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SOME RESULTS ON THE DOMAIN OF ATTRACTION OF STABLE MEASURES ON $C(K)$

BY

EVARIST GINÉ AND MICHAEL B. MARCUS
(COLLEGE STATION, TEXAS)

Abstract. Stable random variables with values in $C(K)$ which can be represented by stochastic integrals with respect to independently scattered random stable measures are studied. A related class of stochastic integrals which are not stable but which also take values in $C(K)$ and which are in the domain of normal attraction of the stable random variables is introduced. Particular attention is paid to the Fourier transform of random measures on $\boldsymbol{R}^N$. These results extend recent work of Araujo and Marcus, and Giné.

1. Introduction. By $C(K)$ we mean, as usual, the Banach space of continuous complex-valued functions on a compact metric space with the sup norm. In the case where $K = [-1/2, 1/2]^N$ a sufficient condition for continuity and a central limit theorem were obtained in [8] for random Fourier series in which the random variables have finite second moments. This was generalized by Fernique [3] to a class of second order stochastic integrals. In [10] the continuity conditions were extended to random Fourier series in which the random variables did not have finite second moments. Moreover, particular attention was paid to the case in which the random variables were real symmetric stable of index p $(1 < p < 2)$. These latter series induce stable measures on $C([-1/2, 1/2]^N)$, and hence were referred to as stable processes with continuous sample paths. In [1], following Fernique's [3] extension of [8], a continuity result was given for stable processes represented by stochastic integrals which included the random Fourier series as a special case. In this paper* we generalize this class of

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stochastic integrals to include continuous processes in the domain of attraction of stable processes. Our main result is given in Theorem 4.1. The statement of the theorem appears cumbersome but we give some interesting examples in Sections 3 and 4 to which these conditions readily apply. We also consider a more general class of stochastic integrals that give rise to continuous processes on $C(K)$ and obtain a central limit theorem for them. This generalizes results of [1] and [6] and is presented in Theorem 4.2.

In Section 2 we include some known results on sums of random variables which have distribution functions dominated by those of stable random variables and a modification of a result of Pisier [14]. In Section 3 we give examples of measures and, consequently, of stochastic integrals which satisfy the conditions of our main results. These are given in Section 4.

2. Preliminaries. We will need two lemmas. The first is generally known but for lack of a suitable reference we will prove it here. We denote by $\{\varepsilon_k\}$ a *Rademacher sequence*, that is a sequence of independent identically distributed symmetric random variables taking on the values ± 1.

LEMMA 2.1. *Let $\{a_k\}$ be a sequence of complex numbers, let $\{\varepsilon_k\}$ be a Rademacher sequence, and let $\{\eta_k\}$ be a sequence of complex-valued random variables not necessarily independent but independent of $\{\varepsilon_k\}$. Assume that for some $p \in (0, 2)$ and $\alpha > 0$*

$$\sup_k t^p P\{|\eta_k| > t\} \leqslant \alpha, \quad t \geqslant 0. \tag{2.1}$$

Then

$$P\{|\sum_k a_k \varepsilon_k \eta_k| > t\} \leqslant \alpha\left(\frac{4-p}{2-p}\right)(\sum_k |a_k|^p) t^{-p}, \quad t \geqslant 0. \tag{2.2}$$

Furthermore, there exist finite constants C and C' depending on α, p, and q such that for $0 < q < p$

$$(\mathrm{E}|\sum_k a_k \varepsilon_k \eta_k|^q)^{1/q} \leqslant C(\sum_k |a_k|^p)^{1/p} \tag{2.3}$$

and

$$(\mathrm{E}[\sum_k |a_k|^2 |\eta_k|^2]^{q/2})^{1/q} \leqslant C'(\sum_k |a_k|^p)^{1/p}. \tag{2.4}$$

Proof. Let $\{\eta_k\}$ be defined on the probability space $(\Omega_1, \mathscr{F}_1, P_1)$, let $\{\varepsilon_k\}$ be defined on the probability space $(\Omega_2, \mathscr{F}_2, P_2)$, and let E_1 and E_2 be the corresponding expectation operators. The series $\sum_k a_k \varepsilon_k \eta_k$ is defined on the product probability space, which we denote by $(\Omega, \mathscr{F}, P)$, with expectation operator E. Let $I_{[A]}$ denote the indicator function of the set A.

We have

$$P\{|\sum_k a_k \varepsilon_k \eta_k| > t\} = \mathrm{E}_1 P_2\{|\sum_k a_k \varepsilon_k \eta_k(\omega_1)| > t\}, \tag{2.5}$$

where $\omega_1 \in \Omega_1$. Clearly,

$$\begin{aligned} P_2\{|\sum_k a_k \varepsilon_k \eta_k(\omega_1)| > t\} &= P_2\{|\sum_k a_k \varepsilon_k \eta_k(\omega_1)| > t, \sup_k |a_k \eta_k(\omega_1)| > t\} + \\ &\quad + P_2\{|\sum_k a_k \varepsilon_k \eta_k(\omega_1)| > t, \sup_k |a_k \eta_k(\omega_1)| \leqslant t\} \\ &\leqslant \sum_k I_{[|a_k\eta_k(\omega_1)| > t]} + \\ &\quad + t^{-2} \sum_k |a_k|^2 |\eta_k(\omega_1)|^2 I_{[|a_k\eta_k(\omega_1)| \leqslant t]}. \end{aligned} \tag{2.6}$$

Applying E_1 to both sides of (2.6) and using (2.5) we get

$$P\{|\sum_k a_k \varepsilon_k \eta_k| > t\} \leqslant \sum_k P\{|a_k \eta_k| > t\} + t^{-2} \sum_k |a_k|^2 \mathrm{E}|\eta_k|^2 I_{[|a_k\eta_k| \leqslant t]}. \tag{2.7}$$

By (2.1) we have

$$\sum_k P\{|a_k \eta_k| > t\} \leqslant \alpha\left(\sum_k |a_k|^p\right) t^{-p}$$

and

$$\begin{aligned} \mathrm{E}|\eta_k|^2 I_{[|a_k\eta_k| \leqslant t]} &= \int_0^{t/|a_k|} u^2\, dP(|\eta_k| \leqslant u) \leqslant 2 \int_0^{t/|a_k|} uP\{|\eta_k| > u\}\, du \\ &\leqslant 2\alpha \int_0^{t/|a_k|} u^{1-p}\, du = 2\alpha(2-p)^{-1}(t/|a_k|)^{2-p}. \end{aligned}$$

Substituting these last two inequalities into (2.7) we get (2.2).

We now obtain (2.3). Without loss of generality we may assume that $\sum_k |a_k|^p = 1$; let

$$\gamma = \alpha\left(\frac{4-p}{2-p}\right).$$

Then

$$\begin{aligned} \mathrm{E}|\sum_k a_k \varepsilon_k \eta_k|^q &= q \int_0^\infty u^{q-1} P\{|\sum_k a_k \varepsilon_k \eta_k| > u\}\, du \\ &\leqslant q \int_0^1 u^{q-1}\, du + \gamma q \int_1^\infty u^{q-1-p}\, du = 1 + \gamma q(p-q)^{-1}, \end{aligned}$$

which implies (2.3).

For (2.4) we use Khintchine's inequality (see, e.g., [7], p. 66). This gives us

$$\mathrm{E}_2 |\sum_k a_k \varepsilon_k \eta_k(\omega_1)|^q \geqslant C''\left(\sum_k |a_k|^2 |\eta_k(\omega_1)|^2\right)^{q/2},$$

where C'' is a constant depending on q. Therefore

$$\Big(\mathrm{E}\big(\sum_k |a_k|^2\,|\eta_k|^2\big)^{q/2}\Big)^{1/q} \leqslant (C'')^{-1/q}\big(\mathrm{E}\,\big|\sum_k a_k \varepsilon_k \eta_k\big|^q\big)^{1/q}$$

and (2.4) follows from (2.3).

Next we give a simple but useful criterion for weak convergence which elaborates on a result of Pisier ([14], Theorem 1.3). Let S be a separable metric space. It is well known that the metric

$$d(X, Y) = \inf\{\varepsilon > 0\colon\ P\{\|X-Y\| > \varepsilon\} \leqslant \varepsilon\} \tag{2.8}$$

(X and Y are S-valued random variables) metrizes the topology of convergence in probability. Note also that $d(X, Y) \leqslant (\mathrm{E}\,\|X-Y\|)^{1/2}$ as can readily be shown using Chebyshev's inequality. As usual, the law of X is denoted by $\mathscr{L}(X)$.

