalent:*

(i) *$[X^k(n)]$ is regular;*

(ii) *the spectral measure Γ of $[X^k(n)]$ is equivalent to the Lebesgue measure and the spectral density $G(s)$ of $[X^k(n)]$ admits a conjugate analytic square root $A(\cdot)$;*

(iii) *there are an integer $d \leqslant T$, a d-dimensional innovation $[\xi_n^k]$, and a sequence A_k, $k = 0, 1, \ldots$, of $T \times d$ matrices such that*

$$[X^p(n)] = \sum_{k=0}^{\infty} A_k [\xi_{n-k}^p]. \tag{18}$$

The series representation (18) is called a *moving average* representation of $[X^k(n)]$. Matrices A_k in (18) are given by $A_k = (1/\sqrt{2\pi})\int_0^{2\pi} e^{ikt} A(t)\,dt$, $k \geqslant 0$, where $A(t)$ is a conjugate analytic square root of $G(s)$. A representation (18) does not guarantee that the orthogonal projection of $[X^q(n)]$ onto $M_X(m)$, $m < n$, is a tail of the series (18), i.e.

$$[(X^p(n) \mid M_X(m))] = \sum_{k=n-m}^{\infty} A_k [\xi_{n-k}^p]. \tag{19}$$

A moving average representation with the property (19) will be called *optimal*. The corresponding property of $A(s)$ is called *outerness*.

DEFINITION 3.2. Functions $f^k(\cdot) \in L_-^2(\mathscr{C}^d)$, $k = 0, \ldots, N$, are said to be *jointly outer* in $L_-^2(\mathscr{C}^d)$ if

$$\overline{\mathrm{sp}}\{e^{-in\cdot} f^k(\cdot)\colon\ k = 0, \ldots, N,\ n \geqslant 0\} = L_-^2(\mathscr{C}^d).$$

A $T \times d$ matrix function $A(s)$ is called *outer* if the rows $A^q(\cdot)$, $q = 0, \ldots, T-1$, of $A(\cdot)$ are jointly outer in $L_-^2(\mathscr{C}^d)$.

Retrieving the coefficients A_k, $k \geqslant 0$, of an optimal moving average representation of $[X^q(n)]$ from its spectrum constitutes a solution to the prediction problem.

PROPOSITION 3.3. *Let $[X^k(n)]$ be a regular T-dimensional stationary sequence of rank r.*

(A) *Matrix coefficients A_k, $k \geqslant 0$, in* (18) *can be chosen so that $d = r$ and*

$$[(X^p(n) \mid M_X(m))] = \sum_{k=n-m}^{\infty} A_k [\xi_{n-k}^p], \qquad n, m \in \mathscr{Z}_T,\ m \leqslant n. \tag{20}$$

(B) *A necessary and sufficient condition for a sequence of $T \times r$-matrices A_k, $k \geqslant 0$, to satisfy* (20) *is that the function* $A(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} A_k e^{-iks}$ *is an outer square root of the spectral density* $G(s)$ *of* $[X^q(n)]$.

The function $A(s)$ in part (B) is unique up to a unitary matrix, that is, if $A(s)$ and $B(s)$ are two outer square roots of $G(s)$, then there is a unitary $r \times r$-matrix D such that $A(s) = B(s) D$, ds-a.e. If A_k, $k \geqslant 0$, satisfy (20), then the *one-step prediction error matrix* is given by

$$G_0 = [(X^k(0) \mid N_X(0))] [(X^k(0) \mid N_X(0))]^* = A_0 A_0^*.$$

In this paper we study the prediction problem for PC sequences. It is rather simple to see that part (A) of Proposition 3.3 holds true for PC sequences, we just need to alter slightly the definition of an innovation.

DEFINITION 3.4. A sequence (ξ_k) of elements of a Hilbert space H is called a 0-1 *innovation* if ξ_k's are mutually orthogonal and $\|\xi_k\| = 1$ or 0 for every $k \in \mathscr{Z}$. The set $S_\xi = \{k \in \mathscr{Z}: \xi_k \neq 0\}$ is called the *support* of a 0-1 innovation (ξ_k). A 0-1 innovation is said to be *T-periodic* if $(\|\xi_k\|)$ is T-periodic.

If $(x(n))$ is PC with period T, then the dimension of $M_x(0) \ominus M_x(-T)$ or, equivalently, the number of non-zero elements in the set $\{(x(n) \mid N_x(n)): n = 0, -1, \ldots, -T+1\}$ will be called the *rank* of a PC sequence $(x(n))$. Hence the rank of a T-periodic 0-1 innovation is the cardinality of the set $S_\xi \cap \{0, -1, \ldots, -T+1\}$. Regular PC sequences of period T and rank T are called *completely regular*.

The following is a slight extension of the construction presented in [10] in the case of completely regular PC sequences.

PROPOSITION 3.5. *Let $(x(n))$ be a PC sequence with period T. The sequence $(x(n))$ is regular iff there are a T-periodic* 0-1 *innovation* (ξ_k) *in* M_x, *and T-periodic (in n) sequences of scalars* (β_k^n), $k \geqslant 0$, *such that*

$$(x(n) \mid M_x(m)) = \sum_{k=n-m}^{\infty} \beta_k^n \xi_{n-k}, \quad n, m \in \mathscr{Z},\ m \leqslant n. \tag{21}$$

If this is the case, then (ξ_k) *and the coefficients* (β_k^n) *can be chosen so that*

(U-1) $\beta_0^n \geqslant 0$ *for all* $n \in \mathscr{Z}$,

(U-2) $\beta_k^{k+m} = 0$ *provided* $m \notin S_\xi$, $k \geqslant 0$,

and then (β_k^n) *and* (ξ_n) *are unique, the rank of* $(x(n))$ *is equal to the rank of* (ξ_n), *and the one-step prediction error* $\|x(n) - (x(n) \mid M_x(n-1))\| = \beta_0^n$, $n \in \mathscr{Z}$.

Proof. If $(x(n))$ satisfies (21), then $M_x(n) \subseteq M_\xi(n)$, $n \in \mathscr{Z}$, and hence $(x(n))$ is regular.

To prove the necessity suppose that $(x(n))$ is regular and define $z_n = (x(n) \mid N_x(n))$, $n \in \mathscr{Z}$. Let $\xi_n = z_n / \|z_n\|$ if $z_n \neq 0$, and zero otherwise. Then (ξ_n) is a T-periodic 0-1 innovation, and $\xi_n \in N_x(n) \subset M_x(n)$, $n \in \mathscr{Z}$. Since

$\bigoplus_n N_x(n) = M_x$, every $y \in M_x$ has an expansion $y = \sum_{j=-\infty}^{\infty} c_j(y)\,\xi_j$, which is unique, provided that $c_j(y) = 0$ if $\xi_j = 0$. In particular, $x(n) = \sum_{j=-\infty}^{n} c_j(x(n))\,\xi_j$, $n \in \mathscr{Z}$. Since $\xi_n \perp M_x(n-1)$, we also have

$$\big(x(n)\,|\,M_x(m)\big) = \sum_{j=-\infty}^{m} c_j\big(x(n)\big)\,\xi_j = \sum_{k=n-m}^{\infty} c_{n-k}\big(x(n)\big)\,\xi_{n-k}, \qquad n, m \in \mathscr{Z},\ m \leqslant n.$$

Define $\beta_k^n = c_{n-k}\big(x(n)\big) = \big(x(n), \xi_{n-k}\big)$. Since the T-shift operator V of $\big(x(n)\big)$ maps $M_x(n)$ onto $M_x(n+T)$, we obtain

$$Vz_j = V\big(x(j) - (x(j)\,|\,M_x(j))\big) = z_{j+T},$$

and hence $V\xi_j = \xi_{j+T}$, $j \in \mathscr{Z}$. Therefore

$$\beta_k^n = \big(x(n), \xi_{n-k}\big) = \big(Vx(n), V\xi_{n-k}\big) = \big(x(n+T), \xi_{n+T-k}\big) = \beta_k^{n+T}, \qquad n \in \mathscr{Z},\ k \geqslant 0,$$

that is (β_k^n) are T-periodic in n. By definition, $\beta_0^n = \big\|\big(x(n)\,|\,N_x(n)\big)\big\| \geqslant 0$, $n \in \mathscr{Z}$, and $\beta_k^{k+m} = \big(x(m+k), \xi_m\big) = 0$ if $\xi_m = 0$, that is, if $m \notin S_\xi$.

To see the uniqueness suppose that

$$\big(x(n)\,|\,M_x(m)\big) = \sum_{k=n-m}^{\infty} \beta_k^n\,\xi_{n-k} = \sum_{k=n-m}^{\infty} \alpha_k^n\,\zeta_{n-k}, \qquad m, n \in \mathscr{Z},\ m \leqslant n,$$

where (ξ_n) and (ζ_n) are 0-1 innovations in M_x, and the sequences (α_k^n) and (β_k^n), $k \geqslant 0$, are T-periodic in n and satisfy (U-1) and (U-2). Then $\big(x(n)\,|\,N_x(n)\big) = \beta_0^n\,\xi_n = \alpha_0^n\,\zeta_n$, $n \in \mathscr{Z}$. Since the rank of $\big(x(n)\big)$ is r, exactly r of $\big(x(n)\,|\,N_x(n)\big)$, $n = 0, 1, \ldots, T-1$, are nonzero, say for n's in the set $S \subseteq \{0, \ldots, T-1\}$. If $[n] \in S$, then $\beta_0^n\,\xi_n = \alpha_0^n\,\zeta_n \neq 0$ and, because of (U-1), $\beta_0^n = \alpha_0^n$ and $\xi_n = \zeta_n$. If $[n] \in S$ and $k > 0$, then $\big(x(n+k)\,|\,N_x(n)\big) = \beta_k^{n+k}\,\xi_n = \alpha_k^{n+k}\,\zeta_n$, and since we have just proved that $\xi_n = \zeta_n$, we conclude that $\beta_k^{n+k} = \alpha_k^{n+k}$, $k \geqslant 0$. Finally, if $[n] \notin S$, then ξ_n's and ζ_n's are zero, because $\bigoplus_k N_x(k) = M_x$, and both innovations are assumed to be in M_x. Therefore $S_\xi = S_\zeta = \{n \in \mathscr{Z}\colon [n] \in S\}$ and, by (U-2), $\beta_k^{k+n} = 0 = \alpha_k^{k+n}$ for all $k \geqslant 0$ and $[n] \notin S$. ∎

Note that if $\big(x(n)\big)$ satisfies (21), (U-1) and (U-2), then the mapping

$$\Phi\colon\ \xi_n \to e^{in\cdot}/\sqrt{2\pi}, \qquad n \in S_\xi,$$

maps $\big(x(n)\big)$ into an equivalent $L^2(\mathscr{C})$-valued PC sequence $\big(y(n)\big)$:

$$y(n)(s) = \frac{e^{ins}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \beta_k^n\,e^{-iks} \tag{22}$$

(in fact, the sum is over the set of all $k \geqslant 0$ such that $n-k \in S_\xi$, but this is taken care of by the assumption that $\beta_k^n = 0$ if $n-k \notin S_\xi$). Putting

$$C(n)(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} \beta_k^n\,e^{-iks}, \qquad n \in \mathscr{Z},$$

we obtain $y(n) = e^{in\cdot} C(n)$, $n \in \mathscr{Z}$, which is a particular form of (17). From Proposition 3.5 we conclude that the spectral densities $g_j(s) = (d\gamma_j/ds)(s)$ of $(x(n))$ are given by

$$g_j(s) = \sum_{q=0}^{T-1} A^q(s-2\pi q/T)\overline{A^{q-j}(s-2\pi q/T)}, \qquad j \in \mathscr{Z}_T, \tag{23}$$

where

$$A^q(s) = (1/T\sqrt{2\pi}) \sum_{k=0}^{\infty} \sum_{n=0}^{T-1} e^{-2\pi i q n/T} \beta_k^n e^{-iks}, \qquad q \in \mathscr{Z}_T. \tag{24}$$

The formulas (23) and (24) allow us to compute the spectrum of $(x(n))$, given the coefficients (β_k^m). The main effort in this paper is the opposite: to construct the coefficients (β_k^n) of the optimal moving average representation (21) of $(x(n))$ from the spectrum (γ_j) of the sequence, similarly as matrices A_k in Proposition 3.3 in the stationary case. We will also obtain an alternative *external* version of an optimal moving average representation for regular PC sequences.

