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MULTIVARIATE LARGE DEVIATIONS WITH STABLE LIMIT LAWS

BY

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Abstract. The large deviation problem for sums of i.i.d. random vectors is considered. It is assumed that the underlying distribution is absolutely continuous and its density is of regular variation. An asymptotic expression for the probability of large deviations is established in the case of a non-normal stable limit law. The role of the maximal summand is also emphasized.

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1. Introduction. Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. random vectors assuming values in R^d. Denote by F the common distribution and consider the partial sums $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2\ldots$ Throughout the paper we assume that there exist sequences $a_n \in R^d$, $b_n \in (0, \infty)$ and a non-degenerate distribution G such that for any G-continuity set A

$$P\left(b_n^{-1}(S_n - a_n) \in A\right) \to G(A), \quad n \to \infty, \tag{1}$$

or, in words, F is attracted by G. The set of distributions attracted by G is called the *domain of attraction* of G. It is known that the domain of attraction of G is not empty iff G is stable in the sense that for all positive numbers b_1, b_2 there exist $a \in R^d$ and a positive number b such that $b_1 X_1 + b_2 X_2$ and $bX + a$ are identically distributed, where X_1, X_2, X have distribution G, and X_1 and X_2 are independent. Up to a shift, each stable distribution is uniquely determined by a number $\alpha \in (0, 2]$ and a probability measure μ defined on the σ-algebra $\mathcal{S}$ of the Borel subsets on the unit sphere S^{d-1} (see, e.g., Samorodnitsky and Taqqu [21]).

Domains of attraction for $d > 1$ were first studied by Rvačeva [20] who gave, in particular, necessary and sufficient conditions for F to be attracted

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by G as follows: F is attracted by $G = G_{\alpha,\mu}$ iff for any $c > 0$

$$\lim_{t\to\infty} \frac{P(|\xi| > t)}{P(|\xi| > ct)} = c^{\alpha},$$

while for any $Q_1, Q_2 \in \mathscr{S}$

$$\lim_{t\to\infty} \frac{P(|\xi| > t, |\xi|^{-1}\xi \in Q_1)}{P(|\xi| > t, |\xi|^{-1}\xi \in Q_2)} = \frac{\mu(Q_1)}{\mu(Q_2)}$$

provided $\mu(Q_2) \neq 0$.

Let us denote by e_x the unit vector of the same direction as x, that is $e_x \doteq |x|^{-1} x$.

Sometimes it is more convenient to use the following form of necessary and sufficient conditions (see e.g. Kalinauskaite [7]): F is attracted by $G_{\alpha,\mu}$ iff for any $c > 0$ and any $Q \in \mathscr{S}$

(2) $$P(|\xi| > t, e_\xi \in Q) = t^{-\alpha} l(t)\big(\mu(Q) + m(Q, t)\big),$$

where $l(t)$ varies slowly at infinity and, given t, $m(Q, t)$ is σ-additive while, given Q, $m(Q, t)$ vanishes as $t \to \infty$.

Let (2) be fulfilled. If, furthermore, S_{n_0} for some $n_0 \geqslant 1$ has a uniformly bounded density, then the local limit theorem takes place, that is

(3) $$\sup_{x\in R^d} \left|b_n^d p_n(x) - g\big(b_n^{-1}(x - a_n)\big)\right| = o(1), \quad n \to \infty,$$

where $p_n(x)$ and $g(x)$ are the densities of S_n and the stable limit law, respectively (see Rvačeva [20]).

A value $x = x_n$ in the range of S_n is called a *large deviation* if

$$b_n^{-1}|x - a_n| \to \infty, \quad n \to \infty.$$

That is why $P(S_n \in A)$ with $A = A_n$, $b_n^{-1}\inf_{x\in A}|x - a_n| \to \infty$, $n \to \infty$, is called a *large deviation probability*. In view of (1), a large deviation probability converges to zero. From (3) it follows that, as $n \to \infty$,

$$b_n^d p_n(x) = g\big(b_n^{-1}(x - a_n)\big) + o(1) \quad \text{if } |x - a_n| = O(b_n),$$

while $b_n^d p_n(x) \to 0$ if x enjoys a large deviation. The basic problem of large deviation theory is to establish a precise asymptotic expression for $P(S_n \in A)$ (or $p_n(x)$) when A (or x) lies in the region of large deviations.

Let

(4) $$\mathscr{A}^{(n)} = \{A \in \mathscr{B}^d\colon \inf_{x\in A}|x| \geqslant c_n b_n\},$$

where $\mathscr{B}^d$ is the σ-algebra of the Borel subsets of R^d and $\lim_{n\to\infty} c_n = \infty$.

