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STABLE PROBABILITY DISTRIBUTIONS AND THEIR DOMAINS OF ATTRACTION: A DIRECT APPROACH

BY

J. L. GELUK AND L. DE HAAN (ROTTERDAM)

Abstract. The theory of stable probability distributions and their domains of attraction is derived in a direct way (avoiding the usual route via infinitely divisible distributions) using Fourier transforms. Regularly varying functions play an important role in the exposition.

Key words: Domain of attraction, (generalised) regular variation, characteristic function.

1. INTRODUCTION AND MAIN RESULTS

Let $X, X_1, X_2, \ldots$ be independent random variables all of them from the same probability distribution with distribution function F. Consider the sequence $S_n := X_1 + X_2 + \ldots + X_n$, $n = 1, 2, \ldots$, and suppose that for some sequences of norming constants $a_n > 0$ and b_n $(n = 1, 2, \ldots)$ the sequence $S_n/a_n - b_n$ has a non-degenerate limit distribution.

In this note we shall find the general form of all the possible limit distributions and for each of these limit distributions we shall give necessary and sufficient conditions for the distribution function F in order that $S_n/a_n - b_n$ converges to that particular distribution function.

The limit distributions are called *stable distributions* and the set of distribution functions F such that $S_n/a_n - b_n$ converges to a particular stable distribution is called its *domain of attraction*. Thus we shall identify all stable distributions and their domains of attraction.

The indicated results have been developed more than sixty years ago. One of the earliest systematic treatments is in Paul Lévy's famous book *Théorie de l'addition des variables aléatoires* [13]. A well-known complete description of the theory is the book by Gnedenko and Kolmogorov [8]. Various standard texts in probability theory offer an exposition of the subject, for example Breiman [2], Feller [6], Dudley [4]. In these texts the theory of stable distributions is treated as part of the (more general and more involved) theory of infinitely divisible distributions. Although infinitely divisible distributions form an interesting and useful subject of probability theory, the stable distributions have attracted far more attention, both in theoretical research (see e.g. the books by Zolotarev [19] and Samorodnitsky and Taqqu [16]) as well as in applied research (see e.g. Fama [5], Kunst [12], Mandelbrot [14], Samuelson [17]).

In contrast to the mentioned references, in this note the theory is developed *ab initio*, independent of results from the theory of infinitely divisible distributions, which is too complicated to be included in a standard course of probability theory. We have tried to present the theory of stable distributions in a sufficiently streamlined form for presentation in such a course.

We now set out to develop some preliminary results that allow us to formulate the two main theorems. We start from the limit relation:

$$S_n/a_n - b_n \xrightarrow{d} Y$$

or, equivalently,

$$\lim_{n\to\infty} P(S_n/a_n - b_n \leqslant x) = G(x) \tag{1}$$

for all continuity points x of G, the distribution function of the non-degenerate random variable Y. The first question is if it is possible to have different limit distributions for different choices of a_n and b_n. Khinchine's convergence to types theorem (Feller [6], Chapter VIII.2, Lemma 1) states that a different choice of the norming constants can only result in a limit distribution function of the form $G(Ax+B)$ with $A > 0$ and B real. The set of all such transforms of G will be called the *type* of G. From now on when we talk about a limit distribution we shall mean the entire type so that no confusion is possible.

DEFINITION 1. Any probability distribution G that can be obtained as a limit in (1) is called a *stable distribution.*

First of all we are going to reformulate the limit relation (1) in terms of the characteristic functions (or Fourier transforms). Define for $s \in \boldsymbol{R}$ the characteristic functions

$$\phi(s) := Ee^{isX} = \int_{-\infty}^{\infty} e^{isx}\, dF(x) \quad \text{and} \quad \psi(s) := Ee^{isY} = \int_{-\infty}^{\infty} e^{isx}\, dG(x)$$

or, what is more convenient in the present setup,

$$\lambda(t) := \phi(1/t) \quad \text{and} \quad g(t) := \psi(1/t)$$

for $t \in [-\infty, \infty]\backslash\{0\}$. By Lévy's continuity theorem for characteristic functions (Feller [6], Chapter XV.3) relation (1) is equivalent to

$$\lim_{n\to\infty} \exp(-ib_n/t)\,\lambda^n(a_n t) = g(t), \quad t \in [-\infty, \infty]\backslash\{0\}, \tag{2}$$

uniformly on neighborhoods of $\pm\infty$. Note that for $t = \pm\infty$ both sides equal 1.