LEMMA 2.2. *Let B be a separable Banach space and $\{X_n\}$ a sequence of B-valued random variables such that for each $\varepsilon > 0$ there exists a sequence $\{Y_{n,\varepsilon}\}$ of B-valued random variables satisfying*

(2.9) *$\{\mathscr{L}(Y_{n,\varepsilon})\}$ converges weakly,*

(2.10) *there exists $n(\varepsilon) < \infty$ such that, for $n > n(\varepsilon)$, $d(X_n, Y_{n,\varepsilon}) < \varepsilon$.*

Then $\{\mathscr{L}(X_n)\}$ converges weakly and the limit is

$$\operatorname*{w-lim}_{n\to\infty} \mathscr{L}(X_n) = \operatorname*{w-lim}_{\varepsilon\to 0}\,[\operatorname*{w-lim}_{n\to\infty} \mathscr{L}(Y_{n,\varepsilon})]. \tag{2.11}$$

Proof. Let $\mathscr{P}(B)$ denote the set of Borel measures on B. The Prokhorov metric on $\mathscr{P}(B)$ is given by

$$\varrho(\mu, \nu) = \inf\{\varepsilon > 0\colon\ \mu(F) \leqslant \nu(F_\varepsilon)+\varepsilon,\ F \subset B,\ F \text{ closed}\},$$

where $\mu, \nu \in \mathscr{P}(B)$ and $F_\varepsilon = \{x\colon\ \|x-y\| < \varepsilon \text{ for some } y \in F\}$. The weak topology of $\mathscr{P}(B)$ is metrizable by ϱ and $(\mathscr{P}(B), \varrho)$ is a complete metric space ([15], Theorem 1.11). Note also that for $X, Y \in B$

$$\varrho\big(\mathscr{L}(X), \mathscr{L}(Y)\big) \leqslant d(X, Y). \tag{2.12}$$

By (2.9), $\{\mathscr{L}(Y_{n,\varepsilon})\}$ converges weakly for each $\varepsilon > 0$. Let Y_ε denote the B-valued random variable associated with $\operatorname*{w-lim}_{n\to\infty} \mathscr{L}(Y_{n,\varepsilon})$. Then, by (2.9), (2.10), (2.12), and the triangle inequality, it is easy to show that

$$\varrho\big(\mathscr{L}(Y_\varepsilon), \mathscr{L}(Y_{\varepsilon'})\big) \leqslant \varepsilon+\varepsilon' \quad \text{and} \quad \varrho\big(\mathscr{L}(X_n), \mathscr{L}(Y_\varepsilon)\big) \leqslant \varepsilon \text{ for } n > n(\varepsilon).$$

This, together with the completeness of $(\mathscr{P}(B), \varrho)$, shows that $\{\mathscr{L}(X_n)\}$ and $\{\mathscr{L}(Y_\varepsilon)\}$ both converge and that their limits coincide.

3. A class of random measures and random integrals. Let $(\Omega, \mathscr{F}, P)$ be a probability space and (S, Σ) a measurable space. Let $M = M(\cdot, \omega)$, $\omega \in \Omega$, be a random measure on (S, Σ). That is, if $A_1, \ldots, A_n \in S$ are disjoint measurable sets, then

$$M\Big(\bigcup_{i=1}^{n} A_i, \omega\Big) = \sum_{i=1}^{n} M(A_i, \omega) \text{ a.s.},$$

and if furthermore $\bigcup_{i=1}^{n} A_i \uparrow A$, then $\sum_{i=1}^{n} M(A_i, \omega)$ converges to $M(A, \omega)$ in probability. We will require M to have the following properties:

(3.1) *Let $A_k \in \Sigma$ $(k = 1, 2, \ldots)$ be disjoint and let $\{\varepsilon_k\}$ be a Rademacher sequence independent of M. Then $\{M(A_k)\}$ and $\{\varepsilon_k M(A_k)\}$ have the same probability law.*

(3.2) *There exists a real positive finite measure m on (S, Σ) such that, for each finite collection of disjoint sets $A_1, \ldots, A_n \in \Sigma$, the random vector $(M(A_1), \ldots, M(A_n))$ is in the domain of normal attraction of $\big((m(A_1))^{1/p}\theta_1, \ldots, (m(A_n))^{1/p}\theta_n\big)$, where the random variables $\theta_1, \ldots, \theta_n$ are independent identically distributed and* $\mathrm{E}\exp[it\theta_1] = \exp[-|t|^p]$, $-\infty < t < \infty$.

(3.3) $$\overline{\lim_{t \to \infty}} \sup_{A \in \Sigma} t^p P\left\{\left|\frac{M(A)}{(m(A))^{1/p}}\right| > t\right\} \leqslant c \text{ for some constant } c.$$

The first of these properties is that M has sign-invariant increments. The next two imply, in an appropriate sense, that M is in the domain of normal attraction of some independently scattered random stable measure with control measure m. (This is defined in Example 1 which follows.) Note that "$(M(A_1), \ldots, M(A_n))$ is in the domain of normal attraction of $\big((m(A_1))^{1/p}\theta_1, \ldots, (m(A_n))^{1/p}\theta_n\big)$" means that

$$\mathscr{L}\Big[k^{-1/p}\sum_{i=1}^{k}(M_i(A_1), \ldots, M_i(A_n))\Big] \xrightarrow{\mathrm{w}} \mathscr{L}\Big[\big((m(A_1))^{1/p}\theta_1, \ldots, (m(A_n))^{1/p}\theta_n\big)\Big],$$

where M_i are independent copies of M. We remark that if M has independent increments, then the condition

(3.2)′ *For each $A \in \Sigma$, $M(A)$ is in the domain of normal attraction of $(m(A))^{1/p}\theta$, where* $\mathrm{E}\exp[it\theta] = \exp[-|t|^p]$, $-\infty < t < \infty$.

implies condition (3.2), but we do not know if the implication is true in the more general situation of sign-invariant increments (it is true in the case $p = 2$).

We give some examples of random measures M satisfying (3.1)-(3.3).

Example 1. *Independently scattered random stable measures.* Let m be a positive finite measure on (S, Σ). For each measurable set $A \in \Sigma$ we define $M(A)$ by

$$\mathrm{E}\exp[itM(A)] = \exp[-m(A)|t|^p], \quad -\infty < t < \infty,\ p \in (0, 2).$$

For $A_1 \cap A_2 = \emptyset$ we take $M(A_1)$ and $M(A_2)$ to be independent. This random set function can be extended to all of (S, Σ) by the Kolmogorov extension theorem. The result is a random measure on (S, Σ) with $\mathcal{L}(M(A)) = \mathcal{L}\big((m(A))^{1/p}\theta\big)$. This shows that (3.2)′ (hence (3.2)) and (3.3) are satisfied; (3.1) is clearly satisfied since $M(A_1), M(A_2), \ldots$ are independent and symmetric for disjoint $A_1, A_2, \ldots \in \Sigma$. The measure m is called the *control measure* of M.

Example 2. *Generalized empirical measures.* Let $\{X_k\}$ be a sequence of random variables with values in (S, Σ). Let $\{\varepsilon_k\}$ be a Rademacher sequence and let $\{\xi_k\}$ be a sequence of real-valued independent random variables such that

(3.4) each ξ_k is in the domain of attraction of θ and

$$\sup_k t^p P\{|\xi_k| > t\} \leqslant c \quad \text{for some constant } c \text{ and } t \geqslant 0.$$

Assume that the sequences $\{\varepsilon_k\}, \{\xi_k\}$, and $\{X_k\}$ are independent of each other. Let E_ε, E_ξ, and E_X denote expectations with respect to these sequences in the product space on which they are defined, as in the proof of Lemma 2.1.