4. INDUCED STATIONARY SEQUENCE

Let $T > 0$ and let $(x(n))$ be a sequence in a Hilbert space. For every $n \in \mathscr{Z}$ and $q \in \mathscr{Z}_T$ let $Z^q(n)$ be an element of $K = M_x^T$ whose p-th coordinate is

$$Z^q(n)(p) = (1/T)\, x(n-p)\, e^{-2\pi i q(n-p)/T}, \qquad p = 0, \ldots, T-1. \tag{25}$$

In other words, for a fixed $n \in \mathscr{Z}$, $Z^q(n)(\cdot)$ is a part of the trajectory of the infinite sequence $(1/T)\, x(n-\cdot)\, e^{-2\pi i q(n-\cdot)/T}$ that is seen in the window $\{0, \ldots, T-1\}$.

PROPOSITION 4.1. *Let $(x(n))$ be a PC sequence with period T, (γ_j) be the spectrum of $(x(n))$, and let $Z^q(n)$ be defined by (25). Then:*

(I-1) *$[Z^q(n)]$ is a T-dimensional stationary sequence in $K = M_x^T$ with the correlation*

$$K^{j,k}(n) = a_{j-k}(n)\, e^{-2\pi i j n/T}, \qquad j, k \in \mathscr{Z}_T,\ n \in \mathscr{Z}, \tag{26}$$

where $a_j(n)$ are defined in (7). The spectrum (γ_j) of $(x(n))$ and the spectral measure $[\Gamma^{j,k}]$ of $[Z^q(n)]$ are related through

$$\Gamma^{j,k}(\Delta) = (1/T)\, \gamma_{j-k}(\Delta + 2\pi j/T), \tag{27}$$

$$\gamma_k(\Delta) = T\Gamma^{j,j-k}(\Delta - 2\pi j/T), \qquad j, k \in \mathscr{Z}_T. \tag{28}$$

(I-2) $x(n) = \sum_{q=0}^{T-1} e^{2\pi i q n/T} Z^q(n)$, $n \in \mathscr{Z}$, *provided M_x is identified with $M_x \oplus \{0\} \oplus \ldots \oplus \{0\}$.*

(I-3) $M_Z = M_x^T$ *and* $M_Z(n) = M_x(n) \oplus M_x(n-1) \oplus \ldots \oplus M_x(n-T+1)$, $n \in \mathscr{Z}$. *Consequently, $(x(n))$ is regular (deterministic) iff $[Z^k(n)]$ is regular (deterministic, respectively).*

(I-4) *For every* $m, n \in \mathscr{Z}$, $n > m$,

$$(29) \qquad \sum_{q=0}^{T-1} e^{2\pi iqn/T} \big(Z^q(n) \mid M_Z(m)\big) = \big(\big(x(n) \mid M_x(m)\big), 0, \ldots, 0\big).$$

Note that despite the fact that $M_Z(m)$ is much larger than $M_x(m)$, the property (29) states that

$$(30) \qquad \big(x(n) \mid M_x(m)\big) = \big(x(n) \mid M_Z(m)\big), \quad n \geqslant m.$$

The idea of an induced sequence comes from [4]. In the above form it appeared in [17] and the properties (I-1) and (I-2) were proved therein. Also in [17] a probabilistic version of an induced sequence was introduced and was used to show that every, not necessarily of finite variance, periodically distributed sequence admits a representation (9). Property (I-3) was proved in [19].

Proofs. (I-1) Since $K_x(m-p, n-p)$ is T-periodic in p, by substituting $r = n-p$ in the second line below we obtain

$$\begin{aligned}
\big(Z^j(m), Z^k(n)\big) &= (1/T^2) \sum_{p=0}^{T-1} e^{-2\pi ij(m-p)/T} e^{2\pi ik(n-p)/T} K_x(m-p, n-p) \\
&= (1/T) e^{-2\pi i(m-n)j/T} (1/T) \sum_{r=0}^{T-1} e^{-2\pi ir(j-k)/T} K_x(m-n+r, r) \\
&= (1/T) e^{-2\pi i(m-n)j/T} a_{j-k}(m-n).
\end{aligned}$$

Hence $[Z^k(n)]$ is a T-dimensional stationary sequence and its correlation equals

$$K^{j,k}(n) = (1/T) e^{-2\pi inj/T} a_{j-k}(n).$$

Writing both sides of the above as the integrals (4) and (13), we obtain (27) and (28).

(I-2) Since $\sum_{q=0}^{T-1} e^{2\pi inq/T} = 0$ if $n \neq 0$ *modulo* T, and $\sum_{q=0}^{T-1} e^{2\pi inq/T} = T$ if $n = 0$ *modulo* T,

$$(31) \qquad \sum_{q=0}^{T-1} e^{2\pi irq/T} Z^q(n) = \big(0, 0, \ldots, \underbrace{x(n-[n-r])}_{([n-r])}, 0, \ldots, 0\big),$$

where the only nonzero entry is at the place $[n-r]$ (recall that $[m]$ is the remainder in division of m by T). If $r = n$, then we obtain (I-2).

(I-3) and (I-4). The inclusion

$$M_Z(m) \subseteq M_x(m) \oplus M_x(m-1) \oplus \ldots \oplus M_x(m-T+1), \quad m \in \mathscr{Z},$$

is obvious. On the other hand, if we fix $j \in \{0, \ldots, T-1\}$ and $m \in \mathscr{Z}$, and substitute $r = n-j$ in (31), then for every $n \leqslant m$ we obtain

$$\sum_{q=0}^{T-1} e^{2\pi i(n-j)q/T} Z^q(n) = (0, 0, \ldots, \underbrace{x(n-j)}_{(j)}, 0, \ldots, 0) \in M_Z(m).$$

Hence

$$M_Z(m) \supseteq \{0\} \oplus \ldots \oplus \underbrace{M_x(m-j)}_{(j)} \oplus \{0\} \oplus \ldots \oplus \{0\}, \qquad j = 0, \ldots, T-1,$$

which proves (I-3). The property (I-4) is an immediate consequence of (I-3). ∎

Clearly, the converse of Proposition 4.1 also holds true.

PROPOSITION 4.2. *If* $[Z^q(n)]$ *in* (25) *is a* T*-dimensional stationary sequence in* M_x^T, *then* $(x(n))$ *is PC with period* T.

Proof. The formula (31) holds true for any (not necessarily PC) sequence $(x(n))$. Therefore, if $[Z^q(n)]$ is stationary and U is its shift, then

$$\begin{aligned} U^T(x(n), 0, \ldots, 0) &= \sum_{q=0}^{T-1} e^{2\pi inq/T} U^T Z^q(n) \\ &= \sum_{q=0}^{T-1} e^{2\pi i(n+T)q/T} Z^q(n+T) = (x(n+T), 0, \ldots, 0), \end{aligned}$$

so $U^T x(n) = x(n+T)$, $n \in \mathscr{Z}$. Hence $(x(n))$ is PC with period T. ∎

DEFINITION 4.3. If $(x(n))$ is PC with period T, then the T-dimensional stationary sequence $[Z^q(n)]$ defined in (25) is said to be *induced* by $(x(n))$.

From Proposition 4.1 it follows that if (γ_j) is the spectrum of a PC sequence with period T, then $[\Gamma^{j,k}] = (1/T)[\gamma_{j-k}(\Delta + 2\pi j/T)]$ is the spectral measure of a T-dimensional stationary sequence. This correspondence is bijective.

COROLLARY 4.4. (A) *If* (γ_j), $j \in \mathscr{Z}_T$, *is the spectrum of a PC sequence, then the* $T \times T$*-matrix measure* $[\Gamma^{j,k}(\Delta)] = (1/T)[\gamma_{j-k}(\Delta + 2\pi j/T)]$ *is the spectral measure of a* T*-dimensional stationary sequence.*

(B) *If the spectral measure* $[\Gamma^{j,k}]$ *of a* T*-dimensional stationary sequence satisfies*

(32) $\Gamma^{j,k}(\Delta) = \Gamma^{j+d,k+d}(\Delta - 2\pi d/T)$ *for every* $j, k, d \in \mathscr{Z}_T$ *and Borel* Δ,

then the measures $\gamma_k(\Delta) = T\Gamma^{p,p-k}(\Delta - 2\pi p/T)$, $k > \mathscr{Z}_T$, *form the spectrum of a PC sequence with period* T.

Proof. Part (A) has already been noticed. Suppose therefore that $[\Gamma^{j,k}]$ is the spectral measure of a T-dimensional stationary sequence $[X^k(n)]$. Define

$$x(n) = \sum_{j=0}^{T-1} e^{2\pi ijn/T} X^j(n), \qquad n \in \mathscr{Z}.$$

Then $(x(n))$ is PC and from (14) we conclude that the spectrum (γ_j) of $(x(n))$ is given by

$$\gamma_k(\Delta) = \sum_{j=0}^{T-1} \Gamma^{j,j-k}(\Delta - 2\pi j/T) = T\Gamma^{p,p-k}(\Delta - 2\pi p/T),$$

because by (32) all the components in the sum above are equal. ∎

Corollary 4.4 is just another form of Gladyshev's Theorem 1 from [5]: *A kernel* $K(m, n) = \sum_{j=0}^{T-1} e^{2\pi i j n/T} a_j(m-n)$ *is the correlation function of a PC sequence with period* T *iff the* $T \times T$*-matrix sequence with entries* $K^{j,k}(n) = a_{j-k}(n) e^{-2\pi i k n/T}$ *is the correlation function of a* T*-dimensional stationary sequence.*

The induced sequence meets all our requirements. It generates the sequence $(x(n))$ (I-2); the spectral characteristics of both sequences are related to each other in a very simple and one-to-one way (I-1); it controls prediction properties of a PC sequence $(x(n))$ and predicting $x(n)$ is equivalent to predicting $[Z^q(n)]$ ((I-3) and (I-4)). The induced sequence has been already used in [19] to relate autoregressive representations of the predictors of $(x(n))$ and $[Z^q(n)]$ to each other. In the next section we will employ the induced sequence technique to study the prediction problem for PC sequences.

It should be pointed out here that the spectral theory of continuous time PC processes is also very well developed (see, for example, [6]–[9], [14]); in particular, a construction of an induced process goes through for continuous time PC processes [14]. A process $[Z^k(t)]$ induced by a continuous parameter locally square integrable PC process $x(t)$ is an infinite-dimensional continuous stationary process with values in $K = L^2([0, T), dt; M_x)$. The main difficulty of retrieving a PC process back from its induced process was recently resolved in [13]. A method of reconstruction of a PC process used in that paper essentially differs from (I-2) and reveals a strong link between induced stationary processes and unitary representations of groups induced from their subgroups. In this framework, part (B) of Corollary 4.4 is equivalent to a special case of Mackey's Imprimitivity Theorem. Please see [13] for details.