We start with a definition and a preliminary result.

DEFINITION 2. A positive measurable function f is *regularly varying* if there exists a constant $\gamma \in \boldsymbol{R}$, the index (or order) such that

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\gamma} \quad \text{for all } x > 0. \tag{3}$$

In this case we will use the notation $f \in RV_{\gamma}$. A function in RV_0 is called *slowly varying*. For positive measurable f the limit in (3) is either identically 0 or of the form given above.

PROPOSITION. *If* (1) *holds, then* $|g(t)|^2 = \exp(-c|t|^{-\alpha})$ *for some* $\alpha \in (0, 2]$ *and* $c > 0$. *Moreover,*

$$\lim_{t\to\infty} \frac{-\log|\lambda(tx)|}{-\log|\lambda(t)|} = x^{-\alpha} \quad \text{for } x > 0, \tag{4}$$

i.e. $-\log|\lambda|$ *is regularly varying with index* $-\alpha$.

Proof. From (2) we have

$$\lim_{n\to\infty} |\lambda(a_n t)|^n = |g(t)|$$

locally uniformly near $\pm\infty$. It follows that

$$\lim_{n\to\infty} -n\log|\lambda(a_n t)| = -\log|g(t)| \tag{5}$$

for each $t \in \boldsymbol{R}$, $t \neq 0$, for which $g(t) \neq 0$. For such t it follows that $\log|\lambda(a_n t)| \to 0$; hence $a_n \to +\infty$ (note that $a_n > 0$ by assumption). Moreover, replacing n with $n+1$ gives

$$\lim_{n\to\infty} -(n+1)\log\left|\lambda\left(\frac{a_{n+1}}{a_n} a_n t\right)\right| = -\log|g(t)|,$$

which in combination with (5) implies $a_{n+1}/a_n \to 1$ as $n \to \infty$ since convergence in (5) is uniform on neighborhoods of infinity. Application of Lemma 9 below then shows that the function $-\log|\lambda|$ is regularly varying and its order, say $-\alpha$, has to be non-positive since $\lim_{t\to\infty} -\log|\lambda(t)| = 0$ by (5). Dividing (5) by its counterpart for $t = 1$ we find

$$\lim_{n\to\infty} \frac{-\log|\lambda(a_n t)|}{-\log|\lambda(a_n)|} = \frac{\log|g(t)|}{\log|g(1)|},$$

whence $\log|g(t)|/\log|g(1)| = t^{-\alpha}$ for $t > 0$. Since $|g(t)|^2 = g(t)g(-t)$ is an even function, we have $\log|g(t)|/\log|g(1)| = |t|^{-\alpha}$ for $t \neq 0$. Note that $|g(t)|^2$ is a characteristic function as a product of two characteristic functions.

The restriction $\alpha > 0$ stems from the fact that Y is non-degenerate. Next we show that necessarily $\alpha \leqslant 2$: the assumption $\alpha > 2$ would lead to a non-constant characteristic function with a vanishing second order derivative at 0, which is a contradiction.

The proposition provides a partition of the class of stable distributions into subclasses indexed by the parameter α.

DEFINITION 3. Fix $\alpha \in (0, 2]$. Any probability distribution function G obtained as a limit in (1) and with characteristic function g satisfying

$$|g(t)|^2 = \exp(-c|t|^{-\alpha}) \tag{6}$$

is called a *stable distribution with index* α or an *α-stable distribution.*

DEFINITION 4. The class of distribution functions F for which (1) holds with a limit distribution G satisfying (6) is called the *α-stable domain of attraction.*

Notation: $F \in D_\alpha$.

The classes of distributions introduced above are useful for the rest of this note. However, the α-stable distributions do not form one type. We shall see that we need another (skewness) parameter to describe the full class of all stable distributions. Note that the characteristic functions $|g(t)|^2$ from (6) represent probability distributions that are symmetric about zero.