We say that the *large deviation problem is solved within* $\mathscr{A}_0^{(n)} \subset \mathscr{A}^{(n)}$ if it is established that

$$\lim_{n\to\infty} \sup_{A\in\mathscr{A}_0^{(n)}} \left|\frac{P(S_n - a_n \in A)}{P(n, A)} - 1\right| = 0,$$

where $P(n, A)$ is an asymptotic expression that involves certain characteristics of both the underlying distribution and A.

So, to each $\mathscr{A}_0^{(n)} \subset \mathscr{A}^{(n)}$ there corresponds a setting of the large deviation problem. The classic theory of large deviations deals with the case when $d = 1$ and $\mathscr{A}_0^{(n)}$ is the class of the half-lines (x_n, ∞) or $(-\infty, -x_n)$ with $b_n^{-1} x_n \to \infty$. An overview of this rather advanced theory can be found e.g. in S. Nagaev [14]. Among those who contributed to the theory studying the case of an infinite variance limit stable law one should mention Zolotarev [25], Aleskjavičene [1], Heyde [5], Tkačuk [22], Kim and A. Nagaev [8], A. Nagaev [11], Rozovskii [19]. Of course, this list is far from being complete.

The first question which arises when we move to higher dimensions is how to choose multidimensional analogues of those half-lines. The simplest way is to define $\mathscr{A}_0^{(n)}$ as follows:

$$\mathscr{A}_0^{(n)} = \{A \in \mathscr{B}^d\colon\ A = tA_0^c,\ t \geqslant c_n b_n\}, \qquad c_n \to \infty, \tag{5}$$

where A_0 contains the origin, $A_0^c = R^d \backslash A_0$. In words, $\mathscr{A}_0^{(n)}$ is the parametric family of the sets homothetic to A_0^c.

In Tkačuk [23] it was shown that under such a setting the large deviation problem can be solved provided the boundary of the generating set A_0 satisfies minimal regularity conditions. Before we present the mentioned result let us assume without loss of generality that $E\xi = 0$ when the expectation exists.

PROPOSITION 1. *Let in* (5)

$$A_0 = \{x \in R^d\colon\ |x| < \varrho(e_x)\},$$

where $\varrho(e)$ is a positive continuous function on S^{d-1}. If F belongs to the domain of attraction of a stable distribution with $\alpha \in (0, 1) \cup (1, 2)$, then

$$\lim_{n \to \infty} \sup_{A \in \mathscr{A}_0^{(n)}} \left| \frac{P(S_n \in A)}{nP(\xi \in A)} - 1 \right| = 0.$$

It is rather easy to show that $\mathscr{A}_0^{(n)}$ in Proposition 1 can be made broader. For instance, for $\mathscr{A}_0^{(n)}$ we may choose

$$\mathscr{A}_0^{(n)} = \{A \in \mathscr{B}^d\colon\ A = tB^c,\ B \in \mathscr{A}_0,\ t \geqslant c_n b_n\},$$

where $\mathscr{A}_0 = \{A \in \mathscr{B}^d\colon\ t_- S^{d-1} \subset A \subset t_+ S^{d-1}\}$ for some fixed t_-, t_+ such that $0 < t_- < t_+ < \infty$. Apparently, this case can be treated as in Pinelis [16].

However, for richer families we have to impose more restrictions either on the underlying distribution or on the order of the considered deviations. For instance, we can deal with

$$\bar{\mathscr{A}}^{(n)} = \{A \in \mathscr{B}^d\colon\ c_n^- b_n \leqslant \inf_{x \in A} |x| \leqslant c_n^+ b_n\},$$

where $c_n^- \to \infty$, $c_n^+ \to \infty$, $c_n^- = o(c_n^+)$.

In order to have an impression on the variety of the (so far considered) settings see e.g. Borovkov and Rogozin [4], Vilkauskas [24], A. Nagaev [9], von Bahr [3], Anorina and A. Nagaev [2], A. Nagaev and Sakojan [12], Osipov [15], Rozovskii [18].

The simplest way to extend the statement of Proposition 1 to $\mathscr{A}^{(n)}$ is to assume that the distribution is absolutely continuous and its density is, asymptotically as $|x| \to \infty$, sufficiently regular. It is worth recalling an assertion proved in Tkačuk [22].

