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CYLINDRICAL MEASURES AND CYLINDRICAL PROCESSES ON LOCALLY CONVEX SPACES

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Abstract. The first part of the paper contains generalizations of results of Gil de Lamadrid [5] and Vakhania and Tarieladze [10] concerning properties of cylindrical processes via the Pettis integral. The second part deals with characterization of cylindrical measures which are scalarly concentrated on compacta.

1. Introduction. In this paper we consider relations between some properties of cylindrical measures and properties of cylindrical processes on locally convex spaces.

Section 2 is devoted to necessary notation and basic facts. In Section 3 we consider the Pettis integral of cylindrical processes which are elements of $E\hat{\otimes}_\varepsilon L^1(\Omega, \mathscr{A}, P)$. The results of Section 3 are used in Section 4. In particular, the results obtained contain those of Gil de Lamadrid [5] and Vakhania and Tarieladze [10]. In Section 4 we show that a gaussian cylindrical measure μ on a complete locally convex space E is scalarly concentrated on compact sets if and only if its cylindrical process belongs to $E\hat{\otimes}_\varepsilon L^2$ (Theorem 4.1). Moreover, in this situation μ has the barycenter in E and the reproducing kernel Hilbert space of μ is a subspace of E with the compact unit ball (this result for gaussian Radon measures is due to Dudley et al. [4]).

2. Notation and preliminaries. In this paper, by a *locally convex space* (l.c.s.) we mean a Hausdorff locally convex space, not equal to $\{0\}$, over the field $\boldsymbol{R}$ of real scalars. If E is an l.c.s., then E' denotes its topological dual, and $\langle x, x'\rangle$ stands for the value of a functional $x' \in E'$ at $x \in E$. For subsets $A \subset E$ and $B \subset E'$, the symbols A° and B° denote the polars with respect to the duality $\langle E, E'\rangle$ and $\langle E', E\rangle$, respectively. Let E'_σ and E'_τ denote the space E' under the topologies $\sigma(E', E)$ and $\tau(E', E)$ (weak and Mackey), respectively. If $\mathfrak{S}$ is the family of all equicontinuous subsets of E' and F

is a Banach space, then the space $L(E'_\tau, F)$ of all $(\tau(E', E), \|\cdot\|_F)$ continuous linear mappings of E' into F with the topology of uniform convergence on the sets $S \in \mathfrak{S}$ will be denoted by $L_e(E'_\tau, F)$.

Let E be an l.c.s., F a Banach space and $E \otimes F$ the tensor product of E and F. We define

$$\varepsilon_U(z) = \sup_{x' \in U^\circ} \sup_{y' \in B^\circ} |\langle z, x' \otimes y' \rangle| \quad \text{for } z \in E \otimes F,$$

where U runs over the base of convex and circled neighborhoods of 0 in E and B is the unit ball in F. The topology generated by the family of seminorms $\varepsilon_U(\cdot)$ is called the *ε-topology*. The completion of $E \otimes F$ with respect to the ε-topology is denoted by $E \hat{\otimes}_\varepsilon F$. In particular, if E is a Banach space, then $E \hat{\otimes}_\varepsilon F$ is also a Banach space with the norm denoted by $\|\cdot\|_\varepsilon$.

PROPOSITION 2.1 (cf. [9], 9.1, p. 167, and [6], p. 166). *Let E be a complete l.c.s. and F a Banach space. The space $E \hat{\otimes}_\varepsilon F$ can be identified with the closure of $E \otimes F$ in $L_e(E'_\tau, F)$. Moreover, if F has the approximation property, then $E \hat{\otimes}_\varepsilon F$ is identical with the space of all continuous linear mappings in $L(E'_\tau, F)$ transforming equicontinuous subsets of E into relatively compact sets in F.*

COROLLARY 2.1. *Let E and F be Banach spaces and suppose that F has the approximation property. Then $E \hat{\otimes}_\varepsilon F$ is norm isomorphic to the space of all $(\tau(E', E), \|\cdot\|_F)$ continuous and compact linear mappings of E' into F, with operator norm topology.*

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let L^p, $0 \leqslant p \leqslant \infty$, denote the space $L^p(\Omega, \mathscr{A}, P)$. If E is an l.c.s., then the linear mapping $T\colon E' \to L^p$ is called the *cylindrical process*. In particular, if $f\colon \Omega \to E$ is a weakly measurable function such that $\langle f, x' \rangle \in L^p$ for each $x' \in E'$, then by T_f we denote the mapping $T_f x' = \langle f, x' \rangle$.