For $\{a_k\} \in l^p$ and $A \in \Sigma$ we define

$$M(A) = \sum_k a_k \varepsilon_k \xi_k I_{[X_k \in A]} \tag{3.5}$$

and refer to it as a *generalized empirical measure*. We will show that it is a random measure and satisfies (3.1)-(3.3). By (2.3) we have

$$\mathrm{E}_\xi \mathrm{E}_\varepsilon |M(A)|^q \leqslant C^q \big(\sum_k |a_k|^p I_{[X_k \in A]}\big)^{q/p}.$$

Therefore, applying E_X we get

$$\mathrm{E}|M(A)|^q \leqslant C^q \big(\sum_k |a_k|^p P\{X_k \in A\}\big)^{q/p}. \tag{3.6}$$

For M as given in (3.5) we define the real positive finite measure m on (S, Σ) by

$$m(A) = \sum_k |a_k|^p P\{X_k \in A\}. \tag{3.7}$$

Thus

$$(\mathrm{E}|M(A)|^q)^{1/q} \leqslant C(m(A))^{1/p}. \tag{3.8}$$

This inequality gives us the desired convergence properties and shows that M is a random measure on (S, Σ). It is easy to see that M satisfies (3.1) since, for $A_1 \cap A_2 = \emptyset$, $I_{[X_k \in A_1]} = 1$ implies $I_{[X_k \in A_2]} = 0$.

Let $P_{\varepsilon\xi}$ denote conditional probability with respect to $\{X_k\}$. By (2.2) we obtain

$$(3.9)\qquad P_{\varepsilon\xi}\{|\sum_k a_k\varepsilon_k\xi_k I_{[X_k\in A]}| > t\} \leqslant c\left(\frac{4-p}{2-p}\right)\left(\sum_k |a_k|^p I_{[X_k\in A]}\right)t^{-p}.$$

Applying E_X to both sides, we get

$$P\{|M(A)| > t\} \leqslant c\left(\frac{4-p}{2-p}\right)(m(A))t^{-p},$$

and since this is true for all $t > 0$, we have

$$P\left\{\left|\frac{M(A)}{(m(A))^{1/p}}\right| > t\right\} \leqslant c\left(\frac{4-p}{2-p}\right)t^{-p},$$

which gives us (3.3).

Finally, note that, for $\{X_k\}$ fixed, $A_1,\ldots,A_n\in\Sigma$ disjoint, and $\alpha_1,\ldots,\alpha_n$ arbitrary, $\sum_{i=1}^{n}\alpha_i\sum_{k=1}^{\infty}a_k\varepsilon_k\xi_k I_{[X_k\in A_i]}$ is in the domain of normal attraction of

$$\left(\sum_k |a_k|^p\left|\sum_i \alpha_i I_{[X_k\in A_i]}\right|^p\right)^{1/p}\theta = \left(\sum_i\sum_k |\alpha_i|^p|a_k|^p I_{[X_k\in A_i]}\right)^{1/p}\theta.$$

This follows from several references (see, e.g., [13], Lemma 5.3 along with the bottom of p. 89, or [6], Proposition 2.4 together with inequality (2.2) here). Therefore

$$(3.10)\qquad \lim_{t\to\infty} t^p P_{\varepsilon\xi}\{|\sum_{i=1}^{n}\alpha_i\sum_{k=1}^{\infty}a_k\varepsilon_k\xi_k I_{[X_k\in A_i]}| > t\} = C\sum_{i=1}^{n}\sum_{k=1}^{\infty}|\alpha_i|^p|a_k|^p I_{[X_k\in A_i]}$$

for the constant

$$C = \lim_{t\to\infty} t^p P\{|\theta| > t\},$$

where θ is as in (3.2). By (2.2) and the dominated convergence theorem we can apply E_X to both sides of (3.10) to get

$$(3.11)\qquad \lim_{t\to\infty} t^p P\{|\sum_{i=1}^{n}\alpha_i M(A_i)| > t\} = C\sum_{i=1}^{n}|\alpha_i|^p m(A_i),$$

which implies, by the Cramér-Wold theorem, that the random measure M satisfies condition (3.2) (as $\sum_{i=1}^{n}\alpha_i(m(A_i))^{1/p}\theta_i$ has the law of $(\sum_{i=1}^{n}|\alpha_i|^p m(A_i))^{1/p}\theta$).

Example 3. *Normed sums of random measures M satisfying* (3.1)-(3.3). Let M be a random measure satisfying (3.1)-(3.3) and let M_i be independent copies of M. Then

$$N_n = n^{-1/p}\sum_{i=1}^{n} M_i$$

is also a random measure satisfying these conditions. Moreover, (3.3) holds uniformly in n, that is

$$\overline{\lim_{t\to\infty}} \sup_{A\in\Sigma, n} t^p P\left\{\left|\frac{N_n(A)}{(m(A))^{1/p}}\right| > t\right\} \leqslant C' \tag{3.12}$$

for some $C' < \infty$.

It is very easy to show this. Indeed, it is immediately seen that N_n satisfies (3.1) and (3.2). Furthermore, inequality (3.12), which implies (3.3), follows from (2.2). Therefore

$$P\left\{\left|\frac{N_n(A)}{(m(A))^{1/p}}\right| > t\right\} = P\left\{\left|\sum_{i=1}^{n} n^{-1/p} M_i(A)(m(A))^{-1/p}\right| > t\right\} \leqslant C'\left(\frac{4-p}{2-p}\right) t^{-p}$$

for some constant $C' > 0$.

We will now define the stochastic processes that will be the object of our study. Let M be a random measure satisfying (3.1)-(3.3) and let m be the real positive finite measure associated with M. Let T be some set and for each $t \in T$ let f_t: $(S, \Sigma) \to C$ be a complex-valued function in $L^p((S, \Sigma), m)$, $0 < p < 2$. We define the stochastic integral

$$X(t) = \int_S f_t(x) M(dx), \quad t \in T. \tag{3.13}$$

The argument is standard. Let f be a *simple function*, that is $f(x) = y_i$ for $x \in A_i$, where $\{A_i\}$ is a disjoint Σ-measurable cover of S. For such a function and for $0 < q < p$, by (2.3) we have

$$\begin{aligned}\left(\mathrm{E}\left|\int_S f(x) M(dx)\right|^q\right)^{1/q} &= \left(\mathrm{E}\left|\sum_i y_i (m(A_i))^{1/p} \varepsilon_i\left(\frac{M(A_i)}{(m(A_i))^{1/p}}\right)\right|^q\right)^{1/q}\\ &\leqslant C\left[\sum_i |y_i|^p m(A_i)\right]^{1/p} = C\left(\int_S |f(x)|^p m(dx)\right)^{1/p}\end{aligned}$$

for some constant C. Since the simple functions are dense in $L^p((S, \Sigma), m)$, we can extend the map $f \to \int f M(dx)$ to all $f \in L^p((S, \Sigma), m)$. Thus for each $t \in T$ and for $X(t)$ given by (3.13) we see that

$$(\mathrm{E}|X(t)|^q)^{1/q} \leqslant C\left(\int_S |f_t(x)|^p m(dx)\right)^{1/p}, \quad 0 < q < p. \tag{3.14}$$

We will particularly be concerned with the case where $S = \boldsymbol{R}^N$, $N < \infty$, $T = [-1/2, 1/2]^N$ and $f_t(x) = e^{i\langle t, x\rangle}$, $t \in T$, $x \in \boldsymbol{R}^N$. (The measurable sets are the Borel sets.) In this case we put

$$Y(t) = \int_{\boldsymbol{R}^N} e^{i\langle t, x\rangle} M(dx), \quad t \in [-1/2, 1/2]^N, \tag{3.15}$$

and we choose $\{Y(t)\colon t \in [-1/2, 1/2]^N\}$ to be separable. By (3.14) we have

$$(\mathrm{E}|Y(t+h) - Y(t)|^q)^{1/q} \leqslant C'\left(\int_{\boldsymbol{R}^N}\left|\sin\frac{\langle h, x\rangle}{2}\right|^p m(dx)\right)^{1/p}, \quad 0 < q < p, \tag{3.16}$$

for some constant C' depending on q.

For a given positive finite measure m we define

$$\sigma_p(h) = \left(\int_{R^N} \left| \sin \frac{\langle h, x \rangle}{2} \right|^p m(dx) \right)^{1/p}. \tag{3.17}$$

Note that $d(t, t+h) = \sigma_p(h)$ is a translation invariant pseudometric on T. One can show this by using the standard sum formula for $\sin(\langle h_1 + h_2, x\rangle/2)$ and the fact that $L^p(R^N, m)$ is a metric space.