5. REGULARITY AND MOVING AVERAGE REPRESENTATION

Since a PC sequence $(x(n))$ is regular iff its induced sequence $[Z^k(n)]$ is regular, and the regularity of the latter depends on the absolutely continuous part of its spectral measure, we will be assuming in this section that the spectral measures (γ_j) of $(x(n))$ are absolutely continuous with respect to Lebesgue measure. Let us put

$$g_j(s) = \frac{d\gamma_j}{ds}(s), \quad j \in \mathscr{Z}_T, \tag{33}$$

$$G^{j,k}(s) = (1/T)\, g_{j-k}(s + 2\pi j/T), \quad j, k \in \mathscr{Z}_T. \tag{34}$$

Also, in order to simplify notation, let E be the $T\times T$-matrix whose j, k-th entry is $e^{2\pi ijk/T}$ and let E^{-1} be its inverse, that is

$$E=[e^{2\pi ijk/T}] \quad \text{and} \quad E^{-1}=(1/T)[e^{-2\pi ijk/T}].$$

Since $G(s)=[G^{j,k}(s)]$ is the spectral density of $[Z^k(n)]$, a simple adaptation of Proposition 3.1 gives the following regularity criteria for PC sequences.

PROPOSITION 5.1. *Let $(x(n))$ be a PC sequence with period T and absolutely continuous spectrum (γ_j). Let $g_j(s)$ and $[G^{j,k}(s)]$ be as in* (33) *and* (34). *The following conditions are equivalent*:

(i) *The sequence $(x(n))$ is regular.*

(ii) *There exist a positive integer $d\leqslant T$ and conjugate analytic functions $A^k(\cdot)\in L^2_-(\mathscr{C}^d)$, $k\in\mathscr{Z}_T$, such that for every $j,k\in\mathscr{Z}_T$*

$$g_k(s)=T\big(A^j(s-2\pi j/T),\ A^{j-k}(s-2\pi j/T)\big),\ ds\text{-a.e.}, \tag{35}$$

where $(\cdot,\cdot)$ is the inner product in $\mathscr{C}^d$.

(iii) *$(x(n))$ is equivalent to an $L^2_-(\mathscr{C}^d)$-valued sequence*

$$y(n)(\cdot)=(1/\sqrt{2\pi})\sum_{k=0}^{\infty}C_k^n e^{i(n-k)\cdot}, \tag{36}$$

where (C_k^n), $k\geqslant 0$, are T-periodic (in n) sequences in $\mathscr{C}^d$ and $d\leqslant T$.

(iv) *There exist $K\supseteq M_x$, a positive integer $d\leqslant T$, a d-dimensional innovation $[\zeta_n^q]$ in K, and $\mathscr{C}^q$-valued T-periodic sequences (C_k^n), $k\geqslant 0$, such that*

$$x(n)=\sum_{k=0}^{\infty}C_k^n[\zeta_{n-k}^q],\qquad n\in\mathscr{Z}. \tag{37}$$

($C_k^n\in\mathscr{C}^d$ above are row-vectors $C_k^n=[C_k^{n,0},C_k^{n,1},\dots,C_k^{n,d-1}]$ and $[\zeta_{n-k}^q]$ are columns as usual, so that $C_k^n[\zeta_{n-k}^q]=\sum_{q=0}^{d-1}C_k^{n,q}\zeta_{n-k}^q$.)

Observe that the right-hand side of (35) does not depend on j. Also note that (36) is equivalent to the representation $y(n)(\cdot)=e^{in\cdot}C(n)(\cdot)$, $n\in\mathscr{Z}$, where $(C(n))$ is a T-periodic sequence in $L^2_-(\mathscr{C}^d)$.

Proof. (i)$\Rightarrow$(ii). If $(x(n))$ is regular, then its induced sequence $[Z^q(n)]$ is regular, and hence its density $G(s)$ admits a conjugate analytic square root $A(\cdot)$ (Proposition 3.1). If $A^k(\cdot)$ is the k-th row of $A(s)$, then

$$\big(A^j(s),A^k(s)\big)=G^{j,k}(s)=(1/T)g_{j-k}(s+2\pi j/T),$$

which gives (35).

(ii)$\Rightarrow$(iii). Let $A(s)=[A^q(s)]$ be the $T\times d$-matrix valued function whose q-th row is $A^q(s)$. From (35) it follows that the $L^2(\mathscr{C}^d)$-valued sequence

$$y(n)(\cdot)=\sum_{q=0}^{T-1}e^{in\cdot}e^{2\pi inq/T}A^q(\cdot) \tag{38}$$

is equivalent to $(x(n))$. Let $C(s) = EA(s)$, $s \in [0, 2\pi)$, $C^q(\cdot)$ be the q-th row of $C(\cdot)$, and let $C(n)(\cdot) = C^{[n]}(\cdot)$, that is

$$C(n)(\cdot) = \sum_{q=0}^{T-1} e^{2\pi i n q/T} A^q(\cdot), \qquad n \in \mathscr{Z}. \tag{39}$$

Then $(C(n))$ is a T-periodic sequence of conjugate analytic functions and $y(n)(\cdot) = e^{in\cdot} C(n)(\cdot)$. If we write $C(n)(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} C_k^n e^{-iks}$, then we obtain (36).

(iii)$\Rightarrow$(iv). Define $\zeta_n^q(s) = (0, \ldots, 0, e^{ins}/\sqrt{2\pi}, 0, \ldots, 0)$, $n \in \mathscr{Z}$, where the only nonzero entry is at the q-th place, $q = 0, \ldots, d-1$. Then $\zeta_n^q \in L^2(\mathscr{C}^d)$ and $[\zeta_n^q]$ is a d-dimensional innovation in $L^2(\mathscr{C}^d)$. If $c = [c_0, \ldots, c_{d-1}] \in \mathscr{C}^d$ is a row vector, then the matrix multiplication of c by $[\zeta_n^q]$ gives

$$c[\zeta_n^q(s)] = \sum_{q=0}^{d-1} c_q \zeta_n^q(s) = (c_0 e^{ins}, c_1 e^{ins}, \ldots, c_{d-1} e^{ins})/\sqrt{2\pi} = ce^{ins}/\sqrt{2\pi}.$$

Therefore (36) can be rewritten as (37).

(iv)$\Rightarrow$(i). Since $M_x(n) \subseteq M_\zeta(n)$ and $[\zeta_n^q]$ is regular, the sequence $(x(n))$ is regular. ∎

Even in the stationary case ($T = 1$ and $d = 1$) the representation (37) is not optimal, unless $C(n)(\cdot) = C(\cdot)$ is outer. In the PC case the situation seems to be yet worse; even if the functions $C(n)(\cdot)$, $n \in \mathscr{Z}$, are jointly outer in $L^2_-(\mathscr{C}^d)$, there is a doubt whether $(y(n) \mid M_y(m)) = (1/\sqrt{2\pi}) \sum_{k=n-m}^{\infty} C_k^n e^{i(n-k)\cdot}$, simply because $M_y(m)$ is much smaller than $\mathscr{M}_m = e^{im\cdot} L^2_-(\mathscr{C}^d)$. Nevertheless, it turns out that due to the property (30) of an induced sequence both projections coincide.

Proposition 5.2. *If $(x(n))$ is a regular PC sequence of rank r, then (C_k^n) in (37) can be chosen so that $d = r$ and*

$$(x(n) \mid M_x(m)) = \sum_{k=n-m}^{\infty} C_k^n [\zeta_{n-k}^q], \qquad n, m \in \mathscr{Z},\ m \leqslant n. \tag{40}$$

Namely, one can take C_k^n to be the $[n]$-th row of the matrix $C_k = EA_k$, where $A(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} A_k e^{-iks}$ is an outer square root of the function $G(s)$ defined in (34).

Proof. Let $[Z^q(n)]$ be induced by $(x(n))$. From Proposition 4.1 (I-3) it follows that $[Z^q(n)]$ is regular, and hence its density $G(s)$ admits an outer square root $A(s)$. Write

$$A(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} A_k e^{-iks}$$

and define

$$C_k = EA_k,\ k \geqslant 0, \qquad C(s) = EA(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} C_k e^{-iks}.$$

Let $C^q(\cdot)$ be the q-th row of $C(\cdot)$ and let C_k^q be the q-th row of C_k. Define

$$y(n)(s) = e^{ins} C^{[n]}(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} C_k^{[n]} e^{i(n-k)s}. \tag{41}$$

Then $(y(n))$ is an $L^2(\mathscr{C}^d)$-valued PC sequence and $y(n) = \sum_{q=0}^{T-1} e^{2\pi inq/T} Y^q(n)$, where $[Y^q(n)]$ is a T-dimensional stationary sequence in $L^2(\mathscr{C}^d)$ defined by $Y^q(n)(\cdot) = e^{in\cdot} A^q(\cdot)$. Since $G(s)$ is the spectral density of $[Y^q(n)]$, the mapping Φ: $Z^q(n) \to Y^q(n)$ extends to an isometry from M_Z onto $M_Y = L^2(\mathscr{C}^d)$. Moreover,

$$\Phi(x(n)) = \Phi\Big(\sum_{q=0}^{T-1} e^{2\pi inq/T} Z^q(n)\Big) = y(n).$$

Consequently, $\Phi(M_x(m)) = M_y(m)$ and, by (30),

$$(y(n) \mid M_y(m)) = \Phi(x(n) \mid M_x(m)) = \Phi(x(n) \mid M_Z(m)) = (y(n) \mid M_Y(m)), \qquad m \leqslant n.$$

Since $A^q(\cdot)$, $q = 0, \ldots, T-1$, are jointly outer in $L^2_-(\mathscr{C}^d)$, $M_Y(m) = \mathscr{M}_m = e^{im\cdot} L^2_-(\mathscr{C}^d)$, and hence

$$(y(n) \mid M_y(m)) = (y(n) \mid \mathscr{M}_m) = (1/\sqrt{2\pi}) \sum_{k=n-m}^{\infty} C_k^{[n]} e^{i(n-k)\cdot}, \qquad m \leqslant n,\ n \in \mathscr{Z}. \tag{42}$$

Applying Φ^{-1} to the above we obtain representation (40) with $\zeta_n^q = \Phi^{-1}(0, \ldots, 0, e^{in\cdot}/\sqrt{2\pi}, 0, \ldots, 0)$.

It remains to prove that $d = \operatorname{rank}([Z^q(n)]) = r$. Denote for simplicity $\eta_p = (x(-p) \mid N_x(-p))$, and $\hat{Z}^p(0) = (Z^p(0) \mid N_Z(0))$, $p = 0, \ldots, T-1$. The rank of $(x(n))$ is equal to the number of nonzero η_p's in the set $\{\eta_0, \ldots, \eta_{T-1}\}$, and the rank of $[Z^q(n)]$ is equal to the dimension of the span of $\{\hat{Z}^q(0)\colon q = 0, \ldots, T-1\}$. From Proposition 4.1 (I-4) it follows that

$$\hat{Z}^q(0) = (1/T)(\eta_0, e^{2\pi iq/T}\eta_1, \ldots, e^{2\pi i(T-1)q/T}\eta_{T-1}).$$

Since

$$\sum_{q=0}^{T-1} e^{-2\pi ikq/T} \hat{Z}^q(0) = (0, \ldots, 0, \underbrace{\eta_k}_{(k)}, 0, \ldots, 0), \qquad k = 0, \ldots, T-1,$$

we conclude that $\operatorname{rank}([Z^q(n)]) \geqslant r$. On the other hand, since each $\hat{Z}^q(0)$ is a linear combination of such vectors, namely

$$\hat{Z}^q(0) = (1/T) \sum_{k=0}^{T-1} e^{2\pi ipk/T} (0, \ldots, 0, \underbrace{\eta_k}_{(k)}, 0, \ldots, 0),$$

$\operatorname{rank}([Z^q(n)]) \leqslant r$. ■

Proposition 5.2 is a version of a prediction formula for PC sequences and provides a description of the predictor coefficients C_k^n in terms of an outer square root of $G(s)$. The moving average representation (40) is *external* in the sense that innovation lives in the space bigger than M_x. In the next lemma we show that the property (32) forces a special structure of the matrices C_k, which links the moving average representations (40) and (21).