We say that a cylindrical process $T\colon E' \to L^p$ $(1 \leqslant p \leqslant \infty)$ is *Pettis integrable* if for each $A \in \mathscr{A}$ there exists $x_A \in E$ such that

$$\langle x_A, x' \rangle = \int_A Tx' \, dP \quad \text{for each } x' \in E',$$

and we write

$$\int_A T dP = x_A.$$

If E is a Banach space and $T\colon E' \to L^p$ a Pettis integrable cylindrical process, then we define the p-th Pettis norm of T as follows:

$$|||T|||_p = \sup_{\|x'\| \leqslant 1} \left(\int_\Omega |Tx'|^p \, dP \right)^{1/p}.$$

Now let f be a function on Ω with values in an l.c.s. E. We say that f is *separably valued* if there exists a set $N \in \mathscr{A}$, $P(N) = 0$, and there exists a separable subspace E_1 of E such that $f(\Omega \backslash N) \subset E_1$. By $\mathscr{L}^p(E)$ $(1 \leqslant p \leqslant \infty)$

we shall denote the space of all separably-valued and Pettis integrable functions f such that $\langle f, x'\rangle \in L^p$ for each $x' \in E'$. We endow $\mathscr{L}^p(E)$ with the p-th Pettis norm $|||\cdot|||_p$ induced by the corresponding cylindrical processes.

3. The Pettis integral of cylindrical processes. In this section we prove that if E is a complete l.c.s., then every $T \in E \hat{\otimes}_\varepsilon L^1(\Omega, \mathscr{A}, P)$ is Pettis integrable. Moreover, we give characterization of the space $\mathscr{L}^1(E)^\wedge$ – the completion of $\mathscr{L}^1(E)$ in topology generated by the family of seminorms

$$|||f|||_U = \sup_{x' \in U^\circ} \int_\Omega |\langle f, x'\rangle| \, dP,$$

where U runs over the base of convex and circled neighborhoods of 0 in E.

A. *The case of a Banach space.* Suppose that E is a Banach space. By Corollary 2.1, elements of $E \hat{\otimes}_\varepsilon L^1$ are identified with compact linear operators in $L(E', L^1)$ which are $(\tau(E', E), \|\cdot\|_{L^1})$ continuous.

PROPOSITION 3.1. *If E is a Banach space, then every $T \in E \hat{\otimes}_\varepsilon L^1$ is Pettis integrable and* $|||T|||_1 = \|T\|_\varepsilon$.

Proof. It is sufficient to show that for every $T \in E \hat{\otimes}_\varepsilon L^1$ there exists $x_0 \in E$ such that

$$\langle x_0, x'\rangle = \int_\Omega Tx' dP \quad \text{for each } x' \in E'.$$

Indeed, if $T \in E \hat{\otimes}_\varepsilon L^1$, then for each $A \in \mathscr{A}$ the operator $T\mathbf{1}_A$, $(T\mathbf{1}_A)x' = Tx'\mathbf{1}_A$, belongs to $E \hat{\otimes}_\varepsilon L^1$.

Now we define a continuous linear mapping of $E \hat{\otimes}_\varepsilon L^1$ into E. For $T \in E \otimes L^1$,

$$T = \sum_{i=1}^{n} x_i \otimes f_i,$$

we set

$$\int_\Omega T dP = \sum_{i=1}^{n} x_i \int_\Omega f_i dP.$$

Then we have

$$\int Tx' dP = \left\langle \int T dP, x' \right\rangle \quad \text{for each } x' \in E',$$

and

$$\left\| \int T dP \right\| \leqslant \sup_{\|x'\| \leqslant 1} \int |Tx'| \, dP = \|T\| = \|T\|_\varepsilon.$$

Therefore, $\int(\cdot) dP$ extends to a continuous linear mapping on $E \hat{\otimes}_\varepsilon L^1$ into E. For $T \in E \hat{\otimes}_\varepsilon L^1$ there exists a sequence $\{T_n\} \subset E \otimes L^1$ such that $\|T_n - T\|_\varepsilon \to 0$. Then we set

$$\int_\Omega T dP = \lim_n \int_\Omega T_n dP.$$

It is easy to verify that the integral defined in such a way coincides with the Pettis integral and that

$$|||T|||_1 = \sup_{\|x'\| \leqslant 1} \int |Tx'|\, dP = \|T\|_\varepsilon .$$

COROLLARY 3.1. *Let E and E_1 be Banach spaces, $S\colon E \to E_1$ a continuous linear operator and S^* its adjoint. If $T \in E \hat{\otimes}_\varepsilon L^1$, then $T \circ S^* \in E_1 \hat{\otimes}_\varepsilon L^1$ and $S \int T\,dP = \int T \circ S^*\, dP$.*

Proof. Using Corollary 2.1 it is easy to verify that $T \circ S^* \in E_1 \hat{\otimes}_\varepsilon L^1$. The second part of Corollary 3.1 is obvious.