Our goal is to obtain conditions under which the processes in (3.13) and (3.15) have continuous versions and satisfy a central limit theorem. To this end we will associate with each process X as given in (3.13) a stable process such that if X satisfies a central limit theorem, then its normed sums will converge to this stable process. We have seen that to each random measure M satisfying (3.1)-(3.3) there is associated by (3.2) a real positive finite measure m. For this m we define $\tilde{M}$ as the independently scattered random stable measure given in Example 1 and the processes $\tilde{X}$ and $\tilde{Y}$, respectively, as

$$\tilde{X}(t) = \int_S f_t(x)\, \tilde{M}(dx), \quad t \in T, \tag{3.18}$$

and

$$\tilde{Y}(t) = \int_{R^N} e^{i\langle t, x\rangle}\, \tilde{M}(dx), \quad t \in [-1/2, 1/2]^N. \tag{3.19}$$

It is easy to see that

$$E \exp[iu\tilde{X}(t)] = \exp\left[-\left(\int_S |f_t(x)|^p m(dx)\right)|u|^p\right], \tag{3.20}$$

so that $\{\tilde{X}(t), t \in T\}$ is a symmetric stable process of order p.

LEMMA 3.1. *The finite-dimensional distributions of the process $\{X(t), t \in T\}$ given in* (3.13) *belong to the domain of normal attraction of the corresponding finite-dimensional distributions of $\{\tilde{X}(t), t \in T\}$ given in* (3.18). *In particular, this holds for the special cases Y and $\tilde{Y}$ given in* (3.15) *and* (3.19).

Proof. It is enough to show that, for all finite sets $\{t_i\}$ and $\{\alpha_i\}$, $\sum_i \alpha_i X(t_i)$ is in the domain of normal attraction of $\sum_i \alpha_i \tilde{X}(t_i)$. This amounts to proving that for every $f \in L^p((S, \Sigma), m)$ the random variable $X = \int_S f(x) M(dx)$ is in the domain of normal attraction of $\tilde{X} = \int_S f(x) \tilde{M}(dx)$. We observe first that there exists a $K > 0$ such that

$$\sup_{t>0} t^p P\left\{ \frac{\left|\int_S f dM\right|}{\left(\int_S |f|^p dm\right)^{1/p}} > t \right\} \leqslant K. \tag{3.21}$$

Indeed, if f is a simple function, then since M satisfies (3.3), we infer from (2.2) that

$$t^p P\left\{\frac{\left|\int_S f dM\right|}{\left(\int_S |f|^p dm\right)^{1/p}} > t\right\} \leqslant c\left(\frac{4-p}{2-p}\right)$$

for the same c as in (3.3). Now, if $f \in L^p((S, \Sigma), m)$, then there exists a sequence of simple functions f_n such that

$$\frac{\int_S f_n dM}{\left(\int_S |f_n|^p dm\right)^{1/p}} \to \frac{\int_S f dM}{\left(\int_S |f|^p dm\right)^{1/p}}$$

in probability, and consequently (3.21) follows.

Now let $\{X_n\}$ be independent copies of X. Then, by (2.2) and (3.21), for all n we obtain

$$P\left\{\left|n^{-1/p}\sum_{i=1}^{n} X_i\right| > t\right\} \leqslant K\left(\frac{4-p}{2-p}\right)t^{-p}\left(\int_S |f|^p dm\right). \tag{3.22}$$

This inequality holds for all $f \in L^p((S, \Sigma), m)$. Consequently, given $\varepsilon > 0$, there exists a simple function $f_\varepsilon \in L^p((S, \Sigma), m)$ such that if $X_\varepsilon = \int_S f_\varepsilon dM$ and if $\{X_{\varepsilon,n}\}$ are independent copies of $\{X_\varepsilon\}$, then for all n

$$d\left(n^{-1/p}\sum_{i=1}^{n} X_i, n^{-1/p}\sum_{i=1}^{n} X_{\varepsilon,i}\right) < \varepsilon, \tag{3.23}$$

where d is as given in (2.8). Now, it follows from (3.2) that

$$\mathscr{L}\left(n^{-1/p}\sum_{i=1}^{n} X_{\varepsilon,i}\right) \xrightarrow{w} \mathscr{L}(\tilde{X}_\varepsilon).$$

Furthermore, by the definition of the stochastic integral, $\mathscr{L}(\tilde{X}_\varepsilon)$ converges weakly to $\mathscr{L}(\tilde{X})$. In view of these remarks and (3.23), Lemma 3.1 follows now from Lemma 2.2.

Remark 3.1. We may consider stochastic integrals of the form (3.13) in which the function f_t is random and independent of M. Let

$$(\Omega, \mathscr{F}, P) = (\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$$

and assume that $f_t = f_t(x, \omega_1)$ depends only on Ω_1 and M on Ω_2. The same argument as above, with an additional integration with respect to P_1, gives:

(i) *The random integral*

$$X(t, \omega_1, \omega_2) = \int_S f_t(x, \omega_1) M(dx, \omega_2) \tag{3.24}$$

can be defined for $f_t \in L^p(S \times \Omega_1, \Sigma \times \mathscr{F}_1, m \times P_1)$.

(ii) *The finite-dimensional distributions of* $X(t, \omega_1, \omega_2)$ *belong to the domain of normal attraction of*

$$\tilde{X}(t, \omega) = \int_{S\times\Omega'_1} f_t(u)\bar{M}(du, \omega), \tag{3.25}$$

where $(P'_1, \mathcal{F}'_1, \Omega'_1)$ *is an independent copy of* $(P_1, \mathcal{F}_1, \Omega_1)$, $\bar{M}$ *is an independently scattered random measure of index p with control measure* $m\times P'_1$ *on* $(S\times\Omega'_1)$, *and* $\omega\in\Omega$.

We will sketch a proof of (ii). Let us consider the random variable $\int_S f(x, \omega_1)M(dx, \omega_2)$ with $\int_S E|f|^p dm < \infty$. By Lemma 3.1 we see that for almost all fixed $\omega_1\in\Omega_1$ this random variable is in the domain of normal attraction of $\int_S f(x, \omega_1)\tilde{M}(dx, \omega_2)$. This latter random variable is stable (for ω_1 fixed) and has the same law as $\left(\int_S |f(x, \omega_1)|^p dm(x)\right)^{1/p}\theta_1$, where θ_1 is given in (3.2). Therefore, for each $\omega_1\in\Omega'_1, \Omega'_1\subset\Omega, P(\Omega'_1) = 1$, we have

$$\lim_{t\to\infty} t^p P_2\left\{\left|\int_S f(x, \omega_1)M(dx, \omega_2)\right| > t\right\} = c\int_S |f(x, \omega_1)|^p dm(x). \tag{3.26}$$

Also, as in (3.21) but with an additional integration, we can show that

$$t^p P\left\{\frac{\left|\int_S f(x, \omega_1)M(dx, \omega_2)\right|}{\left(\int_S \mathrm{E}_1|f|^p dm\right)^{1/p}} > t\right\} \leqslant K$$

for some constant K. Thus we can take expectations in (3.26) to obtain

$$\lim_{t\to\infty} t^p P\left\{\left|\int_S f(x, \omega_1)M(dx, \omega_2)\right| > t\right\} = c\int_S \mathrm{E}|f|^p dm. \tag{3.27}$$

The reader can now check, using (3.20), that $\tilde{X}(t, \omega)$ in (3.25) is stable with the same law as

$$\left(\int_{S\times\Omega'_1} |f_t(x, \omega'_1)|^p dP_1(\omega'_1)dm(x)\right)^{1/p}\theta_1 = \left(\int_S \mathrm{E}|f_t(x)|^p dm(x)\right)^{1/p}\theta_1.$$

Now, using (3.27), the above comment, and repeating the proof of Lemma 3.1, we obtain (ii).

4. Domains of attraction of stable measures. We first consider

$$Y(t) = \int e^{i\langle t,x\rangle} M(dx), \quad t\in[-1/2, 1/2]^N, \tag{4.1}$$

and the associated stable process

$$\tilde{Y}(t) = \int e^{i\langle t,x\rangle}\tilde{M}(dx), \quad t\in[-1/2, 1/2]^N, \tag{4.2}$$

as defined in (3.15) and (3.19). In this case the integral is taken over $\boldsymbol{R}^N$, M is a random measure satisfying (3.1)-(3.3) for a real finite positive measure

m on R^N, and $\tilde{M}$ is an independently scattered random stable measure of index p $(1 < p < 2)$ with control measure m as given in Example 1. We emphasize that in what follows we take $1 < p < 2$. We also have, by (3.16),

$$(4.3)\qquad (\mathrm{E}\,|Y(t+h)-Y(t)|^q)^{1/q} \leqslant C'\sigma_p(h), \quad h\in[-1,1]^N,$$

for $1 < q < p$, where $\sigma_p(h)$ is given in (3.17) and C' is a constant determined by (3.3). Inequality (4.3) also holds with $\tilde{Y}$ replacing Y since $\tilde{Y}$ is only a special case of the class of processes denoted by Y.

