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THE SECOND ORDER OPTIMALITY OF TESTS AND ESTIMATORS FOR MINIMUM CONTRAST FUNCTIONALS. I

BY

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Abstract. The concept of *minimum contrast functionals* is introduced. It is shown that certain statistical procedures (asymptotically similar tests, asymptotically similar confidence procedures and asymptotically median unbiased estimators) derived from the minimum contrast estimators are 2nd order efficient, provided the family of probability measures is rich enough to contain, together with each probability measure, contiguous probability measures which are asymptotically "least favorable". The 2nd order efficiency of statistical procedures based on the maximum likelihood estimator follows by application of these results to parametric families of probability measures. The results are valid only for "continuous" probability measures.

1. Introduction. Let $(X, \mathscr{A})$ be a measurable space and $\mathfrak{P}$ a family of p-measures (probability measures) $P|\mathscr{A}$. Let $\varkappa\colon \mathfrak{P} \to \boldsymbol{R}^k$ be a functional defined on $\mathfrak{P}$. The underlying intuitive idea is that the features of P, relevant for the solution of a certain practical problem, can be summarized by a k-dimensional vector $\varkappa(P)$. Examples of such functionals are the mean, the mode, a quantile, a measure of variance, a measure of concentration (like the Lorenz-measure), a measure of correlation in case of $X = \boldsymbol{R}^2$, etc. For many situations this approach is more natural than the assumption that $\mathfrak{P}$ is a parametrized family. At least, it is more general (see Example 3).

Our aim is to make assertions about $\varkappa(P)$, based on a sample $(x_1, \ldots, x_n) \in X^n$ which is governed by P^n. A successful theory requires certain restrictions on the functional. For an asymptotic theory at the level $o(n^0)$ (leading, for instance, to a normal approximation for estimators), it suffices to assume 1st order "differentiability" of the functional. For an asymptotic theory of higher order, more restrictive assumptions are needed.

As a first step we shall deal here with a special type of functionals to be called *m.c.* (*minimum contrast*) *functionals*: Let $f\colon X \times T$, $T \subset \boldsymbol{R}^k$, be such that $x \to f(x, t)$ is $\mathscr{A}$-measurable and P-integrable for every $P \in \mathfrak{P}$, $t \in T$. Assume that for every $P \in \mathfrak{P}$ the function $t \to P(f(\cdot, t))$ has a unique minimum in T, say $\varkappa(P)$. Then $P \to \varkappa(P)$ defines a functional on $\mathfrak{P}$, assuming its values in $\boldsymbol{R}^k$.

The following examples illustrate the concept of an m.c. functional.

Example 1. If $\mathfrak{P}$ is the family of all p-measures over the Borel algebra of $\boldsymbol{R}$ with finite 1st moment, and $f(x, t) = f_q(x-t)$, where

$$f_q(x) := \begin{cases} -(1-q)x, & x < 0, \\ qx, & x \geq 0, \end{cases} \quad q \in (0, 1),$$

then the m.c. functional is the q-quantile.

Example 2. If $\mathfrak{P}$ is the family of all p-measures over the Borel algebra of $\boldsymbol{R}^k$ with symmetric and unimodal Lebesgue density and $f(x, t) = f_0(x-t)$, where f_0: $\boldsymbol{R}^k \to \boldsymbol{R}$ is neg-unimodal (i.e. bowl-shaped), symmetric about 0, and bounded, then the m.c. estimator is the center of symmetry. This follows easily from Theorem 1 in [1].

Example 3. If $\mathfrak{P} = \{P_t: t \in T\}$, $T \subset \boldsymbol{R}^k$, is a parametrized family with densities $p(\cdot, t)$ and $f(\cdot, t) = -\log p(\cdot, t)$, then the m.c. functional is $\varkappa(P_t) = t$.

Further examples can be found in [7] and [4], p. 724-725.

Let Q_n^x denote the empirical p-measure pertaining to the sample $\boldsymbol{x} = (x_1, \ldots, x_n)$. If $t \to Q_n^x(f(\cdot, t))$ has a unique minimum in T for every $\boldsymbol{x} \in X^n$, this defines an estimator $\varkappa^{(n)}$, the so-called *m.c.* (*minimum contrast*) *estimator*.

The purpose of this paper is to show that, under natural conditions, statistical procedures based on the m.c. estimator are 2nd order efficient.

Organization of the paper. In Section 2 we introduce some basic notions. Section 3 contains the notation. In Section 4 it is shown how asymptotically similar tests of level $\alpha + o(n^{-1/2})$ can be obtained by asymptotic studentization applied to the m.c. estimator. In Section 5 the 2nd order efficiency of these tests is established. Sections 6 and 7 establish the corresponding optimum properties for confidence procedures and median unbiased estimators. Section 8 lists the regularity conditions.

Part II of this paper (see [8]) contains lemmas and the proofs.

2. Basic notions. It will be convenient in this and the following sections to write $(\varkappa_0(P), \varkappa_1(P), \ldots, \varkappa_p(P))$, where $\varkappa_0(P)$ is the parameter under investigation, whereas $\varkappa_i(P)$, $i = 1, \ldots, p$, are nuisance parameters. We first consider the problem of testing the hypothesis $\varkappa_0(P) = t_0$ against alternatives $\varkappa_0(P) > t_0$, using a c.f. (critical function) $\varphi_n(\cdot, t_0)$: $X^n \to [0, 1]$.

Let

$$\mathbb{K} := \{\varkappa_0(P): P \in \mathfrak{P}\}. \tag{2.1}$$

Let $\mathfrak{P}_n \subset \mathfrak{P}$, $n \in N$, denote a sequence of families of p-measures, usually a "shrinking" sequence of neighborhoods of a fixed p-measure P_*.

(2.2) Definition. A sequence of c.f. φ_n: $X^n \times \mathbb{K} \to [0, 1]$, $n \in N$, is *asymptotically [similar] of level* $\alpha + o(n^{-s})$ *for* $\mathfrak{P}_n$, $n \in N$, if

$$P^n(\varphi_n(\cdot, \varkappa_0(P))) \leqslant [=]\, \alpha + o(n^{-s})$$

uniformly for $P \in \mathfrak{P}_n$.

Sequences of hypotheses. In asymptotic theory it is usual to think of a fixed hypothesis and of a sequence of alternatives converging – as the sample size increases – to this hypothesis in such a way that the power under this sequence of alternatives converges to some positive number smaller than 1. If the hypothesis is simple, this appears most natural. If we have a structural parameter θ and a nuisance parameter η (hypothesis: $\{(\theta_0, \eta): \eta \in H\}$), it is natural to consider the sequence of alternatives $(\theta_0 + n^{-1/2} t, \eta_1)$ for some fixed value η_1 of the nuisance parameter.