We are now in a position to formulate the main results. Define

$$U(t) := \operatorname{Re}\lambda(t) \quad \text{and} \quad V(t) := \operatorname{Im}\lambda(t)$$

and for $0 < \alpha < 2$

$$s_\alpha = \int_0^\infty x^{-\alpha} \sin x dx$$

and

$$c_\alpha = \int_1^\infty x^{-\alpha} \cos x dx + \int_0^1 x^{-\alpha}(\cos x - 1) dx.$$

The constants s_α and c_α can also be written in terms of the gamma function. We have for $0 < \alpha < 2$, $\alpha \neq 1$,

$$s_\alpha = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2} \quad \text{and} \quad c_\alpha = \Gamma(1-\alpha)\sin\frac{\alpha\pi}{2} - \frac{1}{1-\alpha}.$$

In the case $\alpha = 1$ one should replace the formulas with the corresponding limit as $\alpha \to 1$: $s_1 = \pi/2$ and $c_1 = \Gamma'(1)$. ($-\Gamma'(1)$ is Euler's constant.)

Further we adopt the convention that the function $(t^\alpha - 1)/a$ is defined for all $t > 0$, $a \in \boldsymbol{R}$ and reads as $\log t$ for $a = 0$ (by continuity). Also the function $\psi_{\alpha,p}(t)$ in formula (7) below is defined to be 1 at $t = 0$ and $(1-\alpha)\tan(\pi/2)$ is defined to be $2/\pi$ at $\alpha = 1$ (by continuity).

THEOREM 1. *Suppose* $0 < \alpha < 2$. *Every α-stable distribution (or rather a distribution type) has a characteristic function of the following form:*

$$\psi_{\alpha,p}(s) = \exp\left(-\left\{|s|^\alpha + is(2p-1)\{(1-\alpha)\tan(\alpha\pi/2)\}\frac{|s|^{\alpha-1}-1}{\alpha-1}\right\}\right) \tag{7}$$

with $0 \leqslant p \leqslant 1$.

The following statements are equivalent:

(i) $F \in D_\alpha$.

(ii) $1-F(t)+F(-t) \in RV_{-\alpha}$ *and there exists a constant* $p \in [0, 1]$ *such that the following tail balance condition holds:*

$$\lim_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} = p.$$

(iii) $1-U(t) \in RV_{-\alpha}$ *and there exists a constant* $p \in [0, 1]$ *such that*

$$\lim_{t\to\infty} \frac{txV(tx)-tV(t)}{t(1-U(t))} = (2p-1)\left\{(1-\alpha)\tan\frac{\alpha\pi}{2}\right\}\frac{|x|^{1-\alpha}-1}{1-\alpha}, \qquad x \in \mathbf{R}\setminus\{0\}. \tag{8}$$

Further, if any of (i), (ii) *or* (iii) *holds, then*

$$\lim_{t\to\infty} \frac{1-U(t)}{1-F(t)+F(-t)} = s_\alpha \tag{9}$$

and

$$\lim_{t\to\infty} \frac{V(t)-t^{-1}\int_0^t \big(1-F(s)-F(-s)\big)\,ds}{1-F(t)+F(-t)} = (2p-1)\,c_\alpha. \tag{10}$$

Remark 1. The parameter α is the same in the three equivalent statements of Theorem 1. The theorem is also true if one keeps α and p fixed in the three statements. Statement (i) then reads: (1) holds with G such that its characteristic function ψ is as in (7).

Remark 2. Unlike in other texts here and in the proof we do not treat the case $\alpha = 1$ separately. However, for $\alpha \neq 1$ the statements of the theorem can be simplified: line (7) reads (remember we need only one member of the type):

$$\psi_{\alpha,p}(s) = \exp\big(-\{|s|^\alpha - is(2p-1)\tan(\alpha\pi/2)\,|s|^{\alpha-1}\}\big).$$

From Lemma 1 below it follows that in the case $0 < \alpha < 1$ (iii) is equivalent to

$$1-U(t) \in RV_{-\alpha} \quad \text{and} \quad V(t) \sim (2p-1)\tan(\alpha\pi/2)\big(1-U(t)\big) \quad \text{as } t \to \infty.$$

If $1 < \alpha < 2$, then (iii) is equivalent to

$$1-U(t) \in RV_{-\alpha}, \quad tV(t) \to \mu \text{ for some constant } \mu$$

and

$$\mu - tV(t) \sim -(2p-1)\tan(\alpha\pi/2)\,t\big(1-U(t)\big) \quad \text{as } t \to \infty.$$

In view of (10) we must have $\mu = EX$, which is finite in this case.