LEMMA 5.3. *Let $G(s)$ be the spectral density of a T-dimensional regular stationary sequence and let $A(s) = (1/\sqrt{2\pi})\sum_{k=0}^{\infty} A_k e^{-iks}$ be a $T \times r$ outer square root of $G(s)$. Suppose that $G(s)$ has the property that $G^{j+d,k+d}(s) = G^{j,k}(s+2\pi d/T)$, ds-a.e. for all $j, k, d \in \mathscr{Z}_T$. Define $C_k = EA_k$, $k \geqslant 0$. Then the rows C_0^q of C_0 are mutually orthogonal elements of $\mathscr{C}^r$ and exactly r of them are nonzero. Moreover, there are T-periodic (in n) scalar sequences (α_k^n), $k \geqslant 0$, such that*

$$C_k^{[n]} = \alpha_k^n C_0^{[n-k]}, \qquad n \in \mathscr{Z},\ k \geqslant 0. \tag{43}$$

Proof. By Corollary 4.4 and (I-3) of Proposition 4.1, $G(s)$ is the spectral density of a stationary sequence $[Z^q(n)]$ induced by some regular PC sequence, say $(x(n))$. From Proposition 5.2 it follows that $(x(n))$ is equivalent to the sequence $(y(n))$ defined in (41). Therefore

$$\begin{aligned}(y(n)\,|\,N_y(m)) &= (y(n)\,|\,M_y(m)) \ominus (y(n)\,|\,M_y(m-1)) \\ &= (1/\sqrt{2\pi})\, C_{n-m}^{[n]} e^{im\cdot}, \qquad n \geqslant m,\end{aligned}$$

that is $C_{n-m}^{[n]} e^{im\cdot} \in N_y(m)$ for every $m \in \mathscr{Z}$ and $n \geqslant m$. Since $N_y(m)$ is at most one-dimensional, we conclude that for every $m \in \mathscr{Z}$ there is at most one-dimensional subspace D_m of $\mathscr{C}^r$, such that

$$C_{n-m}^{[n]} \in D_m \quad \text{for all } n \in \mathscr{Z},\ n \geqslant m. \tag{44}$$

If $n = m$, this implies that $C_0^n \in D_n$, $n = 0, \ldots, T-1$.

We will show that the rows of C_0 are orthogonal, and so are the subspaces $D_0, D_1, \ldots, D_{T-1}$, and that exactly r of them are nonzero. Since $A(s)$ is an outer square root of $G(s)$, the one-step prediction error matrix of $[Z^q(n)]$ is of the form

$$G_0 = [(Z^j(0)\,|\,N_Z(0))]\,[(Z^j(0)\,|\,N_Z(0))]^* = A_0 A_0^*.$$

Hence there is a unitary mapping $\Psi\colon N_Z(0) \to \mathscr{C}^r$ such that $\Psi(\hat{Z}^q(0)) = A_0^q$, where $\hat{Z}^q(0) = (Z^q(0)\,|\,N_Z(0))$ and A_0^q is the q-th row of A_0. Recall that in Proposition 5.2 we have shown that

$$\hat{Z}^q(0) = (1/T)(\eta_0, e^{2\pi i q/T}\eta_1, \ldots, e^{2\pi i (T-1)q/T}\eta_{T-1}),$$

where $\eta_p = (x(-p)\,|\,N_x(-p))$, $p = 0, \ldots, T-1$. Putting $[W^q] = E[\hat{Z}^q(0)]$, we obtain

$$W^{[-q]} = \sum_{j=0}^{T-1} e^{-2\pi i jq/T} \hat{Z}^j(0) = (0, \ldots, 0, \underbrace{\eta_q}_{(q)}, 0, \ldots, 0), \qquad q = 0, \ldots, T-1,$$

and hence the vectors $W^0, \ldots, W^{T-1}$ are orthogonal to each other. Therefore

$$\Psi(W^k) = \sum_{q=0}^{T-1} e^{2\pi ikq/T} \Psi(\hat{Z}^q(0)) = \sum_{q=0}^{T-1} e^{2\pi ikq/T} A_0^q = C_0^k, \qquad k = 1, \ldots, T-1,$$

are mutually orthogonal. By Proposition 5.2, $r = \operatorname{rank}([Z^q(n)])$, and since the latter is equal to $\operatorname{rank}(G_0) = \operatorname{rank}(A_0) = \operatorname{rank}(C_0)$, we conclude that the matrix C_0 has exactly r nonzero rows. Consequently, exactly r out of T-subspaces $D_0, \ldots, D_{T-1}$ are nonzero.

Since $V(M_y(m) \ominus M_y(m-1)) = M_y(mT) \ominus M_y(mT-1)$, where V is the T-shift operator of $(y(n))$ (which in our case is the operator of multiplication by $e^{iT\cdot}$), we conclude that $D_m = D_{m+pT}$ for every $p \in \mathscr{Z}$, and so $D_m = D_{[m]}$, $m \in \mathscr{Z}$. Hence for every $m \in \mathscr{Z}$ the subspace D_m is spanned by the $[m]$-th row $C_0^{[m]}$ of the matrix $C_0 = EA_0$. Substituting $m = n-k$ in (44), we obtain $C_k^{[n]} \in D_{[n-k]}$, and therefore there are scalars α_k^n such that

$$C_k^{[n]} = \alpha_k^n C_0^{[n-k]}, \qquad n \in \mathscr{Z},\ k \geqslant 0. \tag{45}$$

Namely, one can take

$$\alpha_k^n = \frac{(C_k^{[n]}, C_0^{[n-k]})}{\|C_0^{[n-k]}\|^2}$$

(where $0/0 = 0$), and then (α_k^n) is T-periodic in n for every $k \geqslant 0$. ■

Recall that in Proposition 3.5 we proved that any regular PC sequence $(x(n))$ of period T and rank r can be written as

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k}, \qquad n \in \mathscr{Z}, \tag{46}$$

where (ξ_k) is a T-periodic 0-1 innovation in M_x of rank r. Sequences (β_k^n), $k \geqslant 0$, are T-periodic (in n) and can be chosen so that

(U-0) $(x(n) \mid M_x(m)) = \sum_{k=n-m}^{\infty} \beta_k^n \xi_{n-k}$ for all $m, n \in \mathscr{Z}$, $m \leqslant n$,

(U-1) $\beta_0^n \geqslant 0$ for all $n \in \mathscr{Z}$,

(U-2) $\beta_k^{k+m} = 0$ for all $m \notin S_\xi$ and $k \geqslant 0$,

and then (β_k^n) and (ξ_n) are unique. Below we present a "spectral domain" construction of the predictor coefficients (β_k^n).

THEOREM 5.4. *Suppose that $(x(n))$ is a regular PC sequence of period T and rank r, and let (g_j), $j \in \mathscr{Z}$, be the spectral densities of $(x(n))$. Let*

(i) $G^{j,k}(s) = (1/T)\, g_{j-k}(s + 2\pi j/T)$, $j, k \in \mathscr{Z}_T$,

(ii) *$A(s)$ be an outer square root of* $[G^{j,k}(s)]$,

(iii) $A_k = (1/\sqrt{2\pi}) \int_0^{2\pi} e^{ikt} A(t)\, dt$, $C_k = EA_k$, $k \geqslant 0$,

(iv) $\beta_k^n = (C_k^{[n]}, C_0^{[n-k]}) / \|C_0^{[n-k]}\|$, $n \in \mathscr{Z}$, $k \geqslant 0$, *where C_k^q denotes the q-th row of C_k, and the norm and the inner product are in $\mathscr{C}^r$ (we use the convention that $0/0 = 0$).*

Then the sequences (β_k^n), $k \geqslant 0$, *are T-periodic in n and satisfy the conditions* (U-0), (U-1) *and* (U-2) *above. Moreover, the one-step prediction errors are given by*

$$\|x(n)-(x(n)\,|\,M_x(n-1))\| = \beta_0^n, \quad n \in \mathscr{Z}.$$

Proof. From Proposition 5.2 it follows that there is an r-dimensional innovation $[\zeta_n^q]$ in a Hilbert space $K \supseteq M_x$ such that

$$(47) \qquad (x(n)\,|\,M_x(m)) = \sum_{k=n-m}^{\infty} C_k^{[n]}[\zeta_{n-k}^q], \quad m \leqslant n,\ m,n \in \mathscr{Z}.$$

Define

$$\xi_n = \frac{C_0^{[n]}[\zeta_n^q]}{\|C_0^{[n]}\|} = \|C_0^{[n]}\|^{-1} \sum_{q=0}^{r-1} C_0^{[n],q}\zeta_n^q, \quad n \in \mathscr{Z}$$

$(0/0 = 0)$. By Lemma 5.3, $C_k^{[n]} = \alpha_k^n C_0^{[n-k]}$, where $\alpha_k^n = (C_k^{[n]}, C_0^{[n-k]})/\|C_0^{[n-k]}\|^2$. Therefore from (47) we infer that for all $m, n \in \mathscr{Z}$, $m \leqslant n$,

$$(48) \qquad (x(n)\,|\,M_x(m)) = \sum_{k=n-m}^{\infty} \alpha_k^n C_0^{[n-k]}[\zeta_{n-k}^q] = \sum_{k=n-m}^{\infty} \underbrace{\alpha_k^n \|C_0^{[n-k]}\|}_{\beta_k^n}\,\xi_{n-k}.$$

From (48) we obtain $(x(n)\,|\,N_x(n)) = \beta_0^n \xi_n$, $n \in \mathscr{Z}$, and hence (ξ_n) is a 0-1 innovation in M_x. Directly from the construction it also follows that the sequences (β_k^n) satisfy (U-1) and (U-2). ∎

Below we will show that a converse theorem is also true. In order to simplify the formulation we will say that functions $f^k(\cdot) \in L_-^2(\mathscr{C}^T)$, $k = 0, \ldots, N$, are *jointly r-outer* if

$$\overline{\operatorname{sp}}\{e^{-in\cdot} f^k(\cdot)\colon k = 0, \ldots, N,\ n \geqslant 0\} = L_-^2(\mathscr{C}_0 \oplus \ldots \oplus \mathscr{C}_{T-1}),$$

where $\mathscr{C}_k = \mathscr{C}$ or $\mathscr{C}_k = \{0\}$, and exactly r of $\mathscr{C}_k$'s are nonzero. For example, the rows $A^k(\cdot)$ of a $T \times T$ matrix function $A(s) = [A^{k,j}(s)]$ are jointly r-outer if $T-r$ columns of $A(s)$ are zero ds-a.e. and when they are removed from the matrix, the rows of the resulting $T \times r$ matrix function are jointly outer in $L_-^2(\mathscr{C}^r)$.

THEOREM 5.5. *Let* $(x(n))$ *be a regular PC sequence of period T and rank r,* (g_j) *be its spectral densities and let* $G^{j,k}(s)$ *be as in* (34). *Suppose that* (β_k^n), $k \geqslant 0$, *are T-periodic (in n) and satisfy the conditions* (U-0), (U-1) *and* (U-2), *where* (ξ_n) *is a* 0-1 *innovation in* M_x. *Define*

$$\text{(i)} \qquad C_k^q = (0, \ldots, 0, \underbrace{\beta_k^q}_{[q-k]}, 0, \ldots, 0) = \beta_k^q e_{[q-k]}, \quad 0 \leqslant q < T,\ k \geqslant 0,$$

where (e_k) *is the standard basis in* $\mathscr{C}^T$;

(ii) $C^q(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} C_k^q e^{-iks}$

$$\text{and} \quad A^k(s) = (1/T) \sum_{q=0}^{T-1} e^{-2\pi i qk/T} C^q(s), \quad k, q = 0, \ldots, T-1.$$

Then the rank of $(\xi_n) = r$, *the* $\mathscr{C}^T$*-valued functions* $A^k(s)$ *are jointly r-outer, and*

(49) $\quad g_k(s) = T\big(A^j(s-2\pi j/T), A^{j-k}(s-2\pi j/T)\big)$, *ds-a.e.*, $\quad j, k \in \mathscr{Z}_T$.