PROPOSITION 2. *Let $d = 1$. Suppose that F belongs to the domain of normal attraction of a stable distribution G with $\alpha \in (0, 1) \cup (1, 2)$ and its density is such that*

$$p(x) = cx^{-(1+\alpha)}(1+o(1)), \qquad x \to \infty, \; c > 0. \tag{6}$$

Then as $xn^{-1/\alpha} \to \infty$, $n \to \infty$,

$$p_n(x) = np(x)(1+o(1)) = n^{-1/\alpha} g(xn^{-1/\alpha})(1+o(1)).$$

Recall $\mathscr{A}^{(n)}$ as defined in (4) and let $d = 1$, $b_n = n^{1/\alpha}$. Then from Proposition 2 it follows that

$$\lim_{n\to\infty} \sup_{A \in \mathscr{A}^{(n)}} \left| \frac{P(S_n \in A)}{nP(\xi \in A)} - 1 \right| = 0.$$

The conditions of Proposition 2 are rather close to the necessary ones. This means that the local limit theorem provides an approach which enables us to solve the large deviation problem for extremely rich classes of sets. In particular, such an approach was adopted in Borovkov and Rogozin [4] while studying the case of light-tailed distributions.

The basic goal of the paper is to carry over the statement of Proposition 2 to the multidimensional case. The multidimensional analogue of condition (6) may be chosen as follows. Let non-negative functions r_γ and h be defined on $[0, \infty)$ and S^{d-1}, respectively. We say that a *density p belongs to $\mathscr{P}_{\gamma,h}$* if it is uniformly bounded and admits the representation

$$p(x) = r_\gamma(|x|)(h(e_x) + w(x)),$$

where $r_\gamma(t)$ varies regularly at infinity with exponent $-\gamma$, $\gamma > d$, $h(e) \geqslant 0$ is continuous, $w(x) \to 0$ as $|x| \to \infty$ uniformly in e_x (cf. Resnick [17], Section 5.4.2).

In particular, if $p \in \mathscr{P}_{\gamma,h}$, then for any $Q \in \mathscr{S}$

$$P(|\xi| > t, \, e_\xi \in Q) = (\mu(Q) + w_Q(t)) r_{\gamma-d}(t), \tag{7}$$

$\mu(Q) = \left(\int_{S^{d-1}} h(e)\,\lambda(de)\right)^{-1} \int_Q h(e)\,\lambda(de)$, where λ is the surface measure on $\mathscr{S}$ and

$$\sup_{Q \in \mathscr{S}} |w_Q(t)| \to 0, \qquad t \to \infty.$$

One of the simplest examples of $p \in \mathscr{P}_{\gamma,h}$ is given by the function

$$p(x) = \begin{cases} 0 & \text{if } |x| < t_0, \\ c_{\alpha\beta}(|x_1|^\alpha + \ldots + |x_d|^\alpha)^{-\beta} & \text{otherwise.} \end{cases}$$

It is easily seen that $r_\gamma(t) = t^{-\gamma}$, $\gamma = \alpha\beta$, $h(e) = c_{\alpha\beta}(|e_1|^\alpha + \ldots + |e_d|^\alpha)^{-\beta}$ and $w(x) = 0$, $|x| \geqslant t_0$.

It is worth noting that within $\mathscr{P}_{\gamma,h}$ the variables $t^{-1}|\xi|$ and e_ξ are asymptotically independent as $t \to \infty$, given $|\xi| > t$. More precisely, from (7) it follows that

$$P(t^{-1}|\xi| > z,\, e_\xi \in Q \mid |\xi| > t) \to z^{d-\gamma}\mu(Q), \qquad t \to \infty,\ z \geqslant 1.$$

Obviously, (7) implies (2). Therefore, for $\gamma \in (d, d+2)$ the underlying distribution belongs to the domain of attraction of a stable law with exponent $\alpha = \gamma - d$ and measure μ of the mentioned form. If $\alpha = 2$, it belongs to the domain of attraction of a normal law (see Rvačeva [20]). Here we confine ourselves to the case $\alpha \in (0, 1) \cup (1, 2)$. The threshold cases $\alpha = 1$ and $\alpha = 2$ with $\gamma = d+2$ require additional arguments. They will be studied elsewhere. It should be noted that the present paper complements that of A. Nagaev and Zaigraev [13] devoted to the case $\gamma > d+2$ when the limit law is normal.

The paper is organized as follows. The main results are formulated in Section 2 while Section 3 contains their proofs.

2. Main results. It is worth recalling that we confine ourselves to $p \in \mathscr{P}_{\gamma,h}$ and $\alpha \in (0, 1) \cup (1, 2)$. Furthermore, when $\alpha \in (1, 2)$, we assume that $E\xi = 0$. This means that in (1) we may choose $a_n = 0$ while $b_n = \inf\{t\colon nP(|\xi| \geqslant t) \leqslant 1\}$ (see e.g. Rvačeva [20]). If (7) holds, then b_n is determined by the equation $P(|\xi| \geqslant b_n) = n^{-1}$ provided n is sufficiently large.