But how should we choose the sequence of alternatives in a non-parametric set up? Assume that we are given some functional $\varkappa$, defined on a large class of p-measures, say $\mathfrak{P}$, and consider the hypothesis $\{Q^n: Q \in \mathfrak{P}, \varkappa_0(Q) = t_0\}$. We are interested in the power against alternatives P_n^n with $\varkappa_0(P_n) = t_0 + n^{-1/2} \Delta$. But what are reasonable criteria for the choice of the sequence P_n in this case, considering that no p-measure of the hypothesis is distinguished?

This, perhaps, is the right place to remember that, in reality, neither the hypothesis nor the alternatives "move". We have a fixed hypothesis $\{Q^n: Q \in \mathfrak{P}, \varkappa_0(Q) = t_0\}$ and we are interested in the rejection probability of a certain alternative P^n with $\varkappa_0(P) = t_1$. Properly understood, our asymptotic formulas render approximations to the rejection probability which hopefully will be sufficiently accurate if the sample size is adjusted to the interesting alternatives in such a way that the rejection probability is high, but not too close to 1. For the purpose of obtaining such approximations we may as well keep the alternative P^n fixed and consider the rejection power under this alternative for the hypothesis $\{Q^n: Q \in \mathfrak{P}, \varkappa_0(Q) = \varkappa_0(P) - n^{-1/2} \Delta\}$.

As far as the approximation of the rejection power is concerned, this approach serves the same purpose, and it saves us choosing a sequence of alternatives, thereby introducing an arbitrary ingredient into our considerations. Moreover, this modified concept of an asymptotic power function is exactly what we need for the evaluation of confidence procedures and median unbiased estimators.

3. Notation. Let $(X, \mathscr{A})$ be a measurable space.

For $A \subset X$ let $A^c := X \setminus A$.

Let $(X^n, \mathscr{A}^n)$ be the n-fold Cartesian product of $(X, \mathscr{A})$. For a p-measure $P|\mathscr{A}$ let $P^n|\mathscr{A}^n$ denote the n-fold product of P.

Let $\boldsymbol{R}^m$ denote the m-dimensional Euclidean space, $\mathscr{B}^m$ its Borel algebra, and $\|\cdot\|$ the Euclidean norm.

Points $u \in \boldsymbol{R}^{p+1}$ will be denoted by $(u_0, u_1, \ldots, u_p)$.

The P-integral of an $\mathscr{A}$-measurable function h: $X \to \boldsymbol{R}^m$ will be written as $P(h)$ or $\int h(x) P(dx)$, where integration is to be understood componentwise.

For any $\mathscr{A}$-measurable function h: $X \to \boldsymbol{R}^m$ and any p-measure $P|\mathscr{A}$ the *induced measure* $P * h | \mathscr{B}^m$ is defined by $P * h(B) := P(h^{-1}(B))$, $B \in \mathscr{B}^m$.

The *sup-metric* on the space of all p-measures over $\mathscr{A}$ is defined by the distance function

$$d(P, Q) := \sup_{A \in \mathscr{A}} |P(A) - Q(A)|.$$

The topology induced by this metric is the *strong topology.*

Finally,

$$\varphi(u) := (2\pi)^{-1/2} \exp[-\tfrac{1}{2}u^2], \quad \Phi(u) := \int_{r<u} \varphi(r)\, dr, \quad N_\alpha := \Phi^{-1}(\alpha).$$

φ_Σ denotes the Lebesgue density of the multivariate normal distribution with mean vector zero and covariance matrix Σ.

Let $T \subset \boldsymbol{R}^{p+1}$. For a function h: $T \to \boldsymbol{R}$ and a multi-index $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_p)$ which belongs to $(\boldsymbol{N} \cup \{0\})^{p+1}$ let

$$|\alpha| := \sum_{j=0}^{p} \alpha_j \quad \text{and} \quad h^\alpha(t) := \frac{\partial^{|\alpha|}}{\partial t_0^{\alpha_0} \ldots \partial t_p^{\alpha_p}} h(t).$$

For notational convenience we also use

$$h^{(i_0 \ldots i_k)}(t) := \frac{\partial^{k+1}}{\partial t_{i_0} \ldots \partial t_{i_k}} h(t).$$

Furthermore,

$$h^{\cdot}(t) := (h^{(i)}(t))_{i=0,\ldots,p}, \quad h^{\cdot\cdot}(t) := (h^{(ij)}(t))_{i,j=0,\ldots,p}.$$

For $P \in \mathfrak{P}$ and a contrast function f, $q \in \boldsymbol{N}$ and $0 = k_0 < k_1 < k_2 < \ldots < k_{q-1} < k_q$ we define

$$F_{i_1 \ldots i_{k_1}, \ldots, i_{k_{q-1}+1} \ldots i_{k_q}}(P) := P\Big(\prod_{l=1}^{q} f^{(i_{k_{l-1}+1} \ldots i_{k_l})}(\cdot, \varkappa(P))\Big)$$

and for $\boldsymbol{x} \in X^n$

$$F^{(n)}_{i_1 \ldots i_{k_1}, \ldots, i_{k_{q-1}+1} \ldots i_{k_q}}(\boldsymbol{x}) := n^{-1} \sum_{\nu=1}^{n} \prod_{l=1}^{q} f^{(i_{k_{l-1}+1} \ldots i_{k_l})}(x_\nu, \varkappa^{(n)}(\boldsymbol{x})).$$

Furthermore,

$$F_{\cdot\cdot}(P) := \big(P\big(f^{(ij)}(\cdot, \varkappa(P))\big)\big)_{i,j=0,\ldots,p}, \quad F_{\cdot,\cdot}(P) := \big(P\big(f^{(i)} f^{(j)}(\cdot, \varkappa(P))\big)\big)_{i,j=0,\ldots,p}.$$

Let $(A_{ij}(P))_{i,j=0,\ldots,p}$ denote the inverse of the matrix $F_{..}$ and $(A_{ij}^{(n)}(x))_{i,j=0,\ldots,p}$ the inverse of the matrix $(F_{ij}^{(n)}(x))_{i,j=0,\ldots,p}$ whenever defined, and $(\delta_{ij})_{i,j=0,\ldots,p}$ elsewhere.

Convention. If an index occurs in a product more than once, this means summation over all possible values of this index.

For $P \in \mathfrak{P}$ and $j = 0, \ldots, p$ let

$$f_j(\cdot, P) := -A_{ji}(P) f^{(i)}(\cdot, \varkappa(P)),$$

$$\sigma_{ij}(P) := P\big(f_i(\cdot, P) f_j(\cdot, P)\big), \quad \sigma_j(P) := \sigma_{jj}(P)^{1/2}.$$

We remark that $\sigma_{ij} = A_{ik} A_{jl} F_{k,l}$.