In other words, if we remove from the matrix function $A(s) = [A^k(s)]$ the $T-r$ zero columns, then the resulting $T \times r$-matrix function is an outer square root of $[G^{j,k}(s)]$.

Proof. First we will show that the matrix function $A(s) = [A^k(s)]$ is a square root of the density $G(s) = [G^{j,k}(s)]$. Let us put

$$C_k = [C_k^q], \quad A_k = E^{-1} C_k, \quad C(s) = [C^q(s)] = (1/\sqrt{2n}) \sum_{k=0}^{\infty} C_k e^{-iks},$$

and

$$A(s) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} A_k e^{-iks} = E^{-1} C(s).$$

Let $Y^q(n)(s) = e^{ins} A^q(s)$ and let $y(n) = \sum_{q=0}^{T-1} e^{2\pi i qn/T} Y^q(n)$, $n \in \mathscr{Z}$, so that $[Y^q(n)]$ generates $(y(n))$. From the definitions of C_k and A_k it follows that

$$y(n)(\cdot) = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} C_k^{[n]} e^{i(n-k)\cdot} = (1/\sqrt{2\pi}) \sum_{k=0}^{\infty} \beta_k^n e_{[n-k]} e^{i(n-k)\cdot}, \quad n \in \mathscr{Z}.$$

By (U-0) we have

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k}, \quad n \in \mathscr{Z},$$

and hence $(y(n))$ is equivalent to $(x(n))$, where the equivalence is achieved through the mapping $\Phi(\xi_n) = e_{[n]} e^{in\cdot}$, $\xi_n \neq 0$. Therefore (g_j) are the spectral densities of $(y(n))$. Since $[Y^q(n)]$ generates $(y(n))$, we obtain

(50) $$g_q(s) = \sum_{p=0}^{T-1} \big(A^p(s-2\pi p/T), A^{p-q}(s-2\pi p/T)\big), \quad q \in \mathscr{Z}_T$$

(Proposition 3.5). We will show that in fact $[Y^q(n)]$ is equivalent to the sequence induced by $(y(n))$, that is

(51) $$\big(A^p(s), A^q(s)\big) = G^{p,q}(s) = (1/T)\, g_{p-q}(s+2\pi p/T).$$

Because of (50), it is enough to show that functions $\big(A^p(s-2\pi p/T), A^{p-q}(s-2\pi/T)\big)$ do not depend on p. From the definition of A_k it follows that

$$A_k^p = (1/T)\sum_{q=0}^{T-1} e^{-2\pi ijq/T}\beta_k^q e_{[q-k]} = e^{-2\pi iqj/T}\Big(\sum_{r=0}^{T-1} e^{-2\pi iqr/T}\beta_j^{r+j} e_r\Big),$$

$$(A_k^p, A_j^{p-q}) = (1/T^2)e^{-2\pi ip(k-j)/T}e^{-2\pi iqj/T}\sum_{r=0}^{T-1} e^{-2\pi iqr/T}\beta_k^{r+k}\overline{\beta_j^{r+j}},$$

and hence

$$\big(A^p(t), A^{p-q}(t)\big)$$
$$= (1/2\pi T^2)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e^{-2\pi ip(k-j)/T}e^{-2\pi iqj/T}\sum_{r=0}^{T-1} e^{-2\pi iqr/T}\beta_k^{r+k}\overline{\beta_j^{r+j}}e^{-ikt}e^{ijt}.$$

Therefore

$$\big(A^p(t-2\pi p/T), A^{p-q}(t-2\pi p/T)\big)$$
$$= (1/2\pi T^2)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{T-1} e^{-2\pi iq(r+k)/T}\beta_k^{r+k}\overline{\beta_j^{r+j}}e^{-ikt}e^{ijt}$$

does not depend on p. Note that the formula above also gives an explicit expression for the spectral densities (g_q) of $\big(x(n)\big)$ in terms of (β_k^n), namely

$$g_q(t) = (1/2\pi T)\sum_{r=0}^{T-1} e^{-2\pi iqr/T}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\beta_k^r\overline{\beta_j^{r-k+j}}e^{-i(k-j)t}, \tag{52}$$

which is simpler than (23)–(24).

We shall prove that $A(s)$ is an r-outer square root of $G(s)$. From (51) it follows that there is an isometry Φ from M_Z onto M_Y such that $\Phi Z^q(n) = Y^q(n) = e^{in\cdot}A^q(\cdot)$ and, consequently,

$$\Phi x(n) = \sum_{q=0}^{T-1} e^{2\pi iqn/T}\Phi\big(Z^q(n)\big) = y(n).$$

Note that if U is the shift of $[Z^q(n)]$, then from the definition of $[Z^q(n)]$ it follows that for every $(z_0, z_1, \ldots, z_{T-1}) \in M_x^T = M_Z$

$$U(z_0, z_1, \ldots, z_{T-1}) = (Vz_{T-1}, z_0, \ldots, z_{T-2}), \tag{53}$$

where V is the T-shift of $\big(x(n)\big)$. Therefore

$$U\big(N_x(m)\oplus\{0\}\oplus\ldots\oplus\{0\}\big) = \big(\{0\}\oplus N_x(m)\oplus\{0\}\oplus\ldots\oplus\{0\}\big) \subset M_Z(m+1).$$

By iterations we obtain

$$U^p N_x(n-p) \in N_Z(n), \quad n\in\mathscr{Z},\ p\geqslant 0 \tag{54}$$

(recall that $N_x(j)$ is identified with $N_x(j)\oplus\{0\}\oplus\ldots\oplus\{0\} \subset M_Z$). Since Φ is unitary, the same relationship holds for the pair $[Y^q(n)]$ and $\big(y(n)\big)$.

We shall compute $M_Y(0)$. From (U-0) it follows that, for every $n \in \mathscr{Z}$, $N_x(n) = \operatorname{sp}\{\beta_0^n \xi_n\} = \operatorname{sp}\{\xi_n\}$, because if $\beta_0^n = 0$ while $\xi_n \neq 0$, then ξ_n would not be in M_x. Therefore $N_y(m) = \operatorname{sp}\{C_0^{[m]} e^{im\cdot}\}$. Since the shift of $[Y^q(n)]$ is the operator of multiplication by e^{is}, from (54) applied to the pair $[Y^q(n)]$ and $(y(n))$ we conclude that $C_0^{[n-p]} e^{in\cdot} \in N_Y(n)$ for all $p \geqslant 0$, and hence

$$N_Y(n) = \overline{\operatorname{sp}}\{e^{in\cdot} C_0^q \colon q = 0, \ldots, T-1\} = e^{in\cdot}(\mathscr{C}_0 \oplus \ldots \oplus \mathscr{C}_{T-1}),$$

where $\mathscr{C}_k$ is either $\{0\}$ or $\mathscr{C}$ depending on whether β_0^k is zero or not. This shows that $M_Y(0) = L_-^2(\mathscr{C}_0 \oplus \ldots \oplus \mathscr{C}_{T-1})$. Since the rank of $(x(n))$ is r, exactly r terms in the sequence $\beta_0^0, \ldots, \beta_0^{T-1}$ are nonzero, and hence $A^q(\cdot)$, $q = 0, \ldots, T-1$, are jointly r-outer. ■

Theorems 5.4 and 5.5 are PC analogues of Proposition 3.3, part (B). Since the rows of $A(s)$ are r-outer iff the rows of $C(s) = EA(s)$ are r-outer, the proof of Theorem 5.5 yields the following corollary:

COROLLARY 5.6. *Let $(x(n))$ be a regular PC sequence with period T and rank r. Suppose that there is a 0-1 innovation of rank r in M_x such that*

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k}, \qquad n \in \mathscr{Z}, \tag{55}$$

where (β_k^n), $k = 0, 1, \ldots$, are T-periodic in n and satisfy (U-1) *and* (U-2). *Then the representation* (55) *is optimal (i.e.* (U-0) *holds true) iff the functions $C^q(\cdot)$, $q = 0, \ldots, T-1$, defined in Theorem 5.5 are jointly r-outer.*

If a sequence $(x(n))$ is completely regular, then $r = T$ and the phrase *r-outer* in Theorem 5.5 is replaced by a familiar phrase *outer in C^T*. The prediction problem for completely regular PC sequences was studied earlier in [10], where the authors introduced a new notion of a T-complete system: A system of T functions $f_0, \ldots, f_{T-1} \in L_-^2(\mathscr{C})$ is said to be *T-complete* if for every $k = 0, \ldots, T-1$

$$\overline{\operatorname{sp}}\{e^{im\cdot} f_{[m+k]}(\cdot);\ m \leqslant 0\} = L_-^2(\mathscr{C}).$$

If $(x(n))$ is a completely regular PC sequence with period T and (β_k^n) are the coefficients in the optimal moving average representation of $(x(n))$, then the functions $f_n(s) = \sum_{k=0}^{\infty} \beta_k^n e^{-iks}$, $n = 0, \ldots, T-1$, form a T-complete system, and hence the results of this section also characterize T-complete systems.

6. OTHER ASSOCIATED SEQUENCES

Stationary block sequence. Perhaps the most natural stationary sequence associated with a PC sequence is a *block* sequence, which is constructed by partitioning a PC sequence into adjacent blocks of length T. The starting point for a partition can be arbitrary, depending on what predictor is of interest.

DEFINITION 6.1. Let $(x(n))$ be a PC sequence with period T and let $0 \leqslant d < T$. A T-dimensional stationary sequence $[X_d^q(n)]$ defined by $X_d^q(n) = x(d+nT+q)$, $n \in \mathscr{Z}$, $q = 0, \ldots, T-1$, will be called a *block sequence* (at d) associated with $(x(n))$.

A block sequence has excellent properties. It takes values in M_x; for a fixed d the correspondence between $(x(n))$ and $[X_d^q(n)]$ is a bijection from the set of all PC sequences with period T onto the set of all T-dimensional stationary sequences; both sequences are simultaneously regular or deterministic; rank of $[X_d^q(n)]$ is equal to the rank of $(x(n))$; the shift of $[X_d^q(n)]$ is equal to the T-shift operator of $(x(n))$, and, finally, doing prediction of both is the same task because $M_{X_d}(n) = M_x(d+(n+1)T-1)$. A block sequence was successfully used in many papers on PARMA models (e.g. [21], [23]), and also was employed in [19] to obtain the Wold–Cramer decomposition of a PC sequence.