Let c_n be any sequence such that $\lim_{n\to\infty} c_n = \infty$.

THEOREM 1. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{|x| \geqslant c_n b_n} \sup_{e_x \in E_\varepsilon} \left| \frac{p_n(x)}{np(x)} - 1 \right| = 0, \qquad \text{where } E_\varepsilon = \{e \in S^{d-1}\colon h(e) \geqslant \varepsilon\}$$

while

$$\lim_{n\to\infty} \sup_{|x| \geqslant c_n b_n} \sup_{e_x \notin E} \frac{p_n(x)}{nr_\gamma(|x|)} = 0, \qquad \text{where } E = \{e \in S^{d-1}\colon h(e) > 0\}.$$

As in the case $d = 1$ mentioned in Section 1 we easily come to the following statement:

COROLLARY 1. *Under the conditions of Theorem* 1,

$$\lim_{n\to\infty} \sup_{A \in \mathscr{A}(n)} \left| \frac{P(S_n \in A)}{nP(\xi \in A)} - 1 \right| = 0.$$

The assertion of Theorem 1 can be rewritten in the following form:

$$\lim_{n\to\infty} \sup_{|x|\geqslant c_n b_n} \sup_{e_x\in E_\varepsilon} \left| \frac{p_n(x)}{nr_\gamma(|x|)} - h(e_x) \right| = 0.$$

Simple examples show that the continuity of $h(e)$ is essential. Let, for instance, $d = 2$ and $h(e)$ be continuous except for $e = e^{(0)}$. Taking into account the representation $e = e_\phi = (\cos\phi, \sin\phi)$, $0 \leqslant \phi < 2\pi$, assume that $\inf_{e\in S^{d-1}} h(e) > 0$, i.e. $E_\varepsilon = S^{d-1}$, and there exist

$$h^- = \lim_{\phi\uparrow\phi_0} h(e_\phi) \quad \text{and} \quad h^+ = \lim_{\phi\downarrow\phi_0} h(e_\phi)$$

such that $0 < h^- < h^+ < \infty$. It can be shown that for $e_x = e^{(0)}$

$$p_n(x) = h_0\, nr_\gamma(|x|)\,(1+o(1)) \quad \text{with } h_0 = (h^- + h^+)/2$$

while for any fixed $e_x \neq e^{(0)}$

$$p_n(x) = h(e_x)\, nr_\gamma(|x|)\,(1+o(1)).$$

It is of interest to highlight the role played by the maximal summand when a large deviation of the sum takes place. Let j_n be the extreme left point of the set $\arg\max_{1\leqslant j\leqslant n} |\xi_j|$ and $M_n = \xi_{j_n}$.

It is worth noting that, for any $\alpha > 0$, (2) also presents a necessary and sufficient condition for the asymptotic relation

$$P(|M_n| \leqslant tb_n) = \exp(-t^{-\alpha}) + o(1)$$

as well as for

$$P(|M_n| \leqslant tb_n,\ e_{M_n} \in Q) = \exp(-t^{-\alpha})\,\mu(Q) + o(1),$$

where $Q \in \mathscr{S}$ (see Resnick [17], Section 5.4.1).

Moreover, for $\alpha \in (0, 1) \cup (1, 2)$ there exists a limit distribution for $|M_n|^{-1} S_n$ (see e.g. Kalinauskaite [6]). This means that $|M_n|$ and $|S_n|$ are of the same order. Note that in the case of a normal limit distribution $|M_n|/|S_n| \to 0$ in probability.

The next statement shows that the role of M_n even increases when S_n abnormally deviates from the origin.

THEOREM 2. *Let G be the stable limit law in* (1). *Then for any $\varepsilon > 0$ and any G-continuity set A*

$$\lim_{n\to\infty} \sup_{|x|\geqslant c_n b_n} \sup_{e_x\in E_\varepsilon} \left| \frac{P(b_n^{-1}(S_n - M_n) \in A \mid S_n = x)}{G(A)} - 1 \right| = 0.$$

As a corollary we obtain the limit law for the conditional distribution of the projection of M_n onto a deviation direction given the large deviation of the sum. From now on, $\langle\cdot,\cdot\rangle$ denotes the ordinary inner product.