The following proposition was firstly obtained by Gil de Lamadrid ([5], Theorem 6.1) under the assumption that Ω is a locally compact Hausdorff space. However, our proof is simpler than that in [5].

PROPOSITION 3.2. *The space $(\mathscr{L}^1(E)\hat{}, |||\cdot|||_1)$ is norm isomorphic to $E \hat{\otimes}_\varepsilon L^1$.*

Proof. If $f \in \mathscr{L}^1(E)$, then there exists a sequence of simple functions $\{f_n\}$ such that $|||f_n - f|||_1 \to 0$ (cf. [7], Theorem 4.3). Hence $T_{f_n} \in E \otimes L^1$ and

$$|||T_{f_n} - T_{f_m}|||_1 = \|T_{f_n} - T_{f_m}\|_\varepsilon \to 0 \quad \text{as } n, m \to \infty .$$

Therefore, there exists $T \in E \hat{\otimes}_\varepsilon L^1$ such that $\|T_{f_n} - T\|_\varepsilon \to 0$ and, evidently, $T_f = T$. Hence $\mathscr{L}^1(E) \subset E \hat{\otimes}_\varepsilon L^1$. Since for every $T \in E \otimes L^1$ there exists $f \in \mathscr{L}^1(E)$ such that $T = T_f$, we have $\mathscr{L}^1(E)\hat{} = E \hat{\otimes}_\varepsilon L^1$.

B. *The case of a locally convex space.*

THEOREM 3.1. *If E is a complete l.c.s., then each $T \in E \hat{\otimes}_\varepsilon L^1$ is Pettis integrable.*

Proof. To prove this theorem we need the following notation. Let $\{U_\alpha\}_{\alpha \in A}$ be a base of convex circled neighborhoods of 0 in E. The set A is directed under inclusion, i.e. $\alpha \leqslant \beta$ if $U_\beta \subset U_\alpha$. Denote by E_α the completion of the normed space $(E/p_\alpha^{-1}(0), p_\alpha(\cdot))$, where $p_\alpha(\cdot)$ is the Minkowski functional of U_α. Let $g_\alpha\colon E \to E_\alpha$ and $g_{\alpha\beta}\colon E_\beta \to E_\alpha$ $(\alpha \leqslant \beta)$ be quotient (canonical) maps and let h_α and $h_{\beta\alpha}$ be the adjoints of g_α and $g_{\alpha\beta}$, respectively. Since E is complete, it is isomorphic to the projective limit $\varprojlim g_{\alpha\beta} E_\beta$ of the family of Banach spaces E_β ([9], 5.4, p. 53).

Suppose $T \in E \hat{\otimes}_\varepsilon L^1$; then $T \in L(E'_\tau, L^1)$, and since each h_α is Mackey continuous, $T \circ h_\alpha$ is $(\tau(E'_\alpha, E_\alpha), \|\cdot\|_{L^1})$ continuous for each $\alpha \in A$. By Proposition 2.1, there exists a net $\{T_i\} \subset E \otimes L^1$ which converges in $L_e(E'_\tau, L^1)$ to T. Moreover, since

$$\|T_i \circ h_\alpha - T \circ h_\alpha\| = \sup_{x' \in U_\alpha^\circ} \|(T_i \circ h_\alpha)x' - (T \circ h_\alpha)x'\|_{L^1} = \sup_{x' \in U_\alpha^\circ} \|T_i x' - Tx'\|_{L^1},$$

$T \circ h_\alpha$ is a limit in operator norm topology of finite-dimensional operators, and hence $T \circ h_\alpha \in E_\alpha \hat{\otimes}_\varepsilon L^1$. Therefore, by Proposition 3.1, for every $\alpha \in A$

there exists $x_\alpha \in E_\alpha$ such that

$$x_\alpha = \int_\Omega T \circ h_\alpha \, dP.$$

Using Corollary 3.1 we obtain

$$g_{\alpha\beta}(x_\beta) = \int T \circ h_\beta \circ h_{\beta\alpha} \, dP = \int T \circ h_\alpha \, dP = x_\alpha \quad \text{for each } \alpha \leqslant \beta.$$