Let $\mu_{\sigma_p}(\varepsilon) = \lambda\{h\in[-1,1]^N\colon \sigma_p(h) < \varepsilon\}$, where λ is Lebesgue measure. Put

$$\bar{\sigma}_p(u) = \sup\{y\colon \mu_{\sigma_p}(y) < u\}$$

and let

$$\hat{\sigma}_p = \sup_{u\in[-1,1]^N} \sigma_p(u).$$

We see that $\bar{\sigma}_p$ is a non-decreasing function on $[0, 2^N]$ and $0 \leqslant \bar{\sigma}_p \leqslant \hat{\sigma}_p$. Following [11] and [12] we call $\bar{\sigma}_p$ the *non-decreasing rearrangement* of σ_p. We define the integral

$$(4.4)\qquad I(\sigma_p) = \int_0^a \frac{\bar{\sigma}_p(u)}{u(\log(b/u))^{1/2}}\,du,$$

where $a = a_N = 2^N$ and $b = b_N = 4^{N+1}$. The following theorem complements Fernique's theorem in [3]:

THEOREM 4.1. *Let the processes*

$$Y = \{Y(t)\colon t\in[-1/2, 1/2]^N\} \quad \textit{and} \quad \tilde{Y} = \{\tilde{Y}(t)\colon t\in[-1/2, 1/2]^N\}$$

be given as in (4.1) *and* (4.2). *Let* $\{\alpha_k\}$ *be a sequence of positive real numbers increasing to infinity. If* $I(\sigma_p) < \infty$, *then the processes*

$$(4.5)\qquad Y_k(t) = \int_{|x|\leqslant\alpha_k} e^{i\langle t,x\rangle} M(dx), \quad t\in[-1/2, 1/2]^N,$$

have continuous sample paths a.s. and converge uniformly to Y *a.s. Hence* Y *has continuous sample paths a.s., and so does* $\tilde{Y}$ *(as a special case). Furthermore,* Y *is in the domain of normal attraction of* $\tilde{Y}$.

Note. The continuity of $\tilde{Y}$ is shown by Theorem 3.1 in [1]. There are some errors in defining $\tilde{Y}$ in [1]; also the assertion that $\tilde{Y}$ is a stationary process is incorrect. We thank Marek Kanter for pointing this out to us.

Proof. As a first step in the proof we obtain the inequality

$$(4.6)\qquad (\mathrm{E}\sup_{t\in T}|Y(t)|^q)^{1/q} \leqslant D[\hat{m}^{1/p} + I(\sigma_p)],$$

where $T = [-1/2, 1/2]^N$, $\hat{m} = m(R^N)$, and D is a constant. The dependence of D on M will be discussed in the sequel.

Now we shall prove (4.6). By the definition of Y there exist $Y^r = \{Y^r(t)\colon t \in T\}$ of the form

$$Y^r(t) = \sum_{j=1}^{\infty} \exp[i\langle \lambda_{r,j}, t\rangle] M(A_{r,j}),$$

where $\lambda_{r,j} \in A_{r,j} \in \Sigma$ and $\{A_{r,j}\}_{j=1}^{\infty}$ are disjoint for each r and such that $Y^r(t) \to Y(t)$ in probability for each $t \in T$. We use the separability of Y to express the left-hand side of (4.6) as a limit of the same expression for the process Y^r. We have

$$\text{(4.7)} \qquad \mathrm{E}\sup_{t\in T} |Y(t)|^q = q\int_0^{\infty} x^{q-1} P\{\sup_{t\in T} |Y(t)| > x\}\, dx$$
$$= q\int_0^{\infty} x^{q-1} \lim_{\{t_i\}\uparrow} P\{\sup_{t\in\{t_i\}} |Y(t)| > x\}\, dx,$$

where $\{t_i\}$ is a finite set increasing to a dense set of S. By the monotone convergence theorem, this is equal to

$$\text{(4.8)} \qquad \lim_{\{t_i\}\uparrow} q\int_0^{\infty} x^{q-1} P\{\sup_{t\in\{t_i\}} |Y(t)| > x\}\, dx$$
$$\leqslant \lim_{\{t_i\}\uparrow} q\int_0^{\infty} x^{q-1} \varliminf_{r\to\infty} P\left\{\sup_{t\in\{t_i\}} |Y^r(t)| > \frac{x}{1+\delta}\right\} dx$$

for $\delta > 0$, since $Y^r(t) \to Y(t)$ in probability for each $t \in S$. The expression in (4.8) is clearly less than or equal to

$$q\int_0^{\infty} x^{q-1} \varliminf_{r\to\infty} P\left\{\sup_{t\in T} |Y^r(t)| > \frac{x}{1+\delta}\right\} dx$$

which, by Fatou's lemma and a change of variables, is not greater than

$$\varliminf_{r\to\infty} q(1+\delta)^q \int_0^{\infty} x^{q-1} P\{\sup_{t\in T} |Y^r(t)| > x\}\, dx = \varliminf_{r\to\infty} (1+\delta)^q \mathrm{E}[\sup_{t\in T} |Y^r(t)|^q].$$

Since this is true for all $\delta > 0$, we get

$$\text{(4.9)} \qquad \mathrm{E}[\sup_{t\in T} |Y(t)|^q]^{1/q} \leqslant \varliminf_{r\to\infty} \mathrm{E}[\sup_{t\in T} |Y^r(t)|^q]^{1/q}.$$

By condition (3.1) we have

$$\text{(4.10)} \qquad Y^r(t) = \sum_{j=1}^{\infty} \exp[i\langle \lambda_{r,j}, t\rangle] \varepsilon_j M(A_{r,j}),$$

where $\{\varepsilon_j\}$ is a Rademacher sequence independent of M. Let $(\Omega_1, \mathscr{F}_1, P_1)$ be the probability space of M and $(\Omega_\varepsilon, \mathscr{F}_\varepsilon, P_\varepsilon)$ the probability space of $\{\varepsilon_i\}$, and let E_1 and E_ε denote the corresponding expectation operators.

The process in (4.10) is defined on the product of these two spaces. As usual we denote by E the expectation with respect to the product space. For a fixed $\omega_1 \in \Omega_1$ we consider

$$Y^r(t, \omega_1) = \sum_{j=1}^{\infty} \exp\left[i\langle\lambda_{r,j}, t\rangle\right] \varepsilon_j M(A_{r,j}, \omega_1).$$

This is a random Fourier series of the type considered in [11]. It follows from Theorem 1 in [11] and Hölder's inequality that

$$\text{(4.11)}\qquad \mathrm{E}_\varepsilon\left[\sup_{t\in T} |Y^r(t, \omega_1)|^q\right]^{1/q}$$
$$\leqslant C\left[\left(\sum_{i=1}^{\infty} M^2(A_{r,i}, \omega_1)\right)^{1/2} + I\left(\left\{\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^2 M^2(A_{r,i}, \omega_1)\right\}^{1/2}\right)\right],$$

where C is an absolute constant independent of M. Apply $(\mathrm{E}_1|\cdot|^q)^{1/q}$ to each side of (4.11) to get

$$\left(\mathrm{E}\sup_{t\in T} |Y^r(t, \omega_1)|^q\right)^{1/q} \leqslant C\left[\left(\mathrm{E}_1\left|\sum_{i=1}^{\infty} M^2(A_{r,i}, \omega_1)\right|^{q/2}\right)^{1/q} + \right.$$
$$\left. + \left(\mathrm{E}_1\left|I\left(\left\{\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^2 M^2(A_{r,i}, \omega_1)\right\}^{1/2}\right)\right|^q\right)^{1/q}\right].$$

By (2.4) with $\eta_i = M(A_{r,i})/(m(A_{r,i}))^{1/p}$ we obtain

$$\left(\mathrm{E}_1\left[\sum_{i=1}^{\infty} M^2(A_{r,i}, \omega_1)\right]^{q/2}\right)^{1/q} \leqslant C''\left(\sum_{i=1}^{\infty} m(A_{r,i})\right)^{1/p}$$

for some absolute constant C''. It follows from Lemma 5 in [11], a slight generalization of Lemma 6 in [11], and (2.4) that

$$\left(\mathrm{E}_1\left|I\left(\left\{\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^2 M^2(A_{r,i}, \omega_1)\right\}^{1/2}\right)\right|^q\right)^{1/q}$$
$$\leqslant C''\left[C_1\left(\sum_{i=1}^{\infty} m(A_{r,i})\right)^{1/p} + 2I\left(\left\{\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^p m(A_{r,i})\right\}^{1/p}\right)\right],$$

where C_1 is a finite constant. (These arguments can be found in greater detail in [12].) Putting all this together we have

$$\text{(4.12)}\qquad \left(\mathrm{E}\sup_{t\in T} |Y^r(t)|^q\right)^{1/q}$$
$$\leqslant D\left[\left(\sum_{i=1}^{\infty} m(A_{r,i})\right)^{1/p} + I\left(\left\{\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^p m(A_{r,i})\right\}^{1/p}\right)\right],$$

where the only dependence of D upon M comes through the constants in (2.4) and (3.3).