Given $h(\cdot, P)\colon X \to \mathbf{R}^m$, we define $\tilde{h}(\cdot, P)\colon X^n \to \mathbf{R}^m$ by

$$\tilde{h}(\mathbf{x}, P) := n^{-1/2} \sum_{\nu=1}^{n} \big(h(x_\nu, P) - P(h(\cdot, P))\big).$$

For $n \in \mathbf{N}$ let $\mathfrak{Q}_n$ be families of p-measures over $\mathscr{A}$, let $h_n(\cdot, Q)\colon X^n \to \mathbf{R}$ and $g_n(\cdot, Q)\colon X^n \to \mathbf{R}$, $Q \in \mathfrak{Q}_n$ be measurable functions.

We write for $r \geqslant 0$

$$h_n(\cdot, Q) = o_n(r) \text{ with respect to } \mathfrak{Q}_n \tag{3.1}$$

if

$$\sup_{P,Q \in \mathfrak{Q}_n} P^n\{|h_n(\cdot, Q)| > c\} = o(n^{-r}) \quad \text{for every } c > 0.$$

We remark that $h_n(\cdot, Q) = o_n(r)$ implies the existence of a sequence $c_n \downarrow 0$, $n \in \mathbf{N}$, such that

$$\sup_{P,Q \in \mathfrak{Q}_n} P^n\{|h_n(\cdot, Q)| > c_n\} = o(n^{-r}).$$

We write for $r, s \geqslant 0$

$$f_n(\cdot, Q) = g_n(\cdot, Q) + n^{-s} o_n(r)$$

if

$$n^s\big(f_n(\cdot, Q) - g_n(\cdot, Q)\big) = o_n(r).$$

We define

$$U_{n,\delta}(P_*) := \{Q \in \mathfrak{P}\colon d(P_*^n, Q^n) \leqslant 1 - \delta\} \quad \text{for } \delta \in (0, 1), \tag{3.2}$$

where d is the sup-distance.

4. Critical regions for $\varkappa_0$. The results of this and the following sections are obtained under a number of regularity conditions which will be listed in Section 8. Among these we have the mutual absolute continuity of the measures in $\mathfrak{P}$, the continuity of $P \to \varkappa(P)$ and various moment conditions on f and its derivatives, etc.

Moreover, we require that a variant of Cramér's Condition C is fulfilled for the joint distribution induced by certain derivatives of f. In fact, this restricts the applicability of our results to families $\mathfrak{P}$ with Lebesgue densities. It is, however, to be expected that the corresponding 2nd order optimum properties can be obtained for certain randomized estimators without a restriction like Cramér's Condition.

In this section we shall show how c.r. (critical regions) for $\varkappa_0$ can be obtained by applying an *asymptotic studentization procedure* to the m.c. estimator.

Under suitable regularity conditions (see Lemma (9.70) in [8]), the m.c. estimator $\varkappa^{(n)}$ admits a stochastic expansion of type

(4.1) $$n^{1/2}\left(\varkappa_i^{(n)} - \varkappa_i(P)\right) = \tilde{f}_i + n^{-1/2} M_i(\tilde{f}, \tilde{f}^{\cdot}, P) + n^{-1/2} o_n(\tfrac{1}{2}), \qquad i = 0, 1, \ldots, p,$$

with polynomials $M_i(\cdot, \cdot, P)$ defined by (9.74) in [8].

Let $\sigma_0(P) := \sigma_{00}(P)^{1/2}$. Then (4.1) implies, in particular, that

(4.2) $$P^n\{n^{1/2}(\varkappa_0^{(n)} - t_0) > -N_\alpha \sigma_0(P)\} = \alpha + o(n^0)$$

for every $P \in \mathfrak{P}$ with $\varkappa_0(P) = t_0$.

Assume that there exists an estimator-sequence for σ_0, say $\sigma_0^{(n)}$, $n \in N$, which admits a stochastic expansion

(4.3) $$n^{1/2}\left(\sigma_0^{(n)} - \sigma_0(P)\right) = \tilde{k}(\cdot, P) + o_n(\tfrac{1}{2}).$$

By Lemma (9.63) in [8],

$$P^n\{n^{1/2}(\varkappa_0^{(n)} - t_0) > -N_\alpha \sigma_0^{(n)}\} = \alpha + o(n^0)$$

for every $P \in \mathfrak{P}$ with $\varkappa_0(P) = t_0$.

Under suitable conditions there exists $c_\alpha(P) \in \boldsymbol{R}$ such that

(4.4) $$P^n\{n^{1/2}(\varkappa_0^{(n)} - t_0) > -N_\alpha \sigma_0^{(n)} + n^{-1/2} c_\alpha(P)\} = \alpha + o(n^{-1/2}).$$

Using the stochastic expansions (4.1) and (4.3) we obtain

$$P^n\{n^{1/2}(\varkappa_0^{(n)} - t_0) > -N_\alpha \sigma_0^{(n)} + n^{-1/2} c_\alpha(P)\}$$
$$= P^n\{\tilde{f}_0 + N_\alpha \sigma_0(P) + n^{-1/2}(M_0(\tilde{f}, \tilde{f}^{\cdot}, P) + N_\alpha \tilde{k}) > n^{-1/2} c_\alpha(P)\} + o(n^{-1/2}),$$

whence

(4.5) $$c_\alpha = (1 - N_\alpha^2)\sigma_{00}^{-1} A_{0i} A_{0j} A_{0k}\left(\tfrac{1}{6}F_{i,j,k} + A_{lv} F_{j,v}\left(\tfrac{1}{2}A_{qr} F_{k,r} F_{ilq} - F_{lk,i}\right)\right) + A_{0k} A_{ji}\left(F_{kj,i} - \tfrac{1}{2}F_{jkl} A_{lv} F_{i,v}\right) - \sigma_0^{-1} N_\alpha^2 P(f_0 k).$$

Replacing $c_\alpha(P)$ by an estimator $c_\alpha^{(n)}$ for which

(4.6) $$c_\alpha^{(n)} = c_\alpha(P) + o_n(\tfrac{1}{2})$$

we obtain a sequence of c.r. of level $\alpha + o(n^{-1/2})$.

Theorem (4.16) below specifies regularity conditions under which this asymptotic studentization procedure is feasible.

The c.r. obtained by asymptotic studentization are particular instances of a more general class of c.r. $\{F_n(\cdot, t_0) > 0\}$ based on test statistics $F_n(\cdot, t_0)$.