Thus $\tilde{x} = (x_\alpha)_{\alpha \in A} \in \underleftarrow{\lim}\, g_{\alpha\beta} E_\beta$, and hence $\tilde{x} \in E$ (here $\underleftarrow{\lim}\, g_{\alpha\beta} E_\beta$ and E are identified). Consequently, we may set

$$\int_\Omega T dP = \tilde{x}.$$

It can be proved, in a similar way as in Proposition 3.1, that T is Pettis integrable.

Now we introduce some auxiliary notation. Let $f\colon \Omega \to E$ be a weakly measurable function such that $\langle f, x' \rangle \in L^1$ for each $x' \in E'$ and let U be a convex circled neighborhood of 0 in E. We set

$$\nu_{f,U}(A) = \sup_{x' \in U^\circ} \int_A |\langle f, x' \rangle| \, dP \quad \text{for each } A \in \mathscr{A}.$$

The following proposition extends the result of Vakhania and Tarieladze ([10], Theorem 4):

PROPOSITION 3.3. *Let E be a complete l.c.s. and let $f\colon \Omega \to E$ be a weakly measurable and separably-valued function. If for each neighborhood U of 0 in E the set function $\nu_{f,U}$ is absolutely continuous with respect to P, then f is Pettis integrable.*

Proof. We recall that a set function ν on A is *absolutely continuous with respect to P* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $P(A) < \delta$ implies $\nu(A) < \varepsilon$.

Using the same notation as in the proof of Theorem 3.1 we observe that $g_\alpha \circ f$ is weakly measurable for each $\alpha \in A$ and separably valued. Moreover, for each $\varepsilon > 0$ there is $\delta > 0$ such that $P(B) < \delta$ $(B \in \mathscr{A})$ implies

$$\sup_{\|x'\|_{E'_\alpha} \leqslant 1} \int_B |\langle g_\alpha \circ f, x' \rangle| \, dP = \sup_{x' \in U^\circ} \int_B |\langle f, x' \rangle| \, dP = \nu_{f,U}(B) < \varepsilon.$$

Thus, by Theorem 5.3 in [7], $g_\alpha \circ f$ is Pettis integrable. Putting

$$x_\alpha = \int_\Omega g_\alpha \circ f dP$$

we show as in Theorem 3.1 that $(x_\alpha)_{\alpha \in A} \in \underleftarrow{\lim}\, g_{\alpha\beta} E_\beta$, which proves Proposition 3.3.

COROLLARY 3.2. *Let E be a complete l.c.s. and let $f\colon \Omega \to E$ be a weakly measurable and separably-valued function. If, for some $p > 1$, $\langle f, x' \rangle \in L^p$ for each $x' \in E'$, then f is Pettis integrable.*

Proof. From the Hölder inequality it follows that $\nu_{f,U}$ is absolutely continuous with respect to P.

THEOREM 3.2. *If E is an l.c. Fréchet space, then $\mathscr{L}^1(E)^\wedge$ is isomorphic to $E\hat{\otimes}_\varepsilon L^1$.*

The proof of this theorem is analogous to that of Proposition 3.2. We need only a step functions approximation of Pettis integrable functions, which can be obtained in a different way. The following lemma is an extension of Lemma 4.1 in [8] on the case of the l.c. Fréchet space.

LEMMA 3.1. *Let E be an l.c. Fréchet space and let $f\in\mathscr{L}^1(E)$. Then there exists a sequence of finite σ-algebras $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \ldots \subset \mathscr{A}$ such that $\mathrm{E}(f|\mathscr{A}_n)\to f$ a.s. in $\mathscr{L}^1(E)$, where $\mathrm{E}(f|\mathscr{A}_n)$ denotes the weak conditional expectation.*

Proof. Notice first that if $\mathscr{B}$ is a finite sub-σ-algebra of $\mathscr{A}$, then it is generated by atoms $B_1, \ldots, B_n$ and

$$\mathrm{E}(f|\mathscr{B}) = \sum_{i=1}^{n} [P(B_i)]^{-1} \int_{B_i} fdP\mathbf{1}_{B_i},$$

where $\int fdP$ means the Pettis integral and we take $[P(A_i)]^{-1} = 0$ if $P(A_i) = 0$.