COROLLARY 2. *Let $g(x)$ be the density of the stable limit law. Then for any $\varepsilon > 0$ and any $t \in R^1$*

$$\lim_{n\to\infty} \sup_{|x|\geqslant c_n b_n} \sup_{e_x\in E_\varepsilon} \left| \frac{P(|S_n| - \langle M_n, e_x\rangle < tb_n \mid S_n = x)}{\int_{\langle u, e_x\rangle < t} g(u)\,du} - 1 \right| = 0.$$

It is useful to compare Theorem 2 and Corollary 2 with the corresponding assertions in A. Nagaev and Zaigraev [13] (see their Theorem 2 and Corollary 4).

3. Proofs. In what follows it is supposed that $n \to \infty$, $|x| \geqslant c_n b_n$, $c_n \to \infty$. In particular, saying "uniformly in x, $e_x \in A$" we also mean that $|x| \geqslant c_n b_n$, $n \to \infty$.

Proof of Theorem 1. It should be noted that the proof follows the general scheme given in A. Nagaev and Zaigraev [13] (see also A. Nagaev [10]).

It is worth beginning with two relations playing a crucial role in the proof. Let $p \in \mathscr{P}_{\gamma,h}^{-}$ and $|u| = o(|x|)$ as $|x| \to \infty$. Then

(8) $$p(x-u) \sim p(x) \asymp r_\gamma(|x|)$$

uniformly in x, $e_x \in E_\varepsilon$, while

(9) $$p(x-u) = o\big(r_\gamma(|x|)\big), \quad p(x) = o\big(r_\gamma(|x|)\big)$$

uniformly in x, $e_x \notin E$. Here $f(x) \sim g(x)$ and $f(x) \asymp g(x)$ mean, respectively, that

$$f(x)/g(x) \to 1 \text{ as } |x| \to \infty \quad \text{and} \quad 0 < \liminf_{|x| \to \infty} f(x)/g(x) \leqslant \limsup_{|x| \to \infty} f(x)/g(x) < \infty.$$

Let $y_n \to \infty$ as $n \to \infty$. Consider the events

$$A_0 = \{\langle \xi_j, e_x \rangle < y_n,\ j = 1, \ldots, n\},$$
$$A_1 = \{\langle \xi_j, e_x \rangle < y_n,\ j = 1, \ldots, n-1,\ \langle \xi_n, e_x \rangle \geqslant y_n\},$$
$$A_2 = \bigcup_{k,j=1, k \neq j}^{n} \{\langle \xi_k, e_x \rangle \geqslant y_n,\ \langle \xi_j, e_x \rangle \geqslant y_n\}.$$

We choose y_n so that $b_n = o(y_n)$ while $y_n = o(|x|)$. It is easily seen that $y_n = |x|(b_n/|x|)^q$ with $q \in (0, 1)$ satisfies these requirements. In contrast to the mentioned papers, here we have to be more careful when choosing the truncation level.

Obviously, $p_n(x)$ can be represented in the form

(10) $$p_n(x) = p_{n0}(x) + np_{n1}(x) + p_{n2}(x),$$

where

$$p_{nj}(x) = \lim_{|\Delta| \to 0} |\Delta|^{-1} P(x \preccurlyeq S_n \prec x + \Delta,\ A_j), \quad j = 0, 1, 2,$$

and $\prec$ ($\preccurlyeq$) means the componentwise ordering. It should be noted that $p_{n1}(x)$ and $p_{n2}(x)$ are estimated, in essence, as in A. Nagaev and Zaigraev [13] while all the alterations concern the estimation of $p_{n0}(x)$.

Let $\tilde{\xi}$ be a random vector having density

(11) $$\tilde{p}(u) = \begin{cases} \big(P(\langle \xi, e_x \rangle < y_n)\big)^{-1} p(u) & \text{if } \langle u, e_x \rangle < y_n, \\ 0 & \text{otherwise}. \end{cases}$$

Note that due to (7) and the choice of y_n we have

$$nP(\langle \xi, e_x \rangle \geqslant y_n) = O(nr_\alpha(y_n)) = o(1). \tag{12}$$

LEMMA 1. *Under the conditions of Theorem* 1, *for any* $k \geqslant 1$

$$|E\langle \tilde{\xi}, e\rangle^k| = O(y_n^{k-\alpha} l(y_n))$$

uniformly in $e \in S^{d-1}$, *where* $l(t)$ *varies slowly at infinity.*

Proof. First, consider the case $k = 1$. For $\alpha \in (1, 2)$

$$\int\limits_{\langle u,e\rangle < y_n} \langle u, e\rangle p(u)\,du = -\int\limits_{\langle u,e\rangle \geqslant y_n} \langle u, e\rangle p(u)\,du$$

yields

$$|E\langle \tilde{\xi}, e\rangle| = O\Big(\int\limits_{\langle u,e\rangle \geqslant y_n} \langle u, e\rangle p(u)\,du\Big).$$