For simplicity, we say that a sequence of test statistics F_n, $n \in N$, is *asymptotically* [*similar*] *of level* $\alpha + o(n^{-s})$ if the sequence of c.r. $\{F_n > 0\}$ has this property (see Definition (2.2)).

Let $P_* \in \mathfrak{P}$. For $\delta \in (0, 1)$ let $U_{n,\delta}(P_*)$ be defined by (3.2).

(4.7) Definition. A sequence of test statistics $F_n | X^n \times \mathbb{K}$, $n \in N$ (for definition of $\mathbb{K}$ see (2.1)), is of *type S* if it admits an asymptotic expansion of the following kind:

(4.8) $$F_n(\cdot, \varkappa(P)) = c(P) + \tilde{f}_0(\cdot, P) + n^{-1/2} M(\tilde{f}_0(\cdot, P), \tilde{g}(\cdot, P), P) + n^{-1/2} o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$ (see (3.1)), where $M(\cdot, \cdot, P)$ are polynomials and $g(\cdot, P)$: $X \to \mathbb{R}^m$ are measurable functions fulfilling the regularity conditions specified in the sequel.

We consider only test statistics with leading term $\tilde{f}_0$ because test statistics with a different leading term are inefficient, and we investigate the 2nd order efficiency of 1st order efficient test statistics.

(4.9) Remark. Note that under suitable moment conditions on f_0 and g the sequence F_n, $n \in N$, is asymptotically similar of level $\alpha + o(n^0)$ for $U_{n,\delta}(P_*)$ iff $c(P) = N_\alpha \sigma_0(P)$ (as a consequence of Lemma (9.63) in [8] and the Central Limit Theorem).

Regularity conditions required for test statistics of type S:

(4.10) The coefficients of $M(\cdot, \cdot, P)$, considered as functions of P, are continuous at P_*.

(4.11) $P \to g(x, P)$ is continuous at P_* for P_*-a.a. $x \in X$.

(4.12) $f_0(\cdot, P_*)$ and $g_i(\cdot, P_*)$ are P_*-uncorrelated for $i = 1, \ldots, m$.

There exists a strong neighborhood U_* of P_* in $\mathfrak{P}$ such that

(4.13) $P(g(\cdot, P)) = 0$ for $P \in U_*$,

(4.14) $M_3^*(\{P_* * g(\cdot, P): P \in U_*\})$,

(4.15) $C(\{P_* * (f_0(\cdot, P), g(\cdot, P)): P \in U_*\})$.

(4.16) THEOREM. *Assume that* $\sigma_0^{(n)}$ *and* $c_\alpha^{(n)}$ *are estimator-sequences for* σ_0 *and* c_α, *fulfilling* (4.3) *and* (4.6), *where the remainders* $o_n(\frac{1}{2})$ *hold with respect to* $U_{n,\delta}(P_*)$ *for every* $\delta \in (0, 1)$.

Then the sequence of test statistics F_n, $n \in \mathbf{N}$, *obtained from the m.c. estimator by asymptotic studentization, namely*

$$F_n(x, t_0) := n^{1/2}(\varkappa_0^{(n)}(x) - t_0) + N_\alpha \sigma_0^{(n)}(x) - n^{-1/2} c_\alpha^{(n)}(x), \tag{4.17}$$

is of type S and for every $\delta \in (0, 1)$ *asymptotically similar of level* $\alpha + o(n^{-1/2})$ *for* $U_{n,\delta}(P_*)$.

In addition to General Assumptions (8.1)-(8.5) *we need the following regularity conditions*:

(4.18) $P \to k(x, P)$ *is continuous at* P_* *for* P_**-a.a.* $x \in X$.

There exists a strong neighborhood U_* *of* P_* *in* $\mathfrak{P}$ *such that*

(4.19) $K_{3/2}(\varkappa(P_*), U_*)$ *for* $f\colon X \times T \to \mathbf{R}$,

(4.20) $M_3^*(\{P * f^\alpha(\cdot, \varkappa(Q))\colon P, Q \in U_*\})$ *for* $|\alpha| = 1, 2, 3$,

(4.21) $L_{3/2}^*(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X \times T \to \mathbf{R}$ *if* $|\alpha| = 3$,

(4.22) $M_3(\{P * k(\cdot, Q)\colon P, Q \in U_*\})$,

(4.23) $C^*\big(\{P_* * (f_i(\cdot, P), i = 0, \ldots, p, f_0^{(j)}(\cdot, P), j = 0, \ldots, p, k(\cdot, P))\colon P \in U_*\}\big)$.

(4.24) Remark. By repeated applications of the asymptotic studentization procedure one can obtain c.r. which are asymptotically similar of arbitrarily high order. For the case of a 1-dimensional m.c. function, a c.r. which is asymptotically similar of level $\alpha + o(n^{-1})$ was given in [7] (Theorem 3.1, p. 119). This paper also contains numerical computations showing that, for the particular case of the expectation, these critical regions keep the prescribed error type one with sufficient accuracy.

The following proposition gives conditions under which estimator-sequences for σ_0 and c_α fulfilling (4.3) and (4.6) exist:

(4.25) PROPOSITION. (i) *The estimator-sequence*

$$\sigma_0^{(n)} := A_{0i}^{(n)} A_{0j}^{(n)} F_{i,j}^{(n)} \tag{4.26}$$

fulfills (4.3) *with*

$$k(\cdot, P) := \sigma_0(P)^{-1}\big[A_{0j}(P) A_{0l}(P) F_{j,i}(P)\big(F_{lkr}(P) A_{ik}(P) f_r(\cdot, P) - f_i^{(k)}(\cdot, P)\big) + \tfrac{1}{2} f_0^2(\cdot, P) - \big(F_{i,jk}(P) + F_{j,ik}(P)\big) f_k(\cdot, P)\big]. \tag{4.27}$$

(ii) *The estimator-sequence* $c_\alpha^{(n)}$ *obtained from* $c_\alpha(P)$ (see (4.5)) *by replacing* $F_{i_1 \ldots i_{q_1}, \ldots, i_{q_{k-1}+1} \ldots i_l}(P)$ *by* $F_{i_1 \ldots i_{q_1}, \ldots, i_{q_{k-1}+1} \ldots i_l}^{(n)}$ *and* $A_{ij}(P)$ *by* $A_{ij}^{(n)}$ *fulfills* (4.6).

The remainders $o_n(\frac{1}{2})$ *in* (i) *and* (ii) *hold with respect to* $U_{n,\delta}(P_*)$ *for every* $\delta \in (0, 1)$.