It can be assumed that E is separable. Let $U_1 \supset U_2 \supset \ldots$ be a base of neighborhoods of 0 in E and let $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots$ be the corresponding Minkowski functionals. From [8], Lemma 4.1, it follows that for each seminormed space $(E, \|\cdot\|_k)$ there exists a sequence of finite σ-algebras

$$\mathscr{A}_1^k \subset \mathscr{A}_2^k \subset \ldots \subset \mathscr{A}$$

such that

$$P\{\|f-\mathrm{E}(f|\mathscr{A}_n^k)\|_k \leqslant 2^{-n}\} \geqslant 1-2^{-n} \quad \text{for each } n\in N.$$

Since $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots$, we may assume that $\mathscr{A}_i^{k-1} \subset \mathscr{A}_i^k$. We set $\mathscr{A}_n = \mathscr{A}_n^n$. Then

$$P\{\|f-\mathrm{E}(f|\mathscr{A}_n)\|_k \leqslant 2^{-n}\} \geqslant P\{\|f-\mathrm{E}(f|\mathscr{A}_n^n)\|_n \leqslant 2^{-n}\} \geqslant 1-2^{-n}$$

for each fixed k and each $n \geqslant k$. Therefore, the sequence $\{\mathrm{E}(f|\mathscr{A}_n)\}$ converges to f a.s. in each seminorm $\|\cdot\|_k$, and hence it converges a.s. in E.

It remains to show that $\mathrm{E}(f|\mathscr{A}_n)$ converges to f in $\mathscr{L}^1(E)$. If V is a neighborhood of 0 in E, then there is U_k such that $U_k \subset V$, and hence $|||\cdot|||_V \leqslant |||\cdot|||_{U_k}$. Using Proposition 3.3 and the fact that $\{f_n, \mathscr{A}_n\}$ is a martingale, it is easy to show that $|||f_n-f|||_{U_k} \to 0$ as $n\to\infty$.

COROLLARY 3.3. *Let E be an l.c. Fréchet space and let $f\colon \Omega\to E$ be a weakly measurable and separably-valued function. If f is Pettis integrable, then there exists a sequence $\{f_n\}$ of simple functions which converges to f a.s. in $\mathscr{L}^1(E)$.*

4. Applications to cylindrical measures. Let $\mathscr{C}(E)$ denote the algebra of cylindrical sets on an l.c.s. E, and let μ be a cylindrical measure on $(E, \mathscr{C}(E))$. For each $n \in N$ and $x'_1, \ldots, x'_n \in E'$ we denote by $\mu_{x'_1, \ldots, x'_n}$ the probability measure on $(R^n, \mathscr{B}_{R^n})$ such that

$$\mu_{x'_1, \ldots, x'_n}(B) = \mu\{x \in E: (\langle x, x'_1\rangle, \ldots, \langle x, x'_n\rangle) \in B\}, \quad \text{where } B \in \mathscr{B}_{R^n}.$$

A cylindrical measure μ is said to be *gaussian* if, for each $n \in N$ and $x'_1, \ldots, x'_n \in E'$, $\mu_{x'_1, \ldots, x'_n}$ is the gaussian probability on R^n.

Let $T: E' \to L^0(\Omega, \mathscr{A}, P)$ be a cylindrical process. We set

$$\mu(C) = P\{(Tx'_1, \ldots, Tx'_n) \in B\}, \tag{4.1}$$

where $C \in \mathscr{C}(E)$, $C = \{x \in E: (\langle x, x'_1\rangle, \ldots, \langle x, x'_n\rangle) \in B\}$, $B \in \mathscr{B}_{R^n}$. Conversely, if μ is a cylindrical measure on E, then there exist a probability space $(\Omega, \mathscr{A}, P)$ and a cylindrical process T satisfying (4.1) (cf. [1], p. 41).

A cylindrical process T is said to be *gaussian* if, for each $n \in N$, and $x'_1, \ldots, x'_n \in E'$, $(Tx'_1, \ldots, Tx'_n)$ is a gaussian random vector. By (4.1) there is a one-to-one correspondence between gaussian cylindrical measures and gaussian cylindrical processes.