After performing a change of variables $u = Tv$, where T is an orthogonal transformation such that $v_1 = \langle u, e\rangle$, we get

$$\int\limits_{\langle u,e\rangle \geqslant y_n} \langle u, e\rangle p(u)\,du = \int\limits_{y_n}^{\infty} v_1\,dv_1 \int \ldots \int r_\gamma(|v|)\big(h(e_{Tv}) + w(Tv)\big)\,dv_2 \ldots dv_d.$$

The next change of variables $v_2 = v_1 y_1, \ldots, v_d = v_1 y_{d-1}$, fixed v_1, implies

$$\int\limits_{\langle u,e\rangle \geqslant y_n} \langle u, e\rangle p(u)\,du = \int\limits_{y_n}^{\infty} \frac{dv_1}{v_1^\alpha} \int\limits_{R^{d-1}} \frac{l(v_1|\tilde{y}|)}{|\tilde{y}|^\gamma}\big(h(e_{T\tilde{y}}) + w(v_1 T\tilde{y})\big)\,dy,$$

where $\tilde{y} = (1, y)'$, $y \in R^{d-1}$, and the function $l(t)$ varies slowly at infinity. Therefore,

$$\int\limits_{\langle u,e\rangle \geqslant y_n} \langle u, e\rangle p(u)\,du = O\Big(\int\limits_{y_n}^{\infty} r_\alpha(t)\,dt\Big)$$

and

$$|E\langle \tilde{\xi}, e\rangle| = O\Big(\int\limits_{y_n}^{\infty} r_\alpha(t)\,dt\Big) = O(y_n^{1-\alpha} l(y_n))$$

uniformly in $e \in S^{d-1}$.

The same relation holds in the case $\alpha \in (0, 1)$. Indeed, in a similar manner we obtain

$$\Big|\int\limits_{\langle u,e\rangle < y_n} \langle u, e\rangle p(u)\,du\Big| = O\Big(\int\limits_{0}^{y_n} r_\alpha(t)\,dt\Big) = O(y_n^{1-\alpha} l(y_n)).$$

For $k > 1$ in both cases we get

$$\Big|\int\limits_{\langle u,e\rangle < y_n} \langle u, e\rangle^k p(u)\,du\Big| = O\Big(\int\limits_{0}^{y_n} r_{\alpha+1-k}(t)\,dt\Big) = O(y_n^{k-\alpha} l(y_n)).$$

The lemma is proved.

Keeping in mind representation (10), we divide the proof of Theorem 1 into three parts.

LEMMA 2. *Under the conditions of Theorem* 1, *for any* $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{|x|\geq c_n b_n} \sup_{e_x\in E_\varepsilon} \left|\frac{p_{n1}(x)}{p(x)} - 1\right| = 0$$

and

$$\lim_{n\to\infty} \sup_{|x|\geq c_n b_n} \sup_{e_x\notin E} \frac{p_{n1}(x)}{r_\gamma(|x|)} = 0.$$

Proof. Let $\tilde{\xi}_1, \tilde{\xi}_2, \ldots$ be i.i.d. random vectors having density (11) and let us put $\tilde{S}_n = \tilde{\xi}_1 + \ldots + \tilde{\xi}_n$. Let $\tilde{p}_n(x)$ be the density of $\tilde{S}_n$. By definition,

$$p_{n1}(x) = \big(P(\langle \xi, e_x\rangle < y_n)\big)^{n-1} \int_B \tilde{p}_{n-1}(u)\, p(x-u)\, du, \tag{13}$$

where $B = \{u\in R^d \colon \langle u, e_x\rangle \leqslant |x| - y_n\}$. In view of (12), we obtain

$$\big(P(\langle \xi, e_x\rangle < y_n)\big)^n = 1 + o(1). \tag{14}$$

If n is sufficiently large, then we can partition B into

$$B_1 = \{u\in B\colon |u| < y_n\}, \qquad B_2 = \{u\in B\colon |u| \geqslant y_n\}.$$

For $e_x\in E_\varepsilon$, due to (8),

$$\int_{B_1} \tilde{p}_{n-1}(u)\, p(x-u)\, du = P(|\tilde{S}_{n-1}| < y_n)\, p(x)\big(1+o(1)\big), \tag{15}$$

while for $e_x\notin E$ in view of (9) we get

$$\int_{B_1} \tilde{p}_{n-1}(u)\, p(x-u)\, du = o\big(r_\gamma(|x|)\big). \tag{16}$$