In addition to General Assumptions (8.1)-(8.3) *and* (8.5) *we need the following regularity conditions*:

Regularity conditions for (i). *There exists a strong neighborhood* U_* *of* P_* *in* $\mathfrak{P}$ *such that*

(4.28) $K_{3/2}(\varkappa(P_*), U_*))$ *for* $f\colon X\times T\to \boldsymbol{R}$;

(4.29) $M_4^*(\{P*f^\alpha(\cdot, \varkappa(Q))\colon P, Q\in U_*\})$ *for* $|\alpha| = 1, 2$,
$M_3^*(\{P*f^\alpha(\cdot, \varkappa(Q))\colon P, Q\in U_*\})$ *for* $|\alpha| = 3$;

(4.30) $L_3(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 2$,
$L_2(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 3$.

Regularity conditions for (ii). *There exists a strong neighborhood* U_* *of* P_* *in* $\mathfrak{P}$ *such that*

(4.31) $K_1(\varkappa(P_*), U_*)$ *for* $f\colon X\times T\to \boldsymbol{R}$;

(4.32) $M_{9/2}^*(\{P*f^\alpha(\cdot, \varkappa(Q))\colon P, Q\in U_*\})$ *for* $|\alpha| = 1$,
$M_{9/4}^*(\{P*f^\alpha(\cdot, \varkappa(Q))\colon P, Q\in U_*\})$ *for* $|\alpha| = 2$,
$M_{3/2}^*(\{P*f^\alpha(\cdot, \varkappa(Q))\colon P, Q\in U_*\})$ *for* $|\alpha| = 3$;

(4.33) $L_{9/5}(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 1$,
$L_{9/7}(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 2$,
$L_1(\varkappa(P_*), U_*)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 3$.

5. Second order efficiency of the c.r. for $\varkappa_0$. In this section we shall show that all c.r. of type S (in particular, the c.r. obtained from the m.c. estimator by asymptotic studentization; see Theorem (4.16)) are 2nd order efficient.

(5.1) THEOREM. *Let* φ_n, $n\in \boldsymbol{N}$, *be a sequence of c.f. which is asymptotically of level* $\alpha = o(n^{-1/2})$ *for* $U_{n,\delta}(P_*)$ *for every* $\delta\in(0,1)$, *and let* $F_n, n\in \boldsymbol{N}$, *be a sequence of test statistics for* $\varkappa_0$ *of type* S (see (4.7)) *which is asymptotically similar of level* $\alpha+o(n^{-1/2})$ *for* $U_{n,\delta}(P_*)$ *for every* $\delta\in(0,1)$.

Assume that for every $\Delta>0$ *there exists a sequence* $P_{n,\Delta}\in\mathfrak{P}$, $n\in \boldsymbol{N}$, *fulfilling*

(5.2) $$\varkappa_0(P_{n,\Delta}) = \varkappa_0(P_*)-n^{-1/2}\Delta,$$

and admitting a P_**-density*

(5.3) $$P_{n,\Delta} := 1-n^{-1/2}\Delta\sigma_{00}^{-1}f_0(\cdot, P_*)+n^{-1}\Delta^2 h+n^{-3/2}r_{n,\Delta},$$

such that

(5.4) $$M_2^*(P_**h),$$

and for every $\Delta_0 > 0$

(5.5) $M^*_{3/2}\{P_* * r_{n,\Delta}\colon n\in N,\ 0 < \Delta \leqslant \Delta_0\}$ *and* $\sup_{0<\Delta\leqslant\Delta_0} P_*(r^2_{n,\Delta}) = o(n)$.

Then for every $\Delta_0 > 0$

(5.6)
$$P^n_*\big(\varphi_n(\cdot, \varkappa_0(P_*)-n^{-1/2}\Delta)\big) \leqslant P^n_*\{F_n(\cdot, \varkappa_0(P_*)-n^{-1/2}\Delta) > 0\}+o(n^{-1/2}) = \pi_n(\Delta, \alpha)+o(n^{-1/2})$$

uniformly for $0 < \Delta \leqslant \Delta_0$, *where*

(5.7)
$$\pi_n(\Delta, \alpha) = \Phi(N_\alpha+\Delta\sigma_0^{-1})-n^{-1/2}\Delta\sigma_{00}^{-2}\varphi(N_\alpha+\Delta\sigma_0^{-1})\times \times\big[A_{0r}A_{0s}A_{0t}\big(\Delta\sigma_0^{-1}\big(\tfrac{1}{6}F_{r,s,t}+A_{qv}F_{v,r}\big(\tfrac{1}{2}F_{p,t}A_{vp}F_{sqv}-F_{t,sq}\big)\big)-\tfrac{1}{6}N_\alpha F_{r,s,t}\big)\big].$$

In addition to General Assumptions (8.1)-(8.5) *we need the following regularity conditions:*

(5.8) $M^*_{9/2}\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ *for* $|\alpha| = 1$,
$M_4\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ *for* $|\alpha| = 2$,
$M_2\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ *for* $|\alpha| = 3$;

(5.9) $L_4\big(\varkappa(P_*), P_*\big)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 2$,
$L_2\big(\varkappa(P_*), P_*\big)$ *for* $f^\alpha\colon X\times T\to \boldsymbol{R}$ *if* $|\alpha| = 3$.

The basic idea of the proof is as follows. Let $P_{n,\Delta}$, $n\in N$, $0 < \Delta \leqslant \Delta_0$, be sequences of p-measures fulfilling (5.2) and admitting a P_*-density

(5.10)
$$p_{n\Delta} := 1+n^{-1/2}\Delta g+n^{-1}\bar{r}_{n,\Delta}.$$

According to Lemma (9.35) in [8] the sequence of most powerful level α-tests for $P^n_{n\Delta}\colon P^n_*$ has power $\Phi\big(N_\alpha+\Delta P_*(g^2)^{1/2}\big)+o(n^0)$. If a sequence of c.f. φ_n, $n\in N$, is asymptotically of level $\alpha+o(n^0)$ for $U_{n,\delta}(P_*)$ for every $\delta\in(0, 1)$, we obtain

$$P^n_*\big(\varphi_n(\cdot, \varkappa_0(P_*)-n^{-1/2}\Delta)\big) \leqslant \inf_g \Phi\big(N_\alpha+\Delta P_*(g^2)^{1/2}\big)+o(n^0),$$

where the infimum is taken over all g for which p-measures $P_{n,\Delta}$, $n\in N$, $0 < \Delta \leqslant \Delta_0$, with (5.2) and (5.10) belong to $\mathfrak{P}$. From (5.2) and (5.10) we obtain $P_*(gf_0) = -1$.

