PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 2, Fasc. 2 (1982), p. 193-195

ON GL(n, R)-STABLE MEASURES

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Abstract. We give a new proof of Parthasarathy's theorem on GL(n, R)-stable measures.

Let $\mathscr{P}$ be the set of all probability measures defined on Borel subsets of the n-dimensional Euclidean space R^n. We denote by $*$ the convolution operation in $\mathscr{P}$. Given a linear operator A and $\mu \in \mathscr{P}$, we denote by $A\mu$ the measure defined by the formula $A\mu(S) = \mu(A^{-1}(S))$ for all Borel subsets S of R^n.

Let $\mathscr{G} = \mathrm{GL}(n, R)$ be the group of all nonsingular linear operators on R^n. A probability measure μ is said to be *$\mathscr{G}$-stable* if for any $A, B \in \mathscr{G}$ there exist $C \in \mathscr{G}$ and $x \in R^n$ such that

$$(1) \qquad A\mu * B\mu = C\mu * \delta_x,$$

where δ_x is the probability measure concentrated at the point x.

A measure $\mu \in \mathscr{P}$ is said to be *full* if its support is not contained in any $(n-1)$-dimensional hyperplane.

The purpose of this note is to give a simple proof of the following theorem of Parthasarathy ([1] and [2], Theorem 5.6):

THEOREM. *The probability measures on R^n $(n > 1)$ which are $\mathscr{G}$-stable are precisely the full Gaussian and degenerate probability measures.*

Proof. Let μ be a nondegenerate $\mathscr{G}$-stable measure on R^n $(n > 1)$. Passing if necessary to symmetrization of μ, by Cramer's theorem and by elementary properties of full measures it will be sufficient to prove that μ is full Gaussian under the additional assumption that μ is a symmetric distribution. In this case, from $\mathscr{G}$-stability of μ it follows that for each integer N and $A_1, \ldots, A_N \in \mathscr{G}$ there exists a $C \in \mathscr{G}$ such that

$$(2) \qquad A_1 \mu * \ldots * A_N \mu = C\mu.$$

Let M be a subspace of the smallest dimension in R^n for which $\mu(M) = 1$. If $A \in \mathcal{G}$, then from (2) we get $A(M)+M = C(M)$ for some $C \in \mathcal{G}$. Thus $M = R^n$ and, consequently, μ is full.

Choose one-dimensional projectors $P_1, \ldots, P_n$ in R^n such that $P_i P_j = 0$ for $i \neq j$ $(i, j = 1, \ldots, n)$. For every i $(1 \leqslant i \leqslant n)$ we can find a sequence $\{A_{i,j}\}$ of operators from $\mathcal{G}$ such that

$$P_i = \lim_j A_{i,j}.$$

Consequently, as μ is full, applying standard arguments (see [4], p. 120-121) we obtain, by (2), the equality

$$P_1 \mu * \ldots * P_n \mu = T\mu \quad \text{for some } T \in \mathcal{G}.$$

Hence we get

$$Q_1 \mu * \ldots * Q_n \mu = \mu, \tag{3}$$

where $Q_i = T^{-1} P_i T$ $(i = 1, \ldots, n)$. By (2), μ is infinitely divisible and admits the Tortrat representation $\mu = \varrho * \tilde{e}(F)$, where ϱ is a symmetric Gaussian measure and $\tilde{e}(F)$ is a generalized Poisson measure with $F(\{0\}) = 0$ (see [3]). The uniqueness of the Tortrat representation and $\mathcal{G}$-stability of μ imply $\mathcal{G}$-stability of $\tilde{e}(F)$. Consequently, by (3) we have $Q_1 F(S) + \ldots + Q_n F(S) = F(S)$ for all Borel subsets S of $R^n \setminus \{0\}$, which shows that

$$F\big(R^n \setminus \bigcup_{i=1}^{n} \operatorname{Im} Q_i\big) = 0, \tag{4}$$

where $\operatorname{Im} Q_i$ denotes the image of Q_i $(i = 1, \ldots, n)$. Choose an operator A_0 from $\mathcal{G}$ such that

$$\Big(\bigcup_{i=1}^{n} A_0(\operatorname{Im} Q_i)\Big) \cap \Big(\bigcup_{i=1}^{n} \operatorname{Im} Q_i\Big) = \{0\}.$$

Equation (2) implies that

$$A_0 F + F = B_0 F \tag{5}$$

for some $B_0 \in \mathcal{G}$. Put $I = \{i \colon F(\operatorname{Im} Q_i) > 0\}$. From (5) we infer that, for each $i \in I$, $B_0 F\big(A_0(\operatorname{Im} Q_i)\big) > 0$ and $B_0(\operatorname{Im} Q_i) > 0$. But, by (4), the support of $B_0 F$ is equal to the set $\bigcup_{i \in I} B_0(\operatorname{Im} Q_i)$. Hence and from the assumption on A_0 it follows that $I = \emptyset$ and, consequently, F is zero. Thus μ must be Gaussian and the Theorem is proved, since the converse is obvious.

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Received on 23. 5. 1980