Taking the intervals $A_{r,i}$ sufficiently small we can get

$$G_r(h) = \left(\sum_{i=1}^{\infty}\left|\sin\frac{\langle\lambda_{r,i}, h\rangle}{2}\right|^p m(A_{r,i})\right)^{1/p} \leqslant \sigma_p(h)+|h|, \quad h\in[-1,1]^N,$$

from which it follows that the non-increasing rearrangements satisfy

$$\overline{G_r(h)} \leqslant \overline{\sigma_p(h)+|h|}, \quad h\in[-1,1]^N.$$

We will show in the Appendix that

$$I(\sigma_p(h)) < \infty \Leftrightarrow I(\sigma_p(h)+|h|) < \infty. \tag{4.13}$$

Given this we can use (4.9) and the dominated convergence theorem in (4.12) to obtain (4.6). To see that

$$\lim_{r\to\infty}\overline{G_r(h)} = \overline{\sigma_p(h)}$$

one can use either Lemma 2.1 in [4] or Lemma 2.1 in [8]. (Note that these results are given for functions on a compact subset of $\boldsymbol{R}$. It is easy to see that they also hold for functions defined on compact subsets of $\boldsymbol{R}^N$.)

Now consider

$$Y-Y_k = Y(t)-Y_k(t) = \int_{|x|\geqslant\alpha_k} e^{i\langle t,x\rangle} M(dx).$$

By (4.6) we have

$$\left(\mathrm{E}\sup_{t\in T}|Y(t)-Y_k(t)|^q\right)^{1/q} \leqslant D\left[\left(m(|x|\geqslant\alpha_k)\right)^{1/p}+I(\sigma_{p,k})\right], \tag{4.14}$$

where

$$\sigma_{p,k}(h) = \left(\int_{|x|\geqslant\alpha_k}\left|\sin\frac{\langle x, h\rangle}{2}\right|^p m(dx)\right)^{1/p}.$$

Note also that the constant D is independent of k since $Y-Y_k$ is defined for the same M. Since the increments $\{Y_{k+1}-Y_k\}$ are sign-invariant, we can use Lévy's inequality and (4.14) to obtain

$$\mathrm{E}\left[\sup_{j\geqslant k}\sup_{t\in T}|Y(t)-Y_j(t)|\right] \leqslant 2D\left[\left(m(|x|\geqslant\alpha_k)\right)^{1/p}+I(\sigma_{p,k})\right]. \tag{4.15}$$

It is clear from the arguments above relating to dominated convergence that the limit as $k\to\infty$ of the right-hand side of (4.15) is zero.

Finally, we observe that

$$\mathrm{E}\,|Y_k(t+h)-Y_k(t)|^q \leqslant C_k|h|^q \tag{4.16}$$

for some constant C_k. Therefore, by Kolmogorov's theorem ([5], p. 170; note that it also holds for $t\in[-1/2, 1/2]^N$), Y_k has continuous sample paths a.s. Using this fact together with (4.15) we see that Y is the uniform limit of continuous functions a.s.

We proceed to show that Y is in the domain of normal attraction of $\tilde{Y}$. Let M_i be independent copies of M; then

$$(Y-Y_k)_i = \int_{|x|\geqslant \alpha_k} e^{i\langle t,x\rangle} M_i(dx)$$

are independent copies of $(Y-Y_k)$. Consider

$$S_n = n^{-1/p}\sum_{i=1}^{n}(Y-Y_k)_i = \int_{|x|\geqslant\alpha_k} e^{i\langle t,x\rangle}\Big(n^{-1/p}\sum_{i=1}^{n} M_i(dx)\Big).$$

As shown in Example 3, the random measure $n^{-1/p}\sum_{i=1}^{n} M_i$ satisfies (3.1)-(3.3) just like M. Therefore, by (4.14),

$$\mathrm{E}\big[\sup_{t\in[-1/2,1/2]^N}|S_n(t)|\big] \leqslant D\big[(m(|x|\geqslant\alpha_k))^{1/p}+I(\sigma_{p,k})\big]. \tag{4.17}$$

Now let $\{Y_{k,i}\}$ be independent copies of Y_k. We have

$$\Big(\mathrm{E}\Big|n^{-1/p}\sum_{i=1}^{n}\big(Y_{k,i}(t+h)-Y_{k,i}(t)\big)\Big|^q\Big)^{1/q}$$
$$= \Big(\mathrm{E}\Big|\int_{|x|<\alpha_k}\big(e^{i\langle t+h,x\rangle}-e^{i\langle t,x\rangle}\big)\Big(n^{-1/p}\sum_{i=1}^{n}M_i(dx)\Big)\Big|^q\Big)^{1/q}$$
$$\leqslant C\Big(\int_{|x|<\alpha_k}\Big|\sin\frac{\langle x,h\rangle}{2}\Big|^p m(dx)\Big)^{1/p} \leqslant C'|h|.$$

Thus by Theorem 12.3 in [2], which also holds for $t\in[-1/2,1/2]^N$, the sequence $\{n^{-1/p}\sum_{i=1}^{n} Y_{k,i}\}$ is tight, and since the finite-dimensional distributions of this sequence converge weakly to the finite-dimensional distributions of $\int_{|x|<\alpha_k} e^{i\langle t,x\rangle}\tilde{M}(dx)$ by Lemma 3.1, we infer that $\{\mathscr{L}(n^{-1/p}\sum_{i=1}^{n} Y_{k,i})\}$ converges weakly. By (4.17) we have

$$d\Big(n^{-1/p}\sum_{i=1}^{n} Y_i,\, n^{-1/p}\sum_{i=1}^{n} Y_{k,i}\Big)\to 0.$$

Thus, by Lemma 2.2, we see that Y satisfies the central limit theorem with norming constants $n^{-1/p}$ and limiting distribution $\tilde{Y}$.

COROLLARY 4.1. *Let the processes*

$$Y=\{Y(t)\colon t\in[-1/2,1/2]^N\} \quad \textit{and} \quad Y_k=\{Y_k(t)\colon t\in[-1/2,1/2]^N\}$$

be given as in (4.1) *and* (4.5), *where the random measure M now needs only to satisfy* (3.1) *and* (3.3). *Then the process Y_k has continuous sample paths a.s. and converges uniformly to Y a.s. on $[-1/2,1/2]^N$. Therefore, Y has continuous sample paths a.s.*

Proof. Condition (3.2) for M is not used in the proof of the continuity part of Theorem 4.1. (It is used in Lemma 3.1 and needed only to establish the central limit theorem for the finite-dimensional distributions of Y.)

Remark 4.1. As we stated above, $I(\sigma_p) < \infty$ was shown to be a sufficient condition for the continuity of $\tilde{Y}$ in [1]. It was also stated that this condition is not necessary. Recently, G. Pisier and the second-named author have shown that

$$I_p(\sigma_p) = \int_0^a \frac{\bar{\sigma}_p(u)}{u(\log(b/u))^{1/p}}\, du < \infty, \tag{4.18}$$

where a and b are given in (4.4) and $1 < p < 2$, is a sufficient condition for the continuity of $\tilde{Y}$ and as far as we know there are no counterexamples to suggest that this result might not also be necessary. However, the methods used do not yet permit us to replace $I(\sigma_p)$ by $I_p(\sigma_p)$ in Theorem 4.1 when considering either the continuity of Y or the central limit theorem.