To obtain a small upper bound for $P^n_*\big(\varphi_n(\cdot, \varkappa_0(P_*)-n^{-1/2}\Delta)\big)$ we have to minimize $P_*(g^2)$, subject to the condition $P_*(gf_0) = -1$. By Hölder's inequality this minimum is attained for $g_0 = -\sigma_{00}^{-1}f_0$, provided $P_{n,\Delta}$, $n\in N$, belong to $\mathfrak{P}$ for this function. (These p-measures are asymptotically least favorable in the sense that they are asymptotically most difficult to discriminate from P_*.)

Thus we obtain the upper bound

$$P^n_*\big(\varphi_n(\cdot, \varkappa_0(P_*)-n^{-1/2}\Delta)\big) \leqslant \Phi(N_\alpha+\Delta\sigma_0^{-1})+o(n^0).$$

Since this upper bound is attained for every sequence of test functions of type S, this bound is sharp and the test functions are asymptotically optimal up to an error term $o(n^0)$. Theorem (5.1) asserts that this optimality even holds true with an error term $o(n^{-1/2})$.

The purpose of Corollaries (5.11) and (5.15) is to exhibit natural conditions under which the family contains p-measures $P_{n,\Delta}$, $n \in N$, $0 < \Delta \leqslant \Delta_0$, fulfilling (5.2)-(5.5). Corollary (5.15) shows that this always is the case if $\mathfrak{P}$ is a parametrized family and $P_* \in \mathfrak{P}$ an "inner" point.

Another condition – most natural within a non-parametric set up – is that $\mathfrak{P}$ contains all p-measures in a certain neighborhood of P_*. This is the situation considered in Corollary (5.11).

To make the nature of this assumption more transparent, let us first consider two examples where it is not fulfilled.

Assume first that $\mathfrak{P} = \{N_{(\theta,1)}: \theta \in \boldsymbol{R}\}$ and let $f(x, t) = f_*(x-t)$, where f_* is any sufficiently regular $N_{(\theta,1)}$-integrable function which is symmetric about the origin. Then $\varkappa(P_\theta) = \theta$, but the test based on the contrast function f_* is not optimal unless $f_*(x) = cx^2$, because for other contrast functions f_* the least favorable p-measures with $N_{(\theta^*,1)}$-densities

$$p_{n,\Delta}(x) = 1 - n^{-1/2}\Delta N_{(0,1)}\big(f_*'(x)^2\big)^{-1/2} f_*'(x-\theta^*) + n^{-1}\bar{r}_{n,\Delta}(x)$$

do not belong to $\{N_{(\theta,1)}: \theta \in \boldsymbol{R}\}$.

As another example consider the case where $\mathfrak{P}$ is a family with symmetric unimodal densities and $f(x, t) := f_*(x-t)$, f_* being bounded, neg-unimodal, and symmetric about 0. The pertaining m.c. functional is the center of symmetry (see Example 3). Corollary (5.11) requires that the family $\mathfrak{P}$ contains together with P_* certain p-measures with P_*-density $p_{n,\Delta} := 1 + n^{-1/2}\Delta A_{0i} f^{(i)} + n^{-1}\bar{r}_{n,\Delta}$. This, however, is not the case, since such p-measures are not symmetric any more. Hence Corollary (5.11) is not applicable. In fact, it is well known that for 1-dimensional symmetric distributions with unknown shape the center of symmetry can be estimated with the same asymptotic efficiency as in the case of known shape (see, e.g., [9]).

(5.11) COROLLARY. *Assume that there exists a constant $c > 0$ such that any p-measure P admitting a P_*-density p with*

$$\sup_{x \in X} |p(x) - 1| \leqslant c$$

belongs to $\mathfrak{P}$.

Then the c.r. obtained from the m.c. estimator by asymptotic studentization (see (4.17)) *is 2nd order efficient in the sense of* (5.6).

In addition to the assumptions of Theorem (4.16) *we need the following regularity conditions*:

(5.12) $M^*_{9/2}\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ *for* $|\alpha| = 1$,
$M_4\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ *for* $|\alpha| = 2$;

(5.13) $L_4(\varkappa(P_*), P_*)$ *for* f^α: $X \times T \to \boldsymbol{R}$ *if* $|\alpha| = 2$,
$L_2(\varkappa(P_*), P_*)$ *for* f^α: $X \times T \to \boldsymbol{R}$ *if* $|\alpha| = 3$.

(5.14) Remark. For the special case of quantiles, a related result was obtained in [6] (Theorem 3, p. 114) proving the 1st order efficiency of statistical procedures based on the sample quantile.

The following corollary concerns the case of a parametrized family of p-measures, say $\mathfrak{P} = \{P_\theta: \theta \in \Theta\}$ with $\Theta \subset \boldsymbol{R}^{p+1}$. With the likelihood contrast function $f(x, \theta) = -\log p(x, \theta)$, we obtain the m.c. functional $\varkappa(P_\theta) = \theta$. The pertaining m.c. estimator is the *m.l.* (*maximum likelihood*) estimator.

(5.15) Corollary. *Assume that* $\mathfrak{P} = \{P_\theta: \theta \in \Theta\}$ *with* $\Theta \subset \boldsymbol{R}^{p+1}$, *and* θ^* *is an inner point of* Θ.

Then the c.r. obtained from the m.l. estimator by asymptotic studentization is 2nd order efficient in the sense of (5.6).

In addition to the assumptions of Theorem (4.16) *and conditions* (5.12) *and* (5.13) *for* $P_* = P_{\theta^*}$ *and* $f(\cdot, \theta) = -\log p(\cdot, \theta)$ *we need the following regularity conditions*:

(5.16) $M_2^*(P_{\theta^*} * p^\alpha(\cdot, \theta^*)/p(\cdot, \theta^*))$ *for* $|\alpha| = 2$,
$L_2(\theta^*, P_{\theta^*})$ *for* $p^\alpha(\cdot, \theta^*)/p(\cdot, \theta^*)$: $X \times \Theta \to \boldsymbol{R}$ *if* $|\alpha| = 2$.

This result is, of course, well known (see [2], p. 40, Theorem 9.1, and p. 38, Theorem 8.1, and [5], p. 260, Proposition). It is mentioned here because it is of some interest that Theorem (5.1), aiming at non-parametric applications, yields this parametric result as a by-product (of course under somewhat different regularity conditions).

Moreover, we infer from Theorem (5.1) that the 2nd order efficiency is not diminished if the estimator for σ_0 used in the asymptotic studentization procedure is inefficient.