Remark 3. Suppose any of (i), (ii) or (iii) holds. We now indicate how to choose the normalizing constants $a_n > 0$ and b_n in terms of either the distribution function F or the characteristic function ϕ (i.e. in terms of the functions U and V).

The relation (1) holds with G such that the function ψ is exactly as in (7) (i.e. this distribution and not another one of the same type) if we choose a_n and b_n such that

$$\lim_{n\to\infty} ns_\alpha\big(1-F(a_n)+F(-a_n)\big)=1$$

and

$$b_n=\frac{n}{a_n}\int_0^{a_n}\big(1-F(s)-F(-s)\big)\,ds+\frac{2p-1}{s_\alpha}c_\alpha.$$

See (9) and part (iii) $\Rightarrow$ (i) of the proof. Note that the above choice of the sequence a_n is always possible since $1-F(x)+F(-x)$ is regularly varying. By Lemma 1 again, it follows that the above choice of a_n and b_n implies

$$b_n\to(2p-1)\tan(\alpha\pi/2)\qquad\text{for } 0<\alpha<1$$

and

$$b_n-n\mu/a_n\to(2p-1)\tan(\alpha\pi/2)\qquad\text{for } 1<\alpha<2.$$

It follows from relations (9) and (10) that the same limit distribution is obtained with the alternative choices of a_n and b_n:

$$\lim_{n\to\infty} n\big(1-U(a_n)\big)=1\qquad\text{and}\qquad b_n=nV(a_n).$$

Remark 4. The behavior of U and V at $-\infty$ follows from (9) and (10) since U is an even function and V is an odd one.

The case $\alpha=2$ is covered by the following result:

THEOREM 2. *Every 2-stable distribution (or rather a distribution type) has a characteristic function of the following form:*

$$\psi_2(s)=\exp(-s^2),\tag{11}$$

corresponding to the normal distribution.

The following statements are equivalent:

(i) $F\in D_2$.

(ii) *The function* $H_1(t):=\int_0^t u\big(1-F(u)+F(-u)\big)\,du$ *is slowly varying.*

(iii) $1-U(t)\in RV_{-2}$ *and*

$$\frac{\mu-tV(t)}{t\big(1-U(t)\big)}\to 0,\qquad t\to\infty.\tag{12}$$

If (i) *holds, then as* $t\to\infty$

$$1-U(t)\sim H_1(t)/t^2\tag{13}$$

and

$$V(t)-\mu/t=o\big(H_1(t)/t^2\big),\tag{14}$$

where $\mu=EX$.

Remark. The behavior of U and V at $-\infty$ follows from (13) and (14) since U is an even function and V is an odd one.

Using the results of Theorems 1 and 2 one verifies easily that the stable distribution functions are precisely those distribution functions G such that if $Y, Y_1, Y_2, \ldots$ are i.i.d. G, then there exist constants $A_n > 0$ and B_n such that, for $n \geqslant 1$, $(Y_1+Y_2+\ldots+Y_n)/A_n-B_n$ has the same distribution as Y.

2. AUXILIARY RESULTS

Before we prove the theorems we collect some basic facts about regularly varying functions in a sequence of lemmas. Lemmas 1–7 are standard results that are useful in other contexts as well. Lemma 8 (preparing for the use of Lebesgue's theorem on dominated convergence) and Lemma 9 (on replacing a sequence by a continuous variable in the limit relation) are specific for the present setup.

LEMMA 1 (see [7], Theorems 1.9 and 1.10). *Suppose f is a measurable function and there is a positive function a such that for all $x > 0$*

$$\lim_{t\to\infty} \frac{f(tx)-f(t)}{a(t)} = \frac{x^\gamma-1}{\gamma}, \tag{15}$$

where γ is a real parameter. (The right-hand side is interpreted as $\log x$ for $\gamma = 0$.)