The major disadvantage of a block sequence is that its spectrum and the spectrum of a corresponding PC sequence are related in a rather complex way. To see the relation let us put $J_r = [2\pi r/T, 2\pi(r+1)/T)$, $r \in \mathscr{Z}_T$, and let ω: $[0, 2\pi) \to [0, 2\pi)$ be a function defined by

$$\omega(s) = T \sum_{r=0}^{T-1} (s - 2\pi r/T) IND_{J_r}(s), \tag{56}$$

where IND_Δ denotes the indicator of a set Δ. If μ is a measure on $[0, 2\pi)$, then the image of μ under ω will be denoted by $\mathscr{W}\mu$, that is $(\mathscr{W}\mu)(\Delta) = \mu(w^{-1}(\Delta))$. The operator $\mathscr{W}$ splits μ into the sum of its restrictions $\mu_r(\Delta) = \mu(\Delta \cap J_r)$ to J_r, $r = 0, \ldots, T-1$, shifts each μ_r to the left by $2\pi r/T$, adds the shifted measures up and then stretches the resulting sum to the interval $[0, 2\pi)$, that is:

$$(\mathscr{W}\mu)(ds) = \sum_{r=0}^{T-1} \mu_r(ds/T + 2\pi r/T).$$

Note that if f is $\mathscr{W}\mu$-integrable and has period $2\pi/T$, then

$$\int_0^{2\pi} f(s)\, d(\mathscr{W}\mu)(ds) = \int_0^{2\pi} f(Ts)\, d\mu. \tag{57}$$

PROPOSITION 6.2 (cf. [5]). (A) *Let $(x(n))$ be a PC sequence with period $T > 1$ and spectrum (γ_j) and let $[X_d^q(n)]$ be a block sequence of $(x(n))$. Then $[X_d^q(n)]$ is stationary and the p, q-th entry of its spectral measure is given by*

$$\Gamma_d^{p,q}(ds) = \sum_{j=0}^{T-1} e^{2\pi i j(d+q)/T}\, \mathscr{W}\left(e^{i(p-q)s} \gamma_j(ds)\right). \tag{58}$$

(B) *Let $[X^q(n)]$ be a T-dimensional stationary sequence with the spectral measure $[\Gamma^{p,q}]$ and let $x(n) = X^{[n]}(Q(n))$, $n \in \mathscr{Z}$, where $n = Q(n)T + [n]$, $0 \leqslant [n] < T$, $Q(n) \in \mathscr{Z}$. Then $(x(n))$ is PC with period T and the spectrum (γ_j) of*

$(x(n))$ *is uniquely determined by the relations*

$$(59)\qquad \mathscr{W}\big(e^{irs}\gamma_j(ds)\big) = (1/T)\sum_{q=0}^{T-1} e^{-2\pi ijq/T} e^{iQ(q+r)s}\Gamma^{[q+r],q}(ds),$$

$$r = 0, \ldots, T-1,\ j \in \mathscr{Z}_T.$$

Proof. (A) Suppose that $(x(n))$ is PC with period T, and let $[X_d^q(n)]$ be its block sequence (at d). Then from (6), (7), and (57) we obtain

$$\begin{aligned}(X_d^p(n), X_d^q(m)) &= K_x(d+nT+p,\ d+mT+q)\\ &= \sum_{j=0}^{T-1} e^{2\pi i(mT+d+p)j/T} a_j\big((n-m)T+p-q\big)\\ &= \int_0^{2\pi} e^{i(n-m)/Ts}\Big(\sum_{j=0}^{T-1} e^{i(p-q)s} e^{2\pi ij(d+q)/T}\gamma_j(ds)\Big)\\ &= \sum_{r=0}^{T-1}\int_0^{2\pi} e^{i(n-m)s}\Big(\sum_{j=0}^{T-1} e^{i(p-q)(s/T+2\pi r/T)} e^{2\pi ij(d+q)/T}\gamma_j(ds/T+2\pi r/T)\Big).\end{aligned}$$

Hence $[X_d^q(n)]$ is stationary and $\Gamma_d^{p,q}(ds) = \mathscr{W}\big(\sum_{j=0}^{T-1} e^{2\pi ij(d+q)/T} e^{i(p-q)s}\gamma_j(ds)\big)$.

(B) Suppose that $[X^q(n)]$ is a T-dimensional stationary sequence and $x(n) = X^{[n]}(Q(n))$, $n \in \mathscr{Z}$. Then $(x(n))$ is PC with period T and $[X^q(n)]$ is its block sequence at $d = 0$. Substituting $d = 0$ and $p = [q+r]$, $r = 0, \ldots, T-1$, in (58), we obtain $p-q = r-Q(r+q)T$, and hence

$$(60)\qquad \Gamma^{[r+q],q}(ds) = e^{-iQ(r+q)s}\sum_{j=0}^{T-1} e^{2\pi ijq/T}\mathscr{W}\big(e^{irs}\gamma_j(ds)\big).$$

Multiplying both sides by $e^{iQ(r+q)/Ts}$ and taking the inverse discrete Fourier transform, we obtain (59). The relations (59) determine (γ_j) uniquely despite that $\mathscr{W}\mu = \mathscr{W}\nu$ does not imply that $\mu = \nu$. However, if $\mathscr{W}\big(e^{irs}\mu(ds)\big) = \mathscr{W}\big(e^{irs}\nu(ds)\big)$ for $r = 0, \ldots, T-1$, then

$$\int_0^{2\pi} e^{inTs}\mathscr{W}\big(e^{irs}\mu(ds)\big) = \int_0^{2\pi} e^{inTs}\mathscr{W}\big(e^{irs}\nu(ds)\big), \qquad n \in \mathscr{Z},$$

and, by (57), $\int_0^{2\pi} e^{i(nT+r)s}\mu(ds) = \int_0^{2\pi} e^{i(nT+r)s}\nu(ds)$, $n \in \mathscr{Z}$, $r = 0, \ldots, T-1$. Hence $\mu = \nu$. ∎

The relations (58) and (59) are not only complicated, but additionally the operator $\mathscr{W}$ does not transfer conjugate analytic roots of $[d\Gamma^{p,q}/ds]$ into conjugate analytic factors of (γ_j/ds), and this is the main reason that an induced sequence was used in the paper instead of a block sequence. However, the predictors of $(x(n))$ and the predictors of its block sequence are obviously related through a time domain procedure, and therefore the results of Section 5 link the coefficients (β_k^n) of an optimal moving average representation of a PC sequence with matrix coefficients of an optimal moving average representation of its block sequence. Since each T-dimensional stationary sequence arranged in a linear order forms a PC sequence with period T, the author believes that the technique of induced sequences and the results of this paper may also contribute to the prediction theory of T-dimensional stationary sequences.

Generating stationary sequences in M_x. An induced stationary sequence lives in much bigger space than M_x, and this raises the question of whether it is possible to find a generating sequence $[W^q(n)]$ *with values in* M_x, which at least controls the regularity of $(x(n))$. The principal root sequence, so far the only generating sequence of $(x(n))$ constructed in M_x, is always deterministic, and hence does not meet this requirement.

THEOREM 6.3. *A PC sequence $(x(n))$ with period T has a regular generating sequence with values in M_x iff $(x(n))$ is completely regular. Moreover, each regular T-dimensional stationary sequence in M_x that generates a completely regular PC sequence is of rank* 1.

The fact that a completely regular PC sequence admits a regular generating sequence in M_x was first noted in [10].

Proof. ($\Rightarrow$) Assume that $[W^q(n)]$ is a regular T-dimensional sequence in M_x such that

$$x(n) = \sum_{q=0}^{T-1} e^{2\pi inq/T} W^q(n), \qquad n \in \mathscr{Z}.$$

Let r be the rank of $[W^q(n)]$. Then $[W^q(n)]$ is equivalent to an $L^2(\mathscr{C}^r)$-valued sequence $Y^q(n)(t) = e^{itn} A^q(t)$, where $[A^q(\cdot)]$ is a conjugate analytic square root of the spectral density $d\Gamma/dt$ of $[W^q(n)]$ and $\operatorname{rank}(\hat{A}(0)) = r$. Hence the sequence $y(n) = \sum_{q=0}^{T-1} e^{2\pi inq/T} Y^q(n)$ is an $L^2(\mathscr{C}^r)$-valued PC sequence equivalent to $(x(n))$. Note that the shift U of $[Y^q(n)]$ is the operator of multiplication by e^{is} and that $M_Y = M_y$, because by assumption $M_W = M_x$. Since $[Y^q(n)]$ generates $(y(n))$, we conclude that $V = U^T$ is the operator of multiplication by e^{iTs}. Consider the group $V^n = (U^T)^n$, $n \in \mathscr{Z}$, and let $e_1, \ldots, e_r$ be the standard basis in $\mathscr{C}^r$. The functions $u_{k,q} = e^{ik\cdot} e_q$, $k = 0, -1, \ldots, -T+1$, $q = 0, \ldots, r-1$, have the property that the rT subspaces $M_{k,q} = \overline{\operatorname{sp}}\{V^n u_{k,q}, n \in \mathscr{Z}\}$ are orthogonal. Since the spectral type (see [3], pp. 914–918, or [16], Section 2.3) of each vector $u_{k,q}$ is the Lebesgue measure, which is the maximal spectral type of the group V^n, we conclude that the multiplicity of the group V^n in M_Y is rT. On the other hand, the subspaces $V^n(M_y(0) - M_y(-T))$, $n \in \mathscr{Z}$, also span $M_Y = M_y$. Hence the multiplicity of V^n in M_Y is at most $d = \dim(M_y(0) - M_y(-T))$ which is at most T. Comparing this with previously computed multiplicity we infer that

$$\text{multiplicity of } V^n \text{ in } M_Y = rT \leqslant d \leqslant T,$$

which is possible only if $r = 1$ and $d = \dim(M_x(0) \ominus M_x(-T)) = T$, that is if $[W^q(n)]$ is of rank 1 and $(x(n))$ is completely regular. Note that we also proved the "moreover" part.

($\Leftarrow$) Suppose now that $(x(n))$ is completely regular. Then it is regular and from Proposition 3.5 it follows that $(x(n))$ can be written as $x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k}$, $n \in \mathscr{Z}$, where $[\xi_n]$ is an innovation in M_x and the coefficients β_k^n satisfy (21), (U-1) and (U-2) of Proposition 3.5. Define the unitary operator U: $M_x \to M_x$ as a linear extension of the mapping $U\xi_n = \xi_{n+1}$, $n \in \mathscr{Z}$. Since (β_k^n) are T-periodic, $U^T x(n) = x(n+T)$, $n \in \mathscr{Z}$, so U is a T-th root of the T-shift operator V of $(x(n))$. Define

$$p(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{-k}, \qquad W^q(0) = (1/T) \sum_{j=0}^{T-1} e^{-2\pi i j q/T} p(j), \qquad \text{and}$$
$$W^q(n) = U^n W^q(0), \qquad n \in \mathscr{Z}.$$

Then $W^q(n) \in M_\xi(n)$, and so $[W^q(n)]$ is regular. Moreover,

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k} = U^n p(n) = \sum_{q=0}^{T-1} e^{2\pi i q n/T} W^q(n), \qquad n \in \mathscr{Z},$$

and hence $[W^q(n)]$ generates $(x(n))$. ∎

7. PARMA SEQUENCES

In the case of T-dimensional stationary sequences an explicit computation of an outer square root of its spectral density is possible only in few cases. One of them is the case when a sequence is a solution to an ARMA system of equations. A PC analogue of an ARMA sequence is called *PARMA*, and is defined to be a PC sequence $(x(n))$ that satisfies a system

$$x(n) - \sum_{k=1}^{p(n)} \phi_k(n) x(n-k) = \sum_{k=0}^{q(n)} \theta_k(n) \xi_{n-k}, \qquad n \in \mathscr{Z}, \tag{61}$$

where it is assumed that:

(A-1) the sequences $p(n)$, $q(n)$, $\phi_k(n)$, and $\theta_k(n)$ are periodic with the same period T,

(A-2) (ξ_n) is an orthonormal sequence (innovation).

Setting $\phi_k(n) = 0$ and $\theta_j(n) = 0$ for $p(n) < k \leqslant L$ and $q(n) < j \leqslant R$, where $L \geqslant \max\{p(n)\colon p(n) \neq 0, n \in \mathscr{Z}\}$ and $R \geqslant \max\{q(n)\colon q(n) \neq 0, n \in \mathscr{Z}\}$, we may assume that $p(n) \equiv L$, and $q(n) \equiv R$ are constants.

Our goal in this section is to compute spectral densities $g_j = d\gamma_j/ds$, $j \in \mathscr{Z}_T$, of a PARMA sequence $(x(n))$ and identify coefficients in the optimal moving average representation of $(x(n))$, provided that (61) has a unique PC solution in M_ξ. A literature on PARMA sequences is vast and many diverse methods for studying PARMA systems, including procedures for retrieving predictor coefficients and for computing spectral densities, have been already developed (see for example [1], [12], [21]–[23] and references therein). This section should be viewed as an example of an application of an induced process technique rather than a systematic analysis of PARMA models.