Let

$$L(\theta) := \big(P_\theta\big((\log p)^{(i)}(\cdot, \theta)\,(\log p)^{(j)}(\cdot, \theta)\big)\big)_{i,j=0,\dots,p},$$

$$\Lambda(\theta) := L(\theta)^{-1}.$$

In the case of the likelihood contrast function we have $F_{\cdot,\cdot}(P_\theta) = L(\theta)$ and $F_{..}(P_\theta) = -L(\theta)$, whence $A(P_\theta) = -\Lambda(\theta)$, and therefore

$$A_{0i}(P_\theta)\,A_{0j}(P_\theta)\,F_{i,j}(P_\theta) = \Lambda_{00}(\theta).$$

Hence an estimator for σ_0, more natural than the estimator $\sigma_0^{(n)}$ given in Proposition (4.25), is $\Lambda_{00}(\theta^{(n)})^{1/2}$, where $\theta^{(n)}$ is the m.l. estimator. If Λ_{00} is difficult to obtain (as, for instance, in the case of mixtures), one could as well use $(\Lambda_{00}^{(n)})^{1/2}$, where $\Lambda_{00}^{(n)}(\boldsymbol{x})$ is the $(0, 0)$-element of the matrix obtained by inversion of

$$\Big(n^{-1}\sum_{\nu=1}^{n} l^{(i)}(x_\nu, \theta^{(n)}(\mathbf{x}))\, l^{(j)}(x_\nu, \theta^{(n)}(\mathbf{x}))\Big)_{i,j=0,\dots,p}.$$

Using $\sigma_0^{(n)}$ as given by (4.26) rather than $(\Lambda_{00}^{(n)})^{1/2}$ or $\big(\Lambda_{00}(\theta^{(n)})\big)^{1/2}$ leads to a c.r. which is asymptotically similar also for p-measures in the neighborhood of $\{P_\theta\colon \theta\in\Theta\}$, without diminishing the power by more than $o(n^{-1/2})$ if the model $\{P_\theta\colon \theta\in\Theta\}$ is correct.

6. Optimal confidence procedures. In Section 4, critical regions

$$C_{n,\alpha}(t_0) := \{n^{1/2}(\varkappa_0^{(n)}-t_0)-N_\alpha\sigma_0^{(n)}+n^{-1/2}c_\alpha^{(n)} > 0\}$$

have been obtained which are asymptotically similar of level $\alpha+o(n^{-1/2})$ for the hypothesis $\varkappa_0(P) = t_0$. Assume this holds true for any hypothesis $t_0\in\mathbf{K}$ (for $\mathbf{K}$ see (2.1)). In view of their special structure, these c.r. can be inverted immediately to confidence procedures for $\varkappa_0$.

Let

(6.1) $$\varkappa_{0,\alpha}^{(n)} := \varkappa_0^{(n)}-n^{-1/2}N_\alpha\sigma_0^{(n)}+n^{-1}c_\alpha^{(n)}.$$

Then $P^n\big(C_{n,\alpha}(\varkappa_0(P))\big) = \alpha+o(n^{-1/2})$ for all $P\in\mathfrak{P}$ implies

$$P^n\{\varkappa_{0,\alpha}^{(n)} \leqslant \varkappa_0(P)\} = 1-\alpha+o(n^{-1/2}) \quad \text{for all } P\in\mathfrak{P}.$$

(6.2) Definition. A sequence of confidence procedures K_n, $n\in N$, assigning to each $\mathbf{x}\in X^n$ a set $K_n(\mathbf{x})\subset\mathbf{K}$ is *asymptotically [similar] with confidence coefficient* $1-\alpha+o(n^{-s})$ if for every $P_*\in\mathfrak{P}$ and every $\delta\in(0,1)$

$$P^n\{\mathbf{x}\in X^n\colon \varkappa_0(P)\in K_n(\mathbf{x})\} \geqslant [=]\, 1-\alpha+o(n^{-s})$$

uniformly for $P\in U_{n,\delta}(P_*)$.

Hence the confidence procedure $\mathbf{x}\to\big[\varkappa_{0,\alpha}^{(n)}(\mathbf{x}),\infty\big)$ is asymptotically similar with confidence coefficient $1-\alpha+o(n^{-s})$, and the confidence sets, being rays, are of a particularly useful structure. Moreover, the 2nd order efficiency, proved for the c.r. $C_{n,\alpha}(t_0)$ in Section 5, carries over to the confidence procedure derived thereof.

(6.3) THEOREM. (i) *The sequence of confidence procedures* $\mathbf{x}\to\big[\varkappa_{0,\alpha}^{(n)}(\mathbf{x}),\infty\big)$, $n\in N$, *is asymptotically similar with confidence coefficient* $1-\alpha+o(n^{-1/2})$.

(ii) *It is optimal in the following sense provided for every* $\Delta>0$ *there exists a sequence* $P_{n,\Delta}\in\mathfrak{P}$, $n\in N$, *fulfilling* (5.2)-(5.5):

If K_n, $n\in N$, *is any sequence of confidence procedures with asymptotic confidence coefficient* $1-\alpha+o(n^{-1/2})$, *then for every* $P_*\in\mathfrak{P}$ *and every* $\Delta_0>0$

(6.4) $$P_*^n\{\mathbf{x}\in X^n\colon \varkappa_0(P_*)-n^{-1/2}\Delta \geqslant \varkappa_{0,\alpha}^{(n)}(\mathbf{x})\} \leqslant P_*^n\{\mathbf{x}\in X^n\colon \varkappa_0(P_*)-n^{-1/2}\Delta\in K_n(\mathbf{x})\}+o(n^{-1/2})$$

uniformly for $0<\Delta\leqslant\Delta_0$.

Assertions (i) *and* (ii) *are true if the assumptions of Theorem* (4.16) *and* (5.12) *and* (5.13) *hold true for every* $P_* \in \mathfrak{P}$.

Natural conditions on $\mathfrak{P}$ under which sequences $P_{n,\Delta}$, $n \in N$, fulfilling (5.2)-(5.5) exist are given in Corollaries (5.11) and (5.15).

Theorem (6.3) follows immediately from Theorems (4.16) and (5.1) applied for $\varphi_n(\cdot, t_0) := 1_{\{x \in X^n: t_0 \notin K_n(x)\}}$.

7. Optimal median unbiased estimators. In this section we shall show that $\varkappa_{0,1/2}^{(n)}$, the lower confidence bound with confidence coefficient $\frac{1}{2}$, if considered as an estimator for $\varkappa_0(P)$, is asymptotically median unbiased up to an error term $o(n^{-1/2})$, and is 2nd order efficient within this class of estimators.

(7.1) Definition. An estimator-sequence $t_0^{(n)}$, $n \in N$, for $\varkappa_0$ is *asymptotically median unbiased* $o(n^{-s})$ if for every $P_* \in \mathfrak{P}$ and every $\delta \in (0, 1)$

$$P^n\{x \in X^n: \varkappa_0(P) \leqslant t_0^{(n)}(x)\} \geqslant \tfrac{1}{2} + o(n^{-s}),$$

$$P^n\{x \in X^n: \varkappa_0(P) \geqslant t_0^{(n)}(x)\} \geqslant \tfrac{1}{2} + o(n^{-s})$$

uniformly for $P \in U_{n,\delta}(P_*)$.