If (15) holds with $\gamma > 0$, then $a(t) \sim \gamma f(t)$ as $t \to \infty$, both functions tend to infinity, and hence $f \in RV_\gamma$.

If (15) holds with $\gamma < 0$, then $\lim_{t\to\infty} f(t) =: f(\infty)$ exists and

$$a(t) \sim -\gamma\big(f(\infty)-f(t)\big) \to 0 \qquad (t \to \infty).$$

Hence $f(\infty)-f(t)$ is regularly varying of order γ.

If (15) holds with $\gamma = 0$, then $a(t) = o\big(f(t)\big)$ $(t \to \infty)$ and a is regularly varying of order 0, i.e. slowly varying. Also $\lim_{t\to\infty} f(t) =: f(\infty)$ exists (finite or $+\infty$). If $f(\infty) = \infty$, then $f \in RV_0$. If $f(\infty) < \infty$, then $f(\infty)-f(t)$ is slowly varying and $a(t) = o\big(f(\infty)-f(t)\big)$ as $t \to \infty$.

Remark 1. For f measurable the limit in (15), if not identically zero, is necessarily of the given form.

Remark 2. If the limit in (15) exists and is identically 0 for $x > 0$ with $a \in RV_\gamma$, then

for $\gamma > 0$, $f(t) = o\big(a(t)\big)$ as $t \to \infty$,

for $\gamma < 0$, $f(\infty)$ exists and $f(\infty)-f(t) = o\big(a(t)\big)$ as $t \to \infty$.

LEMMA 2 (see [7], Theorem 1.20). *Suppose that the function f is integrable over finite intervals and that (15) holds with $\gamma = 0$.*

(i) *Let* k: $\boldsymbol{R}^+ \to \boldsymbol{R}$ *be a function which is bounded on* $[0, A]$ *for some* $A > 0$. *Then as* $t \to \infty$

$$\int_0^A \frac{f(ts)-f(t)}{a(t)} k(s)\,ds \to \int_0^A \log s\, k(s)\,ds.$$

(ii) *Let* k: $\boldsymbol{R}^+ \to \boldsymbol{R}$ *be a function such that* $\int_A^\infty s^\varepsilon k(s)\,ds < \infty$ *for some* $A, \varepsilon > 0$. *Then*

$$\int_A^\infty \frac{f(ts)-f(t)}{a(t)} k(s)\,ds \to \int_A^\infty \log s\, k(s)\,ds.$$

Lemma 3 (cf. Bingham et al. [1], Chapter 4). *Suppose that the function* g *is integrable over finite intervals and that* (3) *holds with* f *positive. Assume* $g(t)/f(t) \to c \geqslant 0$ *as* $t \to \infty$.

(i) *Suppose* $\gamma > -1$ *in* (3). *Let* k: $\boldsymbol{R}^+ \to \boldsymbol{R}$ *be a function which is bounded on* $[0, A]$ *for some* $A > 0$. *Then as* $t \to \infty$

$$\int_0^A \frac{g(tx)}{f(t)} k(x)\,dx \to c \int_0^A x^\gamma k(x)\,dx.$$

(ii) *Let* k: $\boldsymbol{R}^+ \to \boldsymbol{R}$ *be a function such that* $\int_A^\infty x^{\gamma+\varepsilon} |k(x)|\,dx < \infty$ *for some* $A, \varepsilon > 0$. *Then*

$$\int_A^\infty \frac{g(tx)}{f(t)} k(x)\,dx \to c \int_A^\infty x^\gamma k(x)\,dx.$$

Remark. If the limit in (15) is identically zero, then the limit in Lemma 2 is also identically zero.