On the other hand,

$$\int_{B_2} \tilde{p}_{n-1}(u)\, p(x-u)\, du = O\big(P(|\tilde{S}_{n-1}| \geqslant y_n)\big). \tag{17}$$

It remains to establish the relation

$$P(|\tilde{S}_n| \geqslant y_n) = o(1), \tag{18}$$

which follows from

$$P(|S_n| \geqslant y_n) = \big(P(\langle \xi, e_x\rangle < y_n)\big)^n P(|\tilde{S}_n| \geqslant y_n) + O\big(nP(\langle \xi, e_x\rangle \geqslant y_n)\big)$$

due to (12), (14) and the fact that $P(|S_n| \geqslant y_n) \to 0$ since $b_n = o(y_n)$.

Then the assertion of the lemma follows from (13)–(18).

LEMMA 3. *Under the conditions of Theorem* 1,

$$p_{n2}(x) = o\big(nr(|x|)\big) \quad \textit{uniformly in } x.$$

Proof. By definition,

$$p_{n2}(x) \leqslant n^2 \int\limits_{\langle u, e_x\rangle \geqslant y_n} p(u)\,du \int\limits_{\langle v, e_x\rangle \geqslant y_n} p_{n-2}(x-u-v)\,p(v)\,dv.$$

Clearly,

$$\int\limits_{\langle v, e_x\rangle \geqslant y_n} p_{n-2}(x-u-v)\,p(v)\,dv = O\big(r_\gamma(y_n)\big).$$

Therefore, in view of (7),

$$p_{n2}(x) = O\big(n^2 r_\gamma(y_n) P(\langle \xi, e_x\rangle \geqslant y_n)\big) = O\big(n^2 r_\gamma(y_n) r_\alpha(y_n)\big) = o\big(nr_\gamma(|x|)\big)$$

if we choose $q \in (0, (\gamma+\alpha)^{-1}\alpha)$. The lemma is proved.

LEMMA 4. *Under the conditions of Theorem* 1,

$$p_{n0}(x) = o\big(nr_\gamma(|x|)\big) \quad \textit{uniformly in } x.$$

Proof. Consider the moment generating function

$$f(s) = \int\limits_{\langle u, e_x\rangle < y_n} \exp(s\langle u, e_x\rangle)\,p(u)\,du.$$

Let $\xi(s)$, $\xi_1(s)$, $\xi_2(s)$, ... be i.i.d. random vectors having density

$$p_s(u) = \begin{cases} (f(s))^{-1}\exp(s\langle u, e_x\rangle)\,p(u) & \text{if } \langle u, e_x\rangle < y_n, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $p_s^{(n)}(x)$ the density of $\xi_1(s) + \ldots + \xi_n(s)$. It is easily seen that

$$p_{n0}(u) = \exp(-s\langle u, e_x\rangle)\,f^n(s)\,p_s^{(n)}(u)$$

and, in particular,

$$p_{n0}(x) = \exp(-s|x|)\,f^n(s)\,p_s^{(n)}(x).$$

Let us set $s = y_n^{-1}$. Obviously, $sb_n \to 0$ and $s|x| \to \infty$.

We will prove that

$$f(s) = 1 + o(n^{-1}) \tag{19}$$

and

$$\sup_u p_s^{(n)}(u) = O(b_n^{-d}). \tag{20}$$

If (19) and (20) hold, then the lemma follows since

$$p_{n0}(x) = O\big(b_n^{-d}\exp(y_n^{-1}|x|)\big) = o\big(nr(|x|)\big) \quad \text{uniformly in } x.$$

Let us write

$$f(s) = \int\limits_{\langle u, e_x\rangle < y_n} p(u)\,du + \int\limits_{\langle u, e_x\rangle < y_n} \big(\exp(s\langle u, e_x\rangle) - 1\big)\,p(u)\,du = f_1(s) + f_2(s).$$

By (7), $f_1(s) = 1+o(n^{-1})$. Furthermore, in view of Lemma 1 we obtain

$$(21) \quad |f_2(s)| \leqslant \int_{\langle u,e_x\rangle < y_n} |\exp(s\langle u, e_x\rangle)-1|\, p(u)\, du$$

$$\leqslant \sum_{k=1}^{\infty} \frac{s^k}{k!} \int_{\langle u,e_x\rangle < y_n} |\langle u, e_x\rangle^k|\, p(u)\, du = o(n^{-1})$$

and (19) follows.