We will give some examples of processes (4.1) based on the measures of Example 2 in Section 3. First, let X_k in Example 2 be a random variable associated with a probability measure on R^N which places unit mass at $\lambda_k \in R^N$. Then

$$X(t) = \sum_k a_k \varepsilon_k \xi_k \exp[i\langle\lambda_k, t\rangle], \quad t \in [-1/2, 1/2]^N, \tag{4.19}$$

where $\{a_k\} \in l^p$, $\{\xi_k\}$ satisfies (3.4), and $\{\varepsilon_k\}$ is a Rademacher sequence independent of $\{\xi_k\}$. It is clear that if $I(\sigma_p) < \infty$, then $X(t)$ in (4.19) is in the domain of attraction of

$$\tilde{X}(t) = \sum_k a_k \varepsilon_k \theta_k \exp[i\langle\lambda_k, t\rangle], \quad t \in [-1/2, 1/2]^N, \tag{4.20}$$

where the $\{\theta_k\}$ are as given in (3.2). It is also clear that the control measure m of the stable process $\tilde{X}$ is discrete, with $m(\{\lambda_k\}) = |a_k|^p$.

In keeping with the terminology of second order processes, we will call the measure m that enters in the definition of the stable process $\tilde{Y}$ in (3.19) the *spectrum* of the process. In the example above, the spectrum of $\tilde{X}$ in (4.20) is discrete. One of the problems that motivated our work was to find examples of non-stable processes in the domain of attraction of stable processes with continuous spectra. This is easily done using Example 2 of Section 3. Let us take a continuous probability measure ν on R^N, let $\{X_k\}$ be random variables with distribution ν, and let

$$\sum_k |a_k|^p = 1.$$

Then the measure m in (3.7) is exactly ν, and if $I(\sigma_p) < \infty$ for this ν, then processes of the form (4.1) based on this ν are in the domain of attraction of a stable process with spectrum ν. It is interesting to note that, conditioned on $\{X_k\}$, the process given in (4.1) is

$$Y(t) = \sum_k a_k \varepsilon_k \xi_k \exp[i\langle X_k, t\rangle],$$

i.e. a random Fourier series. Thus Y is a mixture of random Fourier series with frequencies distributed according to the measure ν.

We now consider stochastic integrals

$$X(t) = \int_S f_t(x) M(dx), \quad t \in K, \tag{4.21}$$

and

$$\tilde{X}(t) = \int_S f_t(x) \tilde{M}(dx), \quad t \in K, \tag{4.22}$$

as defined in (3.13) and (3.18), where M is a random measure satisfying (3.1)-(3.3) for a real positive measure m on (S, Σ) and $\tilde{M}$ is an independently scattered random stable measure of index p $(0 < p < 2)$ with control measure m as given in Example 1. Here we take K to be a pseudometric space with pseudometric τ such that

$$\int_0^\infty H_\tau^{1/2}(K, \varepsilon)\, d\varepsilon < \infty, \tag{4.23}$$

where $H_\tau(K, \varepsilon) = \log N_\tau(K, \varepsilon)$ and $N_\tau(K, \varepsilon)$ is the minimum number of open balls of radius ε in the pseudometric τ, with centers in K, that cover K. We assume that, for each fixed $x \in S$, $f_t(x) \in C(K, \tau)$. Also, for each $x \in S$ we define

$$\|f_t(x)\|_\tau = |f_{s_0}(x)| + \sup_{\substack{s,t \in K \\ \tau(s,t) \neq 0}} \frac{|f_s(x) - f_t(x)|}{\tau(s, t)} \quad \text{for a fixed } s_0 \in S,$$

and assume that

$$\int_S \|f_t\|_\tau^p\, dm < \infty. \tag{4.24}$$

(Note that this implies that $f_t \in L^p(S, \Sigma, m)$ so that (4.21) and (4.22) are defined.) We also require the following

(4.25) S is a metric space, Σ is its Borel σ-algebra, m is tight and f_t: $S \to C$ is continuous for every $t \in K$.

Then we have

THEOREM 4.2. *If conditions (4.23)-(4.25) are satisfied, then the processes* $X = \{X(t)\colon t \in K\}$ *and* $\tilde{X} = \{\tilde{X}(t)\colon t \in K\}$ *have continuous sample paths a.s. and* X *is in the domain of normal attraction of* $\tilde{X}$.

Proof. We already know from Lemma 3.1 that the finite-dimensional distributions of X are in the domain of normal attraction of those of $\tilde{X}$. Thus, it is enough to show that if X_i are independent copies of X, then for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$(4.26) \qquad P\{\sup_{\substack{s,t\in K\\ \tau(s,t)<\eta}} \left|n^{-1/p}\sum_{i=1}^{n}(X_i(s)-X_i(t))\right| > \varepsilon\} \leqslant \varepsilon$$

([2], p. 55). Define

$$X^r(t) = \sum_{i=1}^{k_r} f_t(\lambda_{r,i})M(A_{r,i})$$

(k_r finite, $\lambda_{r,i}\in A_{r,i}\subset S$, $\{A_{r,i}\}_{i=1}^{k_r}$ disjoint and measurable, and $r = 1, 2, \ldots$) such that

$$\lim_{r\to\infty}\sum_{i=1}^{k_r} m(A_{r,i})\,\|f_t(\lambda_{r,i})\|_\tau^p = \int_S \|f_t\|_\tau^p\,dm \quad \text{and} \quad X^r(t)\to X(t) \text{ in probability}$$

(such a set of partitions exists because of assumptions (4.24) and (4.25), and by the definition of the stochastic integral). Then, as in (4.9), we infer that, for $0 < q < p$,

$$(4.27) \qquad \Big(\mathrm{E}\sup_{\substack{s,t\in K\\ \tau(s,t)<\eta}} \left|n^{-1/p}\sum_{i=1}^{n}(X_i(s)-X_i(t))\right|^q\Big)^{1/q}$$
$$\leqslant \varliminf_{r\to\infty}\Big(\mathrm{E}\sup_{\substack{s,t\in K\\ \tau(s,t)<\eta}} \left|n^{-1/p}\sum_{i=1}^{n}(X_i^r(s)-X_i^r(t))\right|^q\Big)^{1/q}.$$

Now let $\{\varepsilon_{ij}\}$ be a *Rademacher array* (i.e. a family of independent identically distributed symmetric random variables each one taking on the values ± 1) and let $\{M_i\}$ be independent copies of M and independent of $\{\varepsilon_{ij}\}$. Write

$$Z_n^r = n^{-1/p}\sum_{i=1}^{n}\sum_{j=1}^{k_r} f_t(\lambda_{r,j})\,\varepsilon_{ij}\,M_i(A_{r,j}).$$

Then Z_n^r is equivalent in law to $\sum_{i=1}^{n} X_i^r/n^{1/p}$. Define also

$$G(n,r) = \Big(n^{-2/p}\sum_{i=1}^{n}\sum_{j=1}^{k_r}\|f_t(\lambda_{r,j})\|_\tau^2\,M_i^2(A_{r,j})\Big)^{1/2}$$

and consider the process $Z_n^r(t)/G(n,r)$, $t\in K$. This is a subgaussian process and, clearly,

$$\left(\mathrm{E}_\varepsilon\left|\frac{Z_n^r(t)-Z_n^r(s)}{G(n,r)}\right|^2\right)^{1/2} \leqslant \tau(s,t).$$

Therefore, by Theorem 4.1 in [8] (see Theorem 2.3.1 in [12] for a better proof), we have

$$\mathrm{E}_\varepsilon \sup_{\substack{s,t\in K\\ \tau(s,t)<\eta}} |Z_n^r(s)-Z_n^r(t)| \leqslant C' g(\eta)\, G(n,r),$$

where

$$g(\eta) = \int_0^\eta H_\tau^{1/2}(K,\varepsilon)\,d\varepsilon + \eta(\log\log\eta)^{1/2} \to 0 \quad \text{as } \eta\to 0,$$

and C' is a constant. Note that Theorem 4.1 in [8] is true for (K,τ) without assuming compactness of K; compactness is necessary only if metrics other than τ are considered. Now let r_0 be such that, for $r > r_0$,

$$\sum_{i=1}^{k_r} m(A_{r,i})\, \|f_t(\lambda_{r,i})\|_\tau^p \leqslant 2\int_S \|f_t\|_\tau^p\,dm;$$

then for $r > r_0$ and $q < \min(1,p)$ we obtain

$$\text{(4.28)}\quad \Big(\mathrm{E}\sup_{\substack{s,t\in K\\ \tau(s,t)<\eta}} \Big|n^{-1/p}\sum_{i=1}^n \big(X_i^r(s)-X_i^r(t)\big)\Big|^q\Big)^{1/q} \leqslant C' g(\eta)\big(\mathrm{E}(G(n,r))^q\big)^{1/q}$$

$$\leqslant CC' g(\eta)\sum_{j=1}^{k_r} \|f_t(\lambda_{r,j})\|_\tau^p\, m(A_{r,j}) \leqslant 2CC' g(\eta)\int_S \|f_t\|_\tau^p\,dm,$$

where in the second inequality we use (2.3). Since $g(\eta)\to 0$, (4.27) and (4.28) yield (4.26).