Lemma 4 (see e.g. Ibragimov and Linnik [10], the proof of Lemma 2.6.1). *Suppose* g *is a non-increasing function and* $g(t)/f(t) \to c \in [0, \infty)$ *as* $t \to \infty$ *for some function* $f \in RV_{-\alpha}$ $(0 < \alpha < 2)$. *For any* $\varepsilon > 0$ *there exist constants* A_0 *and* t_0 *such that for all* $t \geqslant t_0$ *and* $A > A_0$

$$\left|\int_A^\infty \frac{g(tx)}{f(t)} \sin x\,dx\right| < \varepsilon \quad \text{and} \quad \left|\int_A^\infty \frac{g(tx)}{f(t)} \cos x\,dx\right| < \varepsilon.$$

Proof. By the second mean value theorem for all $B > A$

$$\int_A^B \frac{g(tx)}{f(t)} \sin x\,dx = \frac{g(tA)}{f(t)} \int_A^\xi \sin x\,dx \qquad \text{for some } \xi \in [A, B],$$

and hence

$$\left|\int_A^B \frac{g(tx)}{f(t)} \sin x\,dx\right| \leqslant 2\,\frac{g(tA)}{f(tA)}\,\frac{f(tA)}{f(t)} \to 2cA^{-\alpha} \qquad \text{as } t \to \infty.$$

The proof of the first statement is complete since the right-hand side tends to zero as $A \to \infty$. The proof of the second statement is similar.

Next we give a version of the monotone density theorem (see e.g. Bingham et al. [1], Chapter I.7.3).

LEMMA 5. *If* $f(t) := \int_0^t \psi(s)\,ds$ *is regularly varying with index* $\alpha > 0$ *and* ψ *is monotone, then* $\psi \in RV_{\alpha-1}$.

In the sequel we need a modification of the above lemma.

LEMMA 6. *Suppose* f *is non-decreasing. If there exists* $\beta \geqslant 0$ *and a positive function* a *such that the function* $\bar{f}$ *defined by* $\bar{f}(t) := t^{-1}\int_0^t f(s)\,ds$ *satisfies*

$$\frac{\bar{f}(tx)-\bar{f}(t)}{a(t)} \to \frac{x^{\beta}-1}{\beta} \quad \text{for } x > 0,\ t \to \infty, \tag{16}$$

then

$$\frac{f(tx)-f(t)}{a(t)} \to \frac{x^{\beta}-1}{\beta} \quad \text{for } x > 0,\ t \to \infty.$$

Proof. Define the function ψ by

$$\psi(t) := tf(t) - \int_0^t f(s)\,ds \qquad (t > 0). \tag{17}$$

It is easy to see that this definition implies

$$\bar{f}(t) = \int_0^t \psi(s)\frac{ds}{s^2}.$$

Hence we have for $x > 0$ and $t \to \infty$

$$\int_1^x \frac{\psi(ts)}{ta(t)}\frac{ds}{s^2} = \frac{\bar{f}(tx)-\bar{f}(t)}{a(t)} \to \frac{x^{\beta}-1}{\beta}.$$

Since ψ is non-decreasing, for $x > 1$ the left-hand side is at least

$$\frac{\psi(t)}{ta(t)}(1-x^{-1}),$$

and hence

$$\limsup_{t\to\infty}\frac{\psi(t)}{ta(t)} \leqslant \frac{x^{\beta}-1}{\beta(1-x^{-1})}.$$

This shows that $\limsup_{t\to\infty}\psi(t)/ta(t) \leqslant 1$ by letting $x \downarrow 1$. Starting with $0 < x < 1$ and applying a similar inequality we get $\liminf_{t\to\infty}\psi(t)/ta(t) \geqslant 1$. It follows that $\psi(t) \sim ta(t)$ $(t \to \infty)$, which combined with (17) gives

$$\lim_{t\to\infty}\frac{f(t)-\bar{f}(t)}{a(t)} = 1.$$

Hence as $t \to \infty$

$$\frac{f(tx)-f(t)}{a(t)} = \frac{\bar{f}(tx)-\bar{f}(t)}{a(t)} + o(1) \to \frac{x^{\beta}-1}{\beta}.$$

Remark. If the limit in (16) is identically zero, the corresponding limit for f is also identically zero.

The next lemma is a special case of Feller [6], Chapter VIII.9, Theorem 2.