It remains to prove (20). Let $\phi_s(t)$ and $\phi(t)$ be the characteristic functions of $\xi(s)$ and ξ, respectively. From (12) and (21) it follows that

$$|\phi_s(t)-\phi(t)| = \Big|(f(s))^{-1} \int_{\langle u,e_x\rangle < y_n} e^{i\langle u,t\rangle} \exp(s\langle u, e_x\rangle)\, p(u)\, du - \int_{R^d} e^{i\langle u,t\rangle}\, p(u)\, du\Big|$$

$$= O\Big(\Big|\int_{\langle u,e_x\rangle < y_n} e^{i\langle u,t\rangle}\big(\exp(s\langle u, e_x\rangle)-1\big)\, p(u)\, du - \int_{\langle u,e_x\rangle \geqslant y_n} e^{i\langle u,t\rangle}\, p(u)\, du\Big|\Big)$$

$$= O\Big(\int_{\langle u,e_x\rangle < y_n} |\exp(s\langle u, e_x\rangle)-1|\, p(u)\, du + \int_{\langle u,e_x\rangle \geqslant y_n} p(u)\, du\Big) = o(n^{-1})$$

uniformly in $t \in R^d$. Therefore,

$$(22) \qquad |\phi_s(t)|^n \leqslant |\phi(t)|^{n-1}\big(|\phi(t)|+w_n(t)\big)$$

uniformly in $t \in R^d$, where $\sup_t |w_n(t)| \to 0$, $n \to \infty$.

By definition, $\sup_u p(u) < \infty$. Hence $\int_{R^d} |\phi(t)|^n\, dt < \infty$, $n \geqslant 2$. Moreover, under the above choice of s we have $\sup_u p_s(u) < \infty$, that is $\int_{R^d} |\phi_s(t)|^n\, dt < \infty$, $n \geqslant 2$. Due to (22) and the inversion formula, we obtain

$$\sup_u p_s^{(n)}(u) = O\Big(\int_{R^d} |\phi_s(t)|^n\, dt\Big) = O\Big(\int_{R^d} |\phi(t)|^n\, dt\Big) = O\Big(b_n^{-d} \int_{R^d} |\phi(b_n^{-1} t)|^n\, dt\Big).$$

Let $g^*(u)$ be the symmetrization of $g(u)$ (see (3)), that is

$$g^*(u) = \int_{R^d} g(u+v)\, g(v)\, dv.$$

By Plancherel's equality and the classical local limit theorem,

$$\frac{1}{2\pi} \int_{R^d} |\phi(b_{2n}^{-1})|^{2n}\, dt = g^*(0)\big(1+o(1)\big).$$

Since $b_n^{-1} b_{2n} \to 2^{1/\alpha}$, the lemma follows.

The assertions of Theorem 1 follow immediately from (10) and Lemmas 2–4.

Proof of Theorem 2. Let A_0, A_1 and A_2 be as above. In view of Lemmas 3 and 4, for any event A' we obtain

$$(23) \qquad \lim_{|\Delta|\to 0} \frac{P(x \preccurlyeq S_n \prec x+\Delta;\ A_j;\ A')}{P(x \preccurlyeq S_n \prec x+\Delta)} \leqslant \frac{p_{nj}(x)}{p_n(x)}, \qquad j = 0, 2.$$

On the other hand, from the proof of Lemma 2 it follows that

$$(24)\qquad P\big(b_n^{-1}(S_n-M_n)\in A \mid S_n = x\big) \sim \big(p(x)\big)^{-1} \int_{(b_nA)\cap B} \tilde{p}_{n-1}(u)\,p(x-u)\,du$$
$$\sim P\big(b_{n-1}^{-1}\tilde{S}_{n-1}\in A\big) \sim G(A)$$

uniformly in x, $e_x \in E_\varepsilon$. The last relation uses arguments similar to those applied for establishing (18). Theorem 2 follows easily from (23) and (24).

Added in proof. In this paper we dealt with so-called *precise* asymptotic of $P(S_n\in A)$. Another class of problems within the large deviations theory unites those related to the asymptotic expression of $\ln P(S_n\in A)$ which is called a *rough* asymptotic expression. It is worth noting that such expressions can be established under rather general conditions, e.g. for random variables taking values in various infinite-dimensional spaces, for dependent random variables, etc. The present state of the work in this direction can be found in Deuschel and Stroock [28], Dembo and Zeitouni [27]. Characterization of stable laws and their domains of attraction in Banach spaces are given e.g. in Jurek and Urbanik [29], Araujo and Gine [26].

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