(7.2) THEOREM. (i) *The sequence* $\varkappa_{0,1/2}^{(n)}$, $n \in N$, *is asymptotically median unbiased* $o(n^{-1/2})$.

(ii) *It is optimal in the following sense provided for every* $\Delta \neq 0$ *there exists a sequence* $P_{n,\Delta} \in \mathfrak{P}$, $n \in N$, *fulfilling* (5.2)-(5.5) (*the supremum in* (5.5) *is taken over* $0 < |\Delta| \leqslant \Delta_0$):

If $t_0^{(n)}$ *is any estimator-sequence for* $\varkappa_0$ *which is asymptotically median unbiased* $o(n^{-1/2})$, *then for every* $P_* \in \mathfrak{P}$ *and every* $\Delta_0 > 0$

$$P_*^n\{x \in X^n: \varkappa_0(P_*) - n^{-1/2}\Delta' < t_0^{(n)}(x) < \varkappa_0(P_*) + n^{-1/2}\Delta''\}$$

$$\leqslant P_*^n\{x \in X^n: \varkappa_0(P_*) - n^{-1/2}\Delta' < \varkappa_{0,1/2}^{(n)}(x) < \varkappa_0(P_*) + n^{-1/2}\Delta''\} + o(n^{-1/2})$$

uniformly for $0 < \Delta', \Delta'' \leqslant \Delta_0$.

Assertions (i) *and* (ii) *are true if the assumptions of Theorem* (4.16) *and* (5.12) *and* (5.13) *hold true for every* $P_* \in \mathfrak{P}$.

From this theorem one can easily derive corollaries corresponding to Corollaries (5.11) and (5.15).

With $o(n^{-1/2})$ replaced by $o(n^0)$ the non-parametric optimality assertion occurs in [6] (Corollary 2, p. 116) for the particular case of the sample quantile. For general m.c. estimators a somewhat different but intuitively related 1st order optimality assertion occurs in [4] (p. 738, Theorem 4.4). See also [3].

Theorem (7.2) follows from Theorem (6.3) and Remarks (9.34) and (9.56) in [8] applied to the confidence procedures $K_n(x) = [t_0^{(n)}(x), \infty)$ and $K_n(x) = (-\infty, t_0^{(n)}(x)]$, respectively.

8. Regularity conditions. In this section we list the regularity conditions which are needed somewhere for the proofs. At the end of this section we specify some "general assumptions", which are made throughout the paper.

Conditions M_r, M_r^*, C, and C^* refer to a family of p-measures $\mathfrak{Q}$ over $\mathscr{B}^{p+1}$.

CONDITION $M_r(\mathfrak{Q})$. $\sup_{Q\in\mathfrak{Q}} \int \|x\|^r Q(dx) < \infty$.

CONDITION $M_r^*(\mathfrak{Q})$. Condition $M_{r+\delta}(\mathfrak{Q})$ is fulfilled for some $\delta > 0$.

CONDITION $C(\mathfrak{Q})$. $\overline{\lim}_{\|t\|\to\infty} \sup_{Q\in\mathfrak{Q}} \left|\int \exp[iu_j t_j] Q(du)\right| < 1$.

CONDITION $C^*(\mathfrak{Q})$. Every $Q\in\mathfrak{Q}$ is concentrated on a k-dimensional affine subspace and there exists a subindex $(i_1, \ldots, i_k)$ of $(0, \ldots, p)$ with $i_1 = 0$ such that Condition $C(\{Q*(\pi_{i_1}, \ldots, \pi_{i_k})\colon Q\in\mathfrak{Q}\})$ is fulfilled.

Let now $\mathfrak{Q}$ be a family of p-measures over $\mathscr{A}$.

CONDITION L_r. $h\colon X\times T\to \boldsymbol{R}$ fulfills Condition $L_r(t, \mathfrak{Q})$ if for some neighborhood $V(t)$ of t

$$|h(x, t')-h(x, t'')| \leqslant g(x)\,\|t'-t''\|$$

for $t', t''\in V(t)$, where g fulfills $M_r(\{Q*g\colon Q\in\mathfrak{Q}\})$.

CONDITION L_r^*. $L_{r+\delta}$ is fulfilled for some $\delta > 0$.

Let $\bar{T}$ denote the closure of T in the one-point compactification of $\boldsymbol{R}^{p+1}$.

CONDITION K_r. $h\colon X\times T\to \boldsymbol{R}$ fulfills $K_r(t, \mathfrak{Q})$ if

(a) $M_r^*(\{Q*h(\cdot, t)\colon Q\in\mathfrak{Q}\})$,

(b) for each $s\in\bar{T}\setminus\{t\}$ there exists a neighborhood $V(s)$ of s in $\bar{T}$ such that

$$M_r^*(\{Q*|\inf\{h(\cdot, u)\colon u\in V(s)\cap T\}|\colon Q\in\mathfrak{Q}\})$$

is fulfilled.

We introduce the following *General Assumptions*:

(8.1) The measures in $\mathfrak{P}$ are mutually absolutely continuous.

(8.2) $t\to f(\cdot, t)$ is three times differentiable on T, and $t\to P(f(\cdot, t))$ can be extended to a continuous function on $\bar{T}$ for every $P\in\mathfrak{P}$.

(8.3) $f\colon X\times T\to \boldsymbol{R}$ fulfills $L_1^*(t, P)$.

(8.4) For each $P\in\mathfrak{P}$ there exist a strong neighborhood $U(P)$ of P and a neighborhood $V(\varkappa(P))$ of $\varkappa(P)$ such that

$$\inf_{Q\in U(P)} \inf_{t\in V(\varkappa(P))} \lambda(Q, t) > 0,$$

where $\lambda(Q, t)$ is the smallest eigenvalue of the matrix $Q\left(f^{(ij)}(\cdot, t)\right)_{i,j=0,\ldots,p}$.

(8.5) For every $P \in \mathfrak{P}$ and $\delta \in (0, 1)$,

$$\sup_{n \in N} \sup_{Q \in U_{n,\delta}(P)} n^{1/2} \|\varkappa(Q) - \varkappa(P)\| < \infty,$$

where $U_{n,\delta}$ is defined by (3.2).

We remark that assumption (8.3) guarantees that the order of differentiation and integration can be interchanged for $t \to P(f(\cdot, t))$.

It will be shown elsewhere that (8.5) holds under certain regularity assumptions.

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