Lemma 7. *Suppose F_0 is a distribution function on $[0, \infty)$. The function U_2 is defined by $U_2(t) := \int_0^t s^2\, dF_0(s)$. Then $U_2 \in RV_0$ if and only if*

$$\frac{t^2(1-F_0(t))}{U_2(t)} \to 0 \qquad (t \to \infty).$$

Remark. An integration by parts shows that the above statements are also equivalent to

$$\frac{t^2(1-F_0(t))}{\int_0^t s(1-F_0(s))\,ds} \to 0 \qquad (t \to \infty).$$

The following result is a modification of a result in Pitman [15].

Lemma 8. *Assume the conditions of Theorem* 1 (iii) (*or Theorem* 2 (iii)) *are satisfied. For every $y > 0$ there is a constant c such that for every $T > 0$ and* $0 \leqslant x \leqslant y$

$$\left|\int_0^T \frac{V(1/t)}{t} \cos txdt\right| \leqslant c, \qquad \left|\int_0^T \frac{V(1/t)}{t^2} \sin txdt\right| \leqslant c$$

and

$$\left|\int_0^T \frac{1-U(1/t)}{t} x \sin txdt\right| \leqslant c.$$

Proof. Since the other statements can be proved similarly, we only prove the first statement. Note that if (8) holds with $0 < \alpha \leqslant 1$, there exists t_0 such that $|V(1/t)| \leqslant t^{\alpha/2}$ for $0 < t < t_0 < 1$. Define

$$\left|\int_0^T \frac{V(1/t)}{t} \cos txdt\right| =: L_1 + L_2, \tag{18}$$

where L_1 and L_2 are the integrals over $(0, t_0)$ and (t_0, T), respectively. It follows that L_1 is bounded if $0 < \alpha \leqslant 1$. For $1 < \alpha < 2$ it follows from (8) that $\lim_{t\to\infty} tV(t) =: \mu$ exists, in the case $\alpha = 2$ this follows from (13). Hence L_1 is bounded. Next we estimate L_2. Integration by parts gives

$$\begin{aligned} V(1/t) &= \int_{-\infty}^{\infty} \sin txdF(x) = \int_{-\infty}^{0} \sin txdF(x) + \int_0^{\infty} \sin txd(F(x)-1) \\ &= t\int_0^{\infty} K(y)\cos tydy, \end{aligned}$$

where $K(y) := 1-F(y)-F(-y)$. Hence

$$L_2 = \int_{t_0}^{T} \int_{0}^{\infty} K(y) \cos ty \cos tx dy dt. \tag{19}$$

Using the second mean value theorem for each $M > 0$ we infer that there exists $\xi \in [0, M]$ such that

$$\begin{aligned}
&\left|\int_0^M (1-F(y)) \cos ty \cos tx dy\right| \\
&\quad = (1-F(0)) \left|\cos tx \int_0^{\xi} \cos ty dy\right| \leqslant 2/t \leqslant 2/t_0 \qquad \text{for } t_0 \leqslant t \leqslant T.
\end{aligned}$$

Note that a similar argument holds for the integral containing $F(-y)$. Hence we may reverse the order of integration in (19) to find

$$\begin{aligned}
L_2 &= \int_0^{\infty} \int_{t_0}^{T} K(y) \cos ty \cos tx dt dy \\
&= \frac{1}{2} \int_0^{\infty} K(y) \left(\frac{\cos T(x+y)}{x+y} + \frac{\cos T(x-y)}{x-y} - \frac{\cos t_0(x+y)}{x+y} - \frac{\cos t_0(x-y)}{x-y} \right) dy.
\end{aligned}$$

The latter integral is bounded since

$$\int_{-\infty}^{\infty} \frac{\cos T(x+y) - \cos t_0(x+y)}{x+y} dy = \int_{-\infty}^{\infty} \frac{\cos Ty - \cos t_0 y}{y} dy$$

exists as a finite (semiconvergent) integral for all real x.

LEMMA 9 (the extension of Kendall [11], cf. Bingham et al. [1], Chapter 1.9). *Suppose*

$$\limsup_{n \to \infty} x_n = \infty, \qquad \limsup_{n \to \infty} x_{n+1}/x_n = 1$$

and f is a continuous function.

1. *Suppose $0 < b < c < \infty$ and for some sequence a_n*

$$a_n f(\lambda x_n) \to \psi(\lambda) \in (0, \infty) \quad \textit{for all } \lambda \in (b, c) \quad \textit{as } n \to \infty;$$

then f varies regularly.