The system (61) may have no solution at all. Even if it does have, it is not obvious whether a solution is PC, whether it belongs to the space M_ξ or is unique, although it is easy to see that if (61) has a unique solution $(x(n))$ in the space M_ξ, then this solution must be PC and its T-shift operator V is equal to U^T, where U: $M_\xi \to M_\xi$ is defined by $U\xi_n = \xi_{n+1}$, $n \in \mathscr{Z}$. By proper partitioning the sums on both sides of (61), Vecchia [23] rewrote (61) in terms of a T-dimensional block sequence $X^q(n) = x(nT+q)$, $n \in \mathscr{Z}$, $q = 0, \ldots, T-1$, and transformed a PARMA system into a T-dimensional ARMA systems involving $[X^q(n)]$. However, due to a complicate relationship between the spectra of a PC sequence and its block sequence, Vecchia's construction is not very suitable for our purpose. Our main tool here will be an induced sequence instead. An induced sequence technique is very similar to the approach used by Sakai in [22] for the same purpose.

Recall that a T-dimensional sequence $[Z^q(n)]$ is said to be *induced* by a sequence $(x(n))$ if $Z^q(n) \in M_x^T$ is defined by

$$Z^q(n)(k) = (1/T)\,x(n-k)\,e^{-2\pi i q(n-k)/T}, \qquad k = 0, \ldots, T-1, \tag{62}$$

$n \in \mathscr{Z}$, $q = 0, \ldots, T-1$. Propositions 4.1 and 4.2 state that $(x(n))$ is PC with period T iff $[Z^q(n)]$ is T-dimensional stationary. Observe that if $[Z_\xi^q(n)]$ is induced by an innovation (ξ_n), then

$$(Z_\xi^p(n), Z_\xi^q(m))_{M_\xi^T} = (1/T^2)\sum_{k=0}^{T-1} e^{-2\pi i p(n-k)/T}\, e^{-2\pi i q(m-k)/T}\,(\xi_{n-k}, \xi_{m-k}) \tag{63}$$

$$= \begin{cases} 1/T & \text{if } n = m \text{ and } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\zeta_n^q(k) = \sqrt{T}\,Z_\xi^q(n)(k) = (1/\sqrt{T})\,\xi_{n-k}\,e^{-2\pi i q(n-k)/T}, \qquad k = 0, \ldots, T-1, \tag{64}$$

satisfies $[\zeta_n^q][\zeta_m^q]^* = I_T\,\delta_{n-m}$, that is $[\zeta_n^q]$ is a T-dimensional innovation in M_ξ^T. Clearly, $M_\zeta = M_\xi^T$ (Proposition 4.1, (I-3)).

Consider a PARMA system

$$x(n) - \sum_{k=1}^{L} \phi_k(n)\,x(n-k) = \sum_{k=0}^{R} \theta_k(n)\,\xi_{n-k}, \qquad n \in \mathscr{Z}, \tag{65}$$

where $\phi_k(n)$, $\theta_k(n)$ and (ξ_n) satisfy (A-1) and (A-2). Let $\boldsymbol{A}_m$ and $\boldsymbol{B}_m$ be

$T\times T$-matrices defined by

$$(66)\qquad A_m^{j,k} = e^{-2\pi imk/T}\tilde{\phi}_m(j-k), \qquad j,k\in\mathscr{Z}_T,\ m=1,\ldots,L,$$

$$(67)\qquad B_m^{j,k} = (1/\sqrt{T})\,e^{-2\pi imk/T}\tilde{\theta}_m(j-k), \qquad j,k\in\mathscr{Z}_T,\ m=1,\ldots,R,$$

where $\tilde{s}$ denotes the discrete Fourier transform (2) of a T-periodic sequence $(s(n))$. Let us put

$$(68)\qquad A(z) = I-\sum_{m=1}^{L} A_m z^m \quad\text{and}\quad B(z) = \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{R} B_m z^m, \qquad z\in\mathscr{C},$$

and consider the ARMA system

$$(69)\qquad [X^j(n)]-\sum_{k=1}^{L} A_k[X^j(n-k)] = \sum_{k=0}^{R} B_k[\zeta_{n-k}^j],$$

where $[\zeta_n^q]$ is the T-dimensional innovation in M_ξ^T defined in (64).

LEMMA 7.1. *The PARMA system* (65) *has a PC solution iff the ARMA system* (69) *has a stationary solution.*

Proof. For convenience rewrite the system (65) as

$$(70)\qquad x(n) = \sum_{k=1}^{L}\phi_k(n)\,x(n-k)+\sum_{k=0}^{R}\theta_k(n)\,\xi_{n-k}, \qquad n\in\mathscr{Z},$$

($\Rightarrow$) Suppose that $(x(n))$ is a PC solution to (70) and let $[Z^q(n)]$ be the sequence induced by $(x(n))$. Then $[Z^q(n)]$ is a T-dimensional stationary sequence and

$$\begin{aligned}
Z^q(n)(p) &= (1/T)\,e^{-2\pi iq(n-p)/T}\Big(\sum_{k=1}^{L}\phi_k(n-p)\,x(n-k-p)+\sum_{k=0}^{R}\theta_k(n-p)\,\xi_{n-k-p}\Big)\\
&= (1/T)\sum_{k=1}^{L} e^{-2\pi iq(n-p)/T}\sum_{j=0}^{T-1} e^{2\pi ij(n-p)/T}\tilde{\phi}_k(j)\,x(n-k-p)\\
&\quad+(1/T)\sum_{k=0}^{R} e^{-2\pi iq(n-p)/T}\sum_{j=0}^{T-1} e^{2\pi ij(n-p)/T}\tilde{\theta}_k(j)\,\xi_{n-k-p}\\
&= (1/T)\sum_{k=1}^{L}\sum_{j=0}^{T-1} e^{-2\pi i(q-j)k/T}\tilde{\phi}_k(j)\,e^{-2\pi i(q-j)(n-k-p)/T}x(n-k-p)\\
&\quad+(1/T)\sum_{k=0}^{R}\sum_{j=0}^{T-1} e^{-2\pi i(q-j)k/T}\tilde{\theta}_k(j)\,e^{-2\pi i(q-j)(n-k-p)/T}\xi_{n-k-p}\\
&= \sum_{k=1}^{L}\sum_{j=0}^{T-1} e^{-2\pi i(q-j)k/T}\tilde{\phi}_k(j)\,Z^{q-j}(n-k)(p)\\
&\quad+(1/\sqrt{T})\sum_{k=0}^{R}\sum_{j=0}^{T-1} e^{-2\pi i(q-j)k/T}\tilde{\theta}_k(j)\,\zeta_{n-k}^{q-j}(p).
\end{aligned}$$

Since all components of the sums above are T-periodic in j, substituting $q-j=r$ we obtain for each $n\in\mathscr{Z}$, $q\in\mathscr{Z}_T$, and $p=0,\ldots,T-1$

$$Z^q(n)(p)=\sum_{k=1}^{L}\sum_{r=0}^{T-1}e^{-2\pi irk/T}\tilde{\phi}_k(q-r)Z^r(n-k)(p)$$
$$+(1/\sqrt{T})\sum_{k=0}^{R}\sum_{r=0}^{T-1}e^{-2\pi irk/T}\tilde{\theta}_k(q-r)\zeta^r_{n-k}(p).$$

Therefore $[Z^j(n)]$ is a T-dimensional stationary solution to (69). Note that if $x(n)$ is a solution in M_ξ, then $M_x\subseteq M_\xi$, and from the property (I-3) of Proposition 4.1 it follows that $M_Z=M_x^T\subseteq M_\xi^T=M_\zeta$, that is $Z^j(n)\in M_\zeta$.

($\Leftarrow$) Suppose now that $[X^j(n)]$ is a T-dimensional stationary sequence that satisfies (69). Define $x(n)=\sum_{q=0}^{T-1}e^{2\pi inq/T}X^q(n)$. Then $(x(n))$ is PC with period T, and

$$x(n)=\sum_{q=0}^{T-1}e^{2\pi inq/T}\Big(\sum_{k=1}^{L}\sum_{r=0}^{T-1}e^{-2\pi irk/T}\tilde{\phi}_k(q-r)X^r(n-k)$$
$$+(1/\sqrt{T})\sum_{k=0}^{R}\sum_{r=0}^{T-1}e^{-2\pi irk/T}\tilde{\theta}_k(q-r)\zeta^r_{n-k}(p)\Big).$$

Writing

$$\tilde{\phi}_k(q-r)=(1/T)\sum_{j=0}^{T-1}e^{-2\pi i(q-r)j/T}\phi_k(j)$$

and

$$\tilde{\theta}_k(q-r)=(1/T)\sum_{j=0}^{T-1}e^{-2\pi i(q-r)j/T}\theta_k(j),$$

and using the fact that $\sum_{q=0}^{T-1}e^{2\pi iq(n-j)/T}=0$ except when $n=j$, we obtain

$$x(n)=\sum_{k=1}^{L}\phi_k(n)\Big(\sum_{r=0}^{T-1}e^{-2\pi ir(n-k)/T}X^r(n-k)\Big)$$
$$+\sum_{k=0}^{R}\theta_k(n)\Big((1/\sqrt{T})\sum_{r=0}^{T-1}e^{-2\pi ir(n-k)/T}\zeta^r_{n-k}\Big)$$
$$=\sum_{k=1}^{L}\phi_k(n)x(n-k)+\sum_{k=0}^{R}\theta_k(n)\xi_{n-k},$$

since $(1/\sqrt{T})\sum_{r=0}^{T-1}e^{-2\pi irn/T}\zeta^r_n=(\xi_n,0,\ldots,0)$ (again M_ξ is identified with $M_\xi\oplus\{0\}\oplus\ldots\oplus\{0\}\subset M_\xi^T$). ∎

From the proof it follows that if $(x(n))$ is a PC solution to (65) and if the system (69) has a unique stationary solution, then the latter must be the sequence induced by $(x(n))$. The conditions for existence and uniqueness of a stationary solution to a T-dimensional ARMA system are well known (e.g. [2]).

LEMMA 7.2. *The system* (69) *has a unique stationary solution iff* $\det A(z)$ *has no zeros on the unit circle. Moreover, if this is the case, then the only stationary solution is given by*

$$(71) \qquad [X^q(n)] = \sum_{k=-\infty}^{\infty} D_k[\zeta_{n-k}^q], \quad n \in \mathscr{Z},$$

where $D_k = (1/\sqrt{2\pi})\int_0^{2\pi} e^{ikt} A(e^{-it})^{-1} B(e^{-it})\,dt$, $k \in \mathscr{Z}$.

By definition the matrices A_k and B_k have the property that, for every $k \geqslant 0$, $e^{2\pi irk/T} A_k^{q,r}$ and $e^{2\pi irk/T} B_k^{q,r}$ depend only on $q-r$. We will show that if $[X^q(n)]$ in (71) is *causal*, then the coefficients D_k also have this property.

LEMMA 7.3. *If* $\det A(z)$ *has no zeros on the unit disc* $\{|z| \leqslant 1\}$, *then the system* (69) *has a unique stationary solution*

$$(72) \qquad [X^q(n)] = \sum_{k=0}^{\infty} D_k[\zeta_{n-k}^q], \quad n \in \mathscr{Z},$$

and the matrices D_k *have the form*

$$(73) \qquad D_k^{q,r} = e^{-2\pi irk/T}\eta_k(q-r), \quad q, r \in \mathscr{Z},$$

where η_k, $k \geqslant 0$, *are functions of* $q-r$.