Remark 4.2. The continuity part of Theorem 4.2 was proved for processes of the form (4.22) in [1] and [6]. If we take $f_t(x) = e^{i\langle t,x\rangle}$ for $x\in \boldsymbol{R}^N$ and $t\in[-1/2,1/2]^N$ in (4.21), we get the processes considered in Theorem 4.1. Theorem 4.2 is weaker than Theorem 4.1 in this case, but it is shown in [1] that, depending upon the smoothness properties of m, it can be quite good.

Remark 4.3. Let τ' be a pseudometric on K. If (K,τ) and (K,τ') are equivalent, then Theorem 4.2 shows that X and $\tilde{X}$ have continuous sample paths a.s. with respect to (K,τ') and that X is in the domain of normal attraction of $\tilde{X}$ in $C(K,\tau')$. In particular, if K is a compact subset of $\boldsymbol{R}^N$ and τ is continuous with respect to the ordinary Euclidean metric, then (4.23)-(4.25) imply that X and $\tilde{X}$ take values in $C(K)$ and that X is in the domain of normal attraction of $\tilde{X}$ in $C(K)$, where $C(K)$ is the space of continuous functions on K with the Euclidean metric.

Remark 4.4. We can extend Theorem 4.2 by taking $f\colon (\Omega,\mathscr{F},P)\to C(K)$ to be a $C(K)$-valued random variable independent of the random measure M and satisfying

$$\text{(4.29)}\quad \int_S \mathrm{E}\,\|f_t(x)\|_\tau^p\, m(dx) < \infty$$

and

(4.30) $f_t(x, \omega)$ is a continuous function of $x \in S$ for every $t \in K$ and almost every $\omega \in \Omega$.

Since (4.29) implies

$$\int_S \mathrm{E}\,|f_t(x)|^p\, m(dx) < \infty \qquad \text{for every } t \in K,$$

we can define

(4.31) $$X(t) = X(t, \omega_1, \omega_2) = \int_S f_t(x, \omega_1) M(dx, \omega_2), \qquad t \in K,$$

as in Remark 3.1. From (ii) in that remark it is obvious that the finite-dimensional distributions of $X(t)$ belong to the domain of normal attraction of the corresponding finite-dimensional distributions of

$$\tilde{X}(t) = \int_{S \times \Omega'_1} f_t\, d\bar{M},$$

where $\bar{M}$ is the independently scattered symmetric stable measure of index $p \in (0, 2)$ with control measure $\bar{m} = m \times P'_1$. Thus, in order to show that X is in the domain of normal attraction of $\tilde{X}$, we need only to prove that $\{\mathscr{L}(n^{-1/p} \sum_{i=1}^{n} X_i)\}$ is uniformly tight, where the processes X_i are independent copies of X. From the proof of Theorem 4.2 it follows that, for almost every $\omega_1 \in \Omega_1$,

(4.32) $$\mathrm{E}_2 \sup_{\substack{s,t \in K \\ \tau(s,t) < \eta}} \Big|n^{-1/p} \sum_{i=1}^{n} \big(X_i(s, \omega_1) - X_i(t, \omega_1)\big)\Big|^q \leqslant \big[2CKg(\eta)\big(\textstyle\int \|f_t(\omega_1)\|_\tau^p\, dm\big)^{1/p}\big]^q, \qquad 0 < q < \min(1, p).$$

Therefore, an inequality analogous to (4.26) for the processes X_i under consideration here follows by taking expectation with respect to E_1 on both sides of (4.32) and applying Chebyshev's inequality.

We now specialize Remark 4.4 to random series. Let $\{\xi_k\}$ be the sequence of independent random variables satisfying (2.1), $\{\varepsilon_k\}$ a Rademacher series independent of $\{\xi_k\}$, and $\{X_k(t), t \in K\}$ a sequence of processes defined on K, independent of $\{\varepsilon_k, \xi_k\}$ and such that

$$\sum_{k=1}^{\infty} \mathrm{E}\,\|X_k\|_\tau^p < \infty,$$

where τ is a pseudometric on K satisfying (4.23). Then the process

$$X(t) = \sum_{k=1}^{\infty} X_k(t)\, \varepsilon_k \xi_k, \qquad t \in (K, \tau),$$

is sample continuous and belongs to the domain of normal attraction of the (sample continuous) stable process

$$\tilde{X}(t) = \int_{N \times \Omega_1'} a_k^{-1} X_k(t, \omega_1') \bar{M}(d(k, \omega_1'), \omega), \qquad t \in (K, \tau),$$

where $\{a_k\} \in l^p$, $a_k > 0$, and $\bar{M}$ is the independently scattered symmetric stable measure of index p and with control measure $\bar{m} = m \times P_1'$, $m\{k\} = a_k^p$. This result follows from Remark 4.4 by observing that if $M\{k\} = a_k \varepsilon_k \xi_k$, $k = 1, 2, \ldots$, then

$$X(t) = \int_S a_k^{-1} X_k(t) M(dk)$$

and that $X_k(t, \omega_1)$ is obviously continuous in k for each t and ω_1.

Appendix. In Theorem 4.1 we assume that $I(\sigma_p) < \infty$. It is elementary to see that $I(h) < \infty$ (i.e. $\sigma_p(h) = h$). We will use the following lemma to show that these two conditions imply $I(\sigma_p + h) < \infty$, that is (4.13).

LEMMA A.1. *Let K be a metric or pseudometric space and let $\tau_i(s, t)$, $i = 1, 2$, be pseudometrics on K. Let $N_{\tau_i}(K, \varepsilon)$, $i = 1, 2$, denote the minimum number of open balls of radius ε in the metric or pseudometric τ_i, with centers in K, necessary to cover K. Then*

$$N_{\tau_1 + \tau_2}(K, 4\varepsilon) \leqslant N_{\tau_1}(K, \varepsilon) N_{\tau_2}(K, \varepsilon). \tag{A.1}$$

Proof. The lemma is obviously true if $N_{\tau_i}(K, \varepsilon) = \infty$ for some i. Thus we assume $N_{\tau_i}(K, \varepsilon) < \infty$, $i = 1, 2$. Given $\varepsilon > 0$, there exists a cover of K by $N_{\tau_1}(K, \varepsilon)$ balls A_j of radius ε with respect to τ_1 and centers $a_j \in K$; similarly, there exists a cover by $N_{\tau_2}(K, \varepsilon)$ balls B_k of radius ε with respect to τ_2 and centers $b_k \in K$. For $1 \leqslant j \leqslant N_{\tau_1}(K, \varepsilon)$ and $1 \leqslant k \leqslant N_{\tau_2}(K, \varepsilon)$ we write $C_{jk} = A_j \cap B_k$ and note that $K = \bigcup_{j,k} C_{jk}$. Let $c_{jk} \in C_{jk}$ if $C_{jk} \neq \emptyset$ and consider

$$D_{jk} = \{u\colon \tau_1(c_{jk}, u) + \tau_2(c_{jk}, u) < 4\varepsilon\}.$$

To obtain (A.1) we show that $C_{jk} \subset D_{jk}$. Let $x \in C_{jk}$; then $\tau_1(a_j, x) < \varepsilon$ and $\tau_2(b_k, x) < \varepsilon$. Therefore $\tau_1(c_{jk}, x) \leqslant \tau_1(c_{jk}, a_j) + \tau_1(a_j, x) < 2\varepsilon$ and, similarly, $\tau_2(c_{jk}, x) < 2\varepsilon$. This completes the proof.

Using Lemma A.1 we have

$$\int_0^\infty (\log N_{\tau_1 + \tau_2}(K, 4\varepsilon))^{1/2} d\varepsilon \leqslant \int_0^\infty (\log N_{\tau_1}(K, \varepsilon))^{1/2} d\varepsilon + \int_0^\infty (\log N_{\tau_2}(K, \varepsilon))^{1/2} d\varepsilon.$$

Taking $K = [-1, 1]^N$ and using Lemma 5 (17) in [11], we get (4.13).

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Texas A&M University
Department of Mathematics
College Station, Texas 77843, U.S.A.

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