Let μ be a cylindrical measure, T_μ its cylindrical process and let $f: E \to R$ be a *tame function*, i.e. $f(x) = g(\langle x, x'_1\rangle, \ldots, \langle x, x'_n\rangle)$ for some Borel measurable function $g: R^n \to R$ and $x'_1, \ldots, x'_n \in E'$. If

$$\int_{R^n} |g(t_1, \ldots, t_n)| \, \mu_{x'_1, \ldots, x'_n}(d\bar{t}) < \infty \quad ((\bar{t} = (t_1, \ldots, t_n)),$$

then we write

$$\int_E f(x)\, \mu(dx) = \int_{R^n} g(t_1, \ldots, t_n)\, \mu_{x'_1, \ldots, x'_n}(d\bar{t}).$$

By (4.1) we obtain

$$\int_E f(x)\, \mu(dx) = \int_\Omega g(T_\mu x'_1, \ldots, T_\mu x'_n)\, dP. \tag{4.2}$$

We say that μ has a *weak p-th order* if

$$\int_E |\langle x, x'\rangle|^p \mu(dx) < \infty.$$

By (4.2), there exists a one-to-one correspondence between cylindrical measures having the weak p-th order and cylindrical processes on E' into L^p.

Now let $\mathfrak{S}$ be a family of subsets of E and μ a cylindrical measure on E. We say that μ is *scalarly concentrated* on $\mathfrak{S}$ if for each $\varepsilon > 0$ there is $A \in \mathfrak{S}$ such that

$$(\mu_{x'})_*(x'(A)) \geq 1 - \varepsilon \quad \text{for each } x' \in E'$$

($*$ denotes the inner measure).

THEOREM 4.1. *Let E be a complete l.c.s. and let $\mathfrak{S}$ be the family of all compact circled subsets of E. Let μ be a gaussian cylindrical measure on E and $T = T_\mu$ its gaussian cylindrical process. Then the following assertions are equivalent:*

(a) *μ is scalarly concentrated on $\mathfrak{S}$.*

(b) *$T \in E \hat{\otimes}_\varepsilon L^2$.*

Proof. (a)$\Rightarrow$(b). If μ is scalarly concentrated on $\mathfrak{S}$, then $T \in L(E'_{\mathfrak{S}}, L^0)$ (cf. [1], p. 21), where $E'_{\mathfrak{S}}$ denotes E' with the topology of uniform convergence on sets $S \in \mathfrak{S}$. Since $T(E')$ is the space of gaussian random variables, then the topologies induced on $T(E')$ by L^0 and by L^2 coincide. Therefore, $T \in L(E'_{\mathfrak{S}}, L^2)$, and hence $T \in L(E'_\tau, L^2)$.

It remains to show that if A is an equicontinuous subset of E', then $T(A)$ is relatively compact in L^2 (Proposition 2.1). Let A be an equicontinuous subset of E' and let $\bar{A}$ denote the closure of A in $\sigma(E', E)$. By [9], 4.3, p. 84, $\bar{A}$ is $\sigma(E', E)$ compact. Now, let $\{y_\alpha\}$ be a net in $T(A)$, $y_\alpha = Tx'_\alpha$, where $x'_\alpha \in A$. Thus there exists a subnet $\{x'_\beta\}$ of $\{x'_\alpha\}$, which converges in E'_σ to some $x'_0 \in \bar{A}$. Therefore, x'_β converges to x'_0 uniformly on each $S \in \mathfrak{S}$ (cf. [9], 4.5, p. 85), and hence $y_\beta = T(x'_\beta)$ converges in L^2.

(b)$\Rightarrow$(a). If $T \in E \hat{\otimes}_\varepsilon L^2$, then $T \in L(E'_\tau, L^2)$ and, for each equicontinuous subset A of E', $T(A)$ is relatively compact in L^2. Let B be the unit ball in L^2. Similarly as in the case where E is a Banach space it can be shown that $T^*(B^\circ)$ is relatively compact in E (cf. [9], 9.4, p. 111). Setting $K = \overline{T^*(B^\circ)}$ we obtain $T(K^\circ) \subset B$. Indeed, let $y \in T(K^\circ)$, $y = Tx'$, where $x' \in K^\circ$. Then

$$\|y\|_{L^2} = \sup_{\|y'\| \leqslant 1} |(Tx', y')| = \sup_{\|y'\| \leqslant 1} |\langle x', T^* y'\rangle| \leqslant 1$$

because $T^* y' \in K$ and $x' \in K^\circ$. Therefore $T \in L(E'_{\mathfrak{S}}, L^2)$, and hence $T \in L(E'_{\mathfrak{S}}, L^0)$. By [1], p. 21, μ is scalarly concentrated on $\mathfrak{S}$.