2. *Suppose $0 < b < c < \infty$, the function a is regularly varying and*

$$\lim_{n \to \infty} \frac{f(\lambda x_n) - f(x_n)}{a(x_n)} \to \psi(\lambda) \quad \textit{for all } \lambda \in (b, c);$$

then there exist constants $c, \gamma \in \mathbf{R}$ such that

$$\frac{f(tx) - f(t)}{a(t)} \to c \frac{x^{\gamma} - 1}{\gamma} \quad \textit{as } t \to \infty, \ x > 0.$$

Proof. The continuity of f is the key assumption.

1. With $V=(b,c)$ there exists a non-empty interval K such that $V\cap u^{-1}V\neq 0$ for all $u\in K$. If t, $ut\in V$, we have

$$\frac{f(x_n ut)}{f(x_n t)}\to\frac{\psi(ut)}{\psi(t)}\quad\text{as } n\to\infty.$$

Hence if we write $f^*(t)=f(ue^t)/f(e^t)$ for $u>0$ fixed and $x_n^*=\log x_n$, then $f^*(t+x_n^*)$ converges as $n\to\infty$ for all t in a non-empty open interval J. Choose $\varepsilon>0$ and define for $k\in Z$, $m\in N$ the closed sets

$$C_{k,m}:=\bigcap_{n\geqslant m}\{t\in\boldsymbol{R};\ f^*(t+x_n^*)\in[k\varepsilon-\varepsilon,\ k\varepsilon+\varepsilon]\}.$$

By Baire's category theorem (see Hewitt and Stromberg [9]), since J is non-empty and open, one of the sets $C_{k,m}$ contains an open interval I. This means that

$$k\varepsilon-\varepsilon\leqslant f^*(t+x_n^*)\leqslant k\varepsilon+\varepsilon\qquad\text{for } n\geqslant m,\ t\in I.$$

Since by assumption $x_n^*\to\infty$, $x_{n+1}^*-x_n^*\to 0$, it follows that $\bigcup_{n\geqslant m}x_n^*+I$ contains an interval of the form $[t_0,\infty]$; hence

$$k\varepsilon-\varepsilon\leqslant f^*(t)\leqslant k\varepsilon+\varepsilon\qquad\text{for all } t\geqslant t_0.$$

Hence $\lim_{t\to\infty}f^*(t)$ exists and is finite and positive for all $u\in K$, i.e.

$$\lim_{t\to\infty}f(ue^t)/f(e^t)$$

exists and is finite for all $u\in K$. It follows that the function f is regularly varying.

2. In a similar way as above, using the fact that a is regularly varying we obtain for $u>0$ fixed and all t in a non-empty open interval

$$\lim_{n\to\infty}\frac{f(x_n tu)-f(x_n u)}{a(x_n u)}=\psi^*(t).$$

Define for $u>0$ fixed the function

$$f^*(t+x_n^*):=\frac{f(x_n ue^t)-f(x_n e^t)}{a(x_n e^t)}$$

(with $x_n^*=\log x_n$ as before). Then in a similar way as above we can show that $\lim_{t\to\infty}f^*(t)$ exists and is finite.

3. PROOF OF THE MAIN THEOREMS

Proof of Theorem 1. We first prove the equivalence of (i), (ii) and (iii). In the part (iii) ⇒ (i) of the proof we obtain the characterization (7) (see (45)).

(i) $\Rightarrow$ (iii). It follows from (2) that for all real $t \neq 0$

$$\lim_{n\to\infty} n \log \lambda(a_n t) - ib_n/t = \log g(t), \tag{20}$$

and hence

$$\lim_{n\to\infty} nR(a_n t) = -\operatorname{Re}(\log g(t)), \tag{21}$$

and

$$\lim_{n\to\infty} nI(a_n t) - b_n/t = \operatorname{Im}(\log g(t)), \tag{22}$$

where $R(t) = -\operatorname{Re}\log\lambda(t)$ and $I(t) = \operatorname{Im}\log\lambda(t)$ (Re and Im denote the real and the imaginary part, respectively). Note that there exists