Proof. The first part is well known. If $\det A(z)$ has no zeros on the unit disc $\{|z| \leqslant 1\}$, then $d(z) = 1/\det A(z)$ is analytic in $\{|z| < 1+\varepsilon\}$ for some $\varepsilon > 0$, and hence $A^{-1}(z)$ and $A^{-1}(z)B(z)$ are analytic. Therefore

$$A^{-1}(z)B(z) = (1/\sqrt{2\pi})\sum_{k=0}^{\infty} D_k z^k, \quad |z| < 1+\varepsilon,$$

which in view of Lemma 7.2 yields (72). Multiplying the above equation by $A(z)$ we obtain

$$(74) \qquad \Big(I - \sum_{k=1}^{L} A_k z^k\Big)\Big(\sum_{k=0}^{\infty} D_k z^k\Big) = \sum_{m=0}^{R} B_m z^m,$$

from which one can recursively compute the coefficients D_k, $k \geqslant 0$:

$$(75) \qquad D_0 = B_0, \quad D_k = B_k + A_1 D_{k-1} + A_2 D_{k-2} + \ldots + A_k D_0, \; k \geqslant 1,$$

where $A_j = 0$ if $j > L$, and $B_k = 0$ if $k > R$. We will show that each D_k has the form (73), given that matrices B_k and A_k do. Clearly, $D_0 = B_0$ does. Suppose that $D_0, \ldots, D_{k-1}$ have the property (73). If $(A_j D_{k-j})^{q,r}$ is the q, r-th entry of the matrix $A_j D_{k-j}$, then

$$(A_j D_{k-j})^{q,r} = \sum_{p=0}^{T-1} A_j^{q,p} D_{k-j}^{p,r} = \sum_{p=0}^{T-1} e^{-2\pi ijp/T}\tilde{\phi}_j(q-p)\, e^{-2\pi i(k-j)r/T}\eta_{k-j}(p-r)$$

$$= e^{-2\pi ikr/T}\sum_{s=0}^{T-1} e^{-2\pi ijs/T}\tilde{\phi}_j(q-r-s)\,\eta_{k-j}(s).$$

Hence, for every $j = 1, \ldots, k$, $A_j D_{k-j}$ has the property (73), and so does $D_k = B_k + A_1 D_{k-1} + A_2 D_{k-2} + \ldots + A_k D_0$. ■

A moving average representation (72) does not have to be optimal. It will be if $M_X(m) = M_\zeta(m)$, that is, if $[X^k(n)]$ is *invertible*. By changing the roles of $[X^k(n)]$ and $[\zeta_n^k]$ in the previous lemmas it is clear that $[X^k(n)]$ is invertible if $\det B(z)$ has no zeros in the unit disk. Therefore, if $\det A(z)$ and $\det B(z)$ have no zeros in the closed unit disc $\{|z| \leqslant 1\}$, then $D(t) = (1/\sqrt{2\pi})\sum_{k=0}^{\infty} D_k e^{-ikt}$ is an outer square root of the density of $[X^q(n)]$.

The following theorem is the main result in this section.

THEOREM 7.4. *Suppose that $\phi_k(n)$, $\theta_k(n)$ and (ξ_n) satisfy* (A-1) *and* (A-2), $A(z)$ *and* $B(z)$ *are defined by* (66), (67) *and* (68), *and that* $\det A(z)$ *and* $\det B(z)$ *have no zeros in the closed unit disc* $\{|z| \leqslant 1\}$. *Let D_k, $k \geqslant 0$, be defined by the equation* $A^{-1}(z)B(z) = (1/\sqrt{2\pi})\sum_{k=0}^{\infty} D_k z^k$, $|z| < 1+\varepsilon$. *Then:*

(i) *the system* (65) *has a unique PC solution $\big(x(n)\big)$ in M_ξ;*

(ii) *the solution is given by*

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n \xi_{n-k},\ n \in \mathscr{Z}, \quad \text{where } \beta_k^n = \sum_{j=0}^{T-1} e^{2\pi ijn/T} D_k^{j,0};$$

(iii) *the moving average representation above is optimal, that is*

$$\big(x(n) \,|\, M_x(m)\big) = \sum_{k=n-m}^{\infty} \beta_k^n \xi_{n-k}, \quad n, m \in \mathscr{Z},\ m \leqslant n; \tag{76}$$

(iv) *the spectral measures (γ_j) of $\big(x(n)\big)$ are absolutely continuous with respect to the Lebesgue measure and their densities are given by*

$$\frac{d\gamma_j}{ds}(s) = T \sum_{q=0}^{T-1} D^{0,q}(s)\overline{D^{T-j,q}(s)},\ ds\text{-a.e.}, \quad j = 0, \ldots, T-1, \tag{77}$$

where $D(t) = A(e^{-it})^{-1} B(e^{-it}) = (1/\sqrt{2\pi})\sum_{k=0}^{\infty} D_k e^{-itk}$.

Proof. Consider an associated system (69), where $\zeta_n^q \in M_\xi^T$ is defined by (64). From Lemmas 7.2 and 7.3 it follows that the system (69) has a unique solution $[X^q(n)]$, which is in $M_\zeta = M_\xi^T$ and is given by (72). Since $\det B(z) \neq 0$ in the unit disc,

$$\big[\big(X^q(n) \,|\, M_X(m)\big)\big] = \sum_{k=n-m}^{\infty} D_k[\zeta_{n-k}^q], \quad n, m \in \mathscr{Z},\ m \leqslant n. \tag{78}$$

Define $x(n) = \sum_{q=0}^{T-1} e^{2\pi iqn/T} X^q(n)$, $n \in \mathscr{Z}$. Then $\big(x(n)\big)$ is a PC sequence with period T, and from the proof of Lemma 7.1 it follows that $\big(x(n)\big)$ satisfies (65) with

$$\xi_n = \sum_{j=0}^{T-1} e^{2\pi ijn/T} \zeta_n^j = (\xi_n, 0, \ldots, 0).$$

We will show that $x(n) \in M_\xi \oplus \{0\} \oplus \ldots \oplus \{0\}$. From (72) and (73) we obtain

$$\begin{aligned} x(n) &= \sum_{q=0}^{T-1} e^{2\pi iqn/T} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} D_k^{q,j} \zeta_{n-k}^j = \sum_{q=0}^{T-1} e^{2\pi iqn/T} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} e^{-2\pi ikj/T} \eta_k(q-j) \zeta_{n-k}^j \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} e^{-2\pi ikj/T} \Big(\sum_{p=0}^{T-1} e^{2\pi i(p+j)n/T} \eta_k(s) \Big) \zeta_{n-k}^j \\ &= \sum_{k=0}^{\infty} \Big(\sum_{p=0}^{T-1} e^{2\pi ipn/T} \eta_k(p) \Big) \Big(\sum_{j=0}^{T-1} e^{2\pi ij(n-k)/T} \zeta_{n-k}^j \Big), \end{aligned}$$

where $\eta_k(p)$ are as in (73). Therefore

$$x(n) = \sum_{k=0}^{\infty} \beta_k^n (\xi_{n-k}, 0, \ldots, 0), \qquad \text{where } \beta_k^n = \sum_{j=0}^{T-1} e^{2\pi ijn/T} \eta_k(j),$$

and hence $x(n) \in M_\xi = M_\xi \oplus \{0\} \oplus \ldots \oplus \{0\}$. Note that we have also proved part (ii), because by definition $\eta_k(j) = D_k^{j,0}$, $j \in \mathscr{Z}_T$, $k \geqslant 0$. To see uniqueness, assume that $(z(n))$ is a PC solution to (65) in M_ξ different than $(x(n))$. The sequences induced by $(x(n))$ and $(z(n))$ are then two different stationary solutions to (69), but by Lemma 7.2 this is impossible.

To prove part (iii) note that from the uniqueness of a stationary solution to (69) and from the proof of Lemma 7.1 it follows that $[X^q(n)]$ is the sequence induced by $(x(n))$. Therefore from Proposition 4.1, (I-4), we obtain

$$\begin{aligned} \big((x(n) \mid M_x(m)), 0, \ldots, 0 \big) &= \sum_{q=0}^{T-1} e^{2\pi iqn/T} \big(X^q(n) \mid M_X(m) \big) \\ &= \sum_{q=0}^{T-1} e^{2\pi iqn/T} \sum_{k=n-m}^{\infty} \sum_{j=0}^{T-1} D_k^{q,j} \zeta_{n-k}^j. \end{aligned}$$

The same computation as in the proof of part (ii) gives (76).

Since $[X^q(n)]$ is induced by a PC sequence $(x(n))$, from (28) we infer that $(d\gamma_k/ds)(s) = TG^{0,-k}(s)$, ds-a.e., $k \in \mathscr{Z}_T$, where $\boldsymbol{G}(s)$ is the spectral density of $[X^q(n)]$. Since $\boldsymbol{D}(t) = \boldsymbol{A}(e^{-it})^{-1} \boldsymbol{B}(e^{-it})$ is a square root of $\boldsymbol{G}(t)$, we have

$$G^{0,-k}(s) = \sum_{q=0}^{T-1} D^{0,q}(s) \overline{D^{-k,q}(s)}, \qquad k \in \mathscr{Z}_T. \blacksquare$$

The matrices $\boldsymbol{D}_k$ can be obtained recursively from (75), and hence the theorem produces an algorithm for computing an optimal moving average representation and the spectrum of a PARMA sequence $(x(n))$.

Lemmas 7.1 and 7.2 also produce a certain condition for existence of a PC solution to (65) in terms of the discrete Fourier transforms of the coefficients of the system. For example, if $T = 2$ and $(x(n))$ is a PAR(1) sequence that satisfies

$$(79) \qquad x(n) = \phi(n) x(n-1) + \xi_n,$$

then

$$A_1 = \begin{bmatrix} \tilde{\phi}(0) & -\tilde{\phi}(1) \\ \tilde{\phi}(1) & -\tilde{\phi}(0) \end{bmatrix}, \quad A(z) = I - A_1 z, \quad \text{and} \quad \det A(z) = 1 - Pz^2,$$

where $P = \tilde{\phi}(0)^2 - \tilde{\phi}(1)^2$. Hence $\det A(z)$ has no zeros in the unit disc iff $|P| < 1$. Since $A(z)^{-1} = (\det A(z))^{-1}(I + A_1 z)$, provided $|P| < 1$, $D_{2k} = P^k I$ and $D_{2k+1} = P^k A_1$, $k \geqslant 0$. Consequently,

$$(80) \quad \beta^n_{2k} = P^k \quad \text{and} \quad \beta^n_{2k+1} = P^k\big(\tilde{\phi}(0) + (-1)^n \tilde{\phi}(1)\big), \quad n \in \mathscr{Z},\ k \geqslant 0.$$

This is consistent with the solution obtained for example in [23] or [11]. To see this write $\tilde{\phi}(0) = (1/2)(\phi(0) + \phi(1))$ and $\tilde{\phi}(1) = (1/2)(\phi(0) - \phi(1))$. Then $P = \phi(0)\phi(1)$, and

$$(81) \qquad \beta^n_{2k} = P^k \quad \text{and} \quad \beta^n_{2k+1} = \begin{cases} P^k \phi(0) & \text{if } n \text{ is even}, \\ P^k \phi(1) & \text{if } n \text{ is odd}. \end{cases}$$

Although in this example the condition $|P| = |\phi(0)\phi(1)| < 1$, that guarantees the existence of a unique causal PC solution to (79), can be easier obtained by Vecchia's block sequence approach or by direct solving the system, it seems likely that for some special PARMA systems (for example, if $\phi_k(n)$ and $\theta_k(n)$ have only few nonzero harmonics) phrasing the solution in terms of $\tilde{\phi}_k(n)$ and $\tilde{\theta}_k(n)$ may be advantageous.

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Department of Mathematics, Hampton University
Hampton, VA 23668, USA
Phone: (757) 727-5200, Fax: (757) 727-5832
E-mail: andrzej.makagon@hamptonu.edu

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