Remark. The assumption that μ is gaussian cannot be omitted. We remark that even if μ is a Radon measure on a Banach space E having the weak second order, then in general $T \notin E \hat{\otimes}_\varepsilon L^2$. Note also that the implication (b)$\Rightarrow$(a) is true for any cylindrical measure having the weak second order.

Let E be an l.c.s. and μ a cylindrical measure on E. If μ has the weak second order, then there exist $m_\mu \in E'^*$ and a linear mapping $R_\mu \colon E' \to E'^*$ such that

$$\langle m_\mu, x'\rangle = \int_E \langle x, x'\rangle \mu(dx),$$

$$\langle R_\mu x', y'\rangle = \int_E \langle x, x'\rangle \langle x, y'\rangle \mu(dx) - \langle m_\mu, x'\rangle \langle m_\mu, y'\rangle$$

(E'^* denotes the algebraic dual of E'). We say that m_μ is the *mean* (or *barycenter*) of μ, and R_μ is called the *covariance operator* of μ.

On the subspace $R_\mu(E')$ of E'^* we define the inner product. For $h_1, h_2 \in R_\mu(E')$, $h_1 = R_\mu x_1'$, $h_2 = R_\mu x_2'$, we set

$$(h_1, h_2)_\mu = \langle R_\mu x_1', x_2' \rangle.$$

Let H_μ denote the completion of $R_\mu(E')$ in the norm $\|\cdot\|_\mu = (\cdot,\cdot)_\mu^{1/2}$. If μ is a gaussian cylindrical measure, then H_μ is called the *reproducing kernel Hilbert space* of μ (many results concerning H_μ of cylindrical measures can be found in [3]).

The forthcoming theorem extends results of Dudley et al. ([4], Theorem 4) and Borell ([2], Theorem 2.1 and Corollary 2.3) on the case of cylindrical gaussian measures on a complete l.c.s. In the above-mentioned papers it was assumed that μ is a gaussian Radon measure.

THEOREM 4.2. *Let E be a complete l.c.s. and $\mathfrak{S}$ the family of all compact circled subsets of E. Suppose that μ is a cylindrical gaussian measure on E, scalarly concentrated on $\mathfrak{S}$. Let m_μ denote the barycenter of μ and R_μ its covariance operator. Then:*

(a) $m_\mu \in E$;

(b) R_μ: $E' \to E$ *and* $R_\mu(U^\circ)$ *is a compact subset of* E *for each neighborhood* U *of* 0 *in* E;

(c) *the canonical injection* θ: $H_\mu \to E$ *is continuous and* $\theta(\gamma_H) = \mu_0$, *where* γ_H *denotes the canonical gaussian measure on* H_μ *and* $\mu_0(\cdot) = \mu(\cdot + m_\mu)$;

(d) $\{h \in H_\mu\colon \|h\|_\mu \leqslant 1\}$ *is a compact subset of* E.

Proof. Let $T = T_\mu$ be the cylindrical gaussian process of μ.

(a) By Theorem 4.1, $T \in E \hat{\otimes}_\varepsilon L^2$, so $T \in E \hat{\otimes}_\varepsilon L^1$. Therefore, by Theorem 3.1 and (4.2),

$$m_\mu = \int_\Omega T dP \in E.$$

(b) Note that if $T \in E \hat{\otimes}_\varepsilon L^2$ and $f \in L^2$, then the linear mapping $f \cdot T$: $E' \to L^1$, $(f \cdot T)x' = f \cdot Tx'$, belongs to $E \hat{\otimes}_\varepsilon L^1$. Indeed, let $\varepsilon_U(\cdot)$ be a seminorm on $E \hat{\otimes}_\varepsilon L^1$, i.e.

$$\varepsilon_U(S) = \sup_{x' \in U^\circ} \|Sx'\|_{L^1} \quad \text{for each } S \in E \hat{\otimes}_\varepsilon L^1,$$

where U is a circled convex neighborhood of 0 in E. Since $T \in E \hat{\otimes}_\varepsilon L^2$, there exists a net $\{T_\alpha\} \subset E \otimes L^2$ such that

$$\sup_{x' \in U^\circ} \|T_\alpha x' - Tx'\|_{L^2} \to 0.$$

Therefore $f \cdot T_\alpha \in E \otimes L^1$ and

$$\varepsilon_U(f \cdot T_\alpha - f \cdot T) \leqslant \sup_{x' \in U^\circ} \|f\|_{L^2} \|T_\alpha x' - Tx'\|_{L^2} \to 0.$$

Now let $x'_0 \in E'$ be fixed. We consider the operator $S_{x'_0}\colon E' \to L^1$, $S_{x'_0} = Tx'_0 \cdot T$. Since $S_{x'_0} \in E \hat{\otimes}_\varepsilon L^1$, there exists $x_0 \in E$ such that

$$\langle x_0, y' \rangle = \int_\Omega S_{x'} y' \, dP = \int_\Omega Tx' \cdot Ty' \, dP \quad \text{for each } y' \in E'.$$

Therefore $R_\mu x'_0 = x_0 - \langle m_\mu, x' \rangle m_\mu \in E$.

The second part of (b) follows from the fact that $R_\mu = T^* \circ T$ and from Proposition 2.1.

(c) Let U be a convex circled neighborhood of 0 in E, $p_U(\cdot)$ its gauge and let $\hat{E}_U$ be the completion of the normed space $(E/p_U^{-1}(0), p_U(\cdot))$. Let $\Phi_U\colon E \to \hat{E}_U$ denote the quotient map. Since E is complete, it suffices to show, by [9], 5.4, p. 53, that $\Phi_U \circ \theta$ is continuous for each U.

Suppose $x \in R_\mu(E')$, $x = R_\mu x'$. We have

$$p_U\big((\Phi_U \circ \theta) x'\big) = p_U\big(\Phi_U(\theta(R_\mu x'))\big) = \sup_{y' \in U^\circ} |\langle R_\mu x', y' \rangle|$$
$$= \sup_{y' \in U^\circ} |\langle Tx', Ty' \rangle| \leqslant \|Tx'\|_{L^2} \varepsilon_U^{(2)}(T),$$

where $\varepsilon_U^{(2)}(\cdot)$ denotes the seminorm on $E \hat{\otimes}_\varepsilon L^2$. Therefore, the mapping θ is continuous and a simple calculation proves the second part of (c).

(d) Denote by $\mathscr{H}$ the closure of $T_0(E')$ in L^2, where $T_0 = T - m_\mu$. Observe that $H_\mu = T_0^*(\mathscr{H})$. Indeed, the map T_0^* is the norm isomorphism of $T_0(E')$ onto $R_\mu(E')$, and thus extends to the norm isomorphism of $\mathscr{H}$ onto H_μ.

So, if B is the unit ball in $\mathscr{H}$, then $B_\mu = T_0^*(B)$ is also a closed unit ball in H_μ. As was pointed out in the proof of Theorem 4.1, $B_\mu = T_0^*(B)$ is relatively compact in E. Thus it suffices to show that B_μ is closed in E.

Let $\{x_\alpha\}$ be a net in B which converges in E to some $x \in E$. Since B is $\sigma(H_\mu, H'_\mu)$ compact, there exists a subnet $\{x_\beta\}$ such that x_β converges to some $z \in B_\mu$ in $\sigma(H_\mu, H'_\mu)$. By (c) the canonical injection $\theta\colon H_\mu \to E$ is continuous, so it is weakly continuous. Therefore x_β converges to z in $\sigma(E, E')$, and hence $x = z$. This completes the proof of Theorem 4.2.

Now we give some modification of Theorem 4.2 for non-gaussian cylindrical measures.

THEOREM 4.3. *Let E be a complete l.c.s. and μ a cylindrical measure on E such that $T_\mu \in E \hat{\otimes}_\varepsilon L^2$. Then:*

(a) $m_\mu \in E$;

(b) $R_\mu\colon E' \to E$ *and* $R_\mu(U^\circ)$ *is relatively compact in* E *for each convex neighborhood* U *of* 0 *in* E;

(c) *the canonical injection* $\theta\colon H_\mu \to E$ *is continuous*;

(d) $\{h \in H_\mu\colon \|h\|_\mu \leqslant 1\}$ *is compact in* E.

The proof of this theorem is similar to that of Theorem 4.2.

Remark. Note that the conditions of Theorem 4.3 are satisfied if μ is a Radon measure on E having the strong second order. This extends the results of Vakhania and Tarieladze (cf. [10], Theorem 7 and Proposition 6).

Added in proof. The proof of Theorem 3.1 follows easily from Corollary 1 of the author's paper *Remarks on Pettis integrability of cylindrical processes*, Lecture Notes in Math. 828 (1980), p. 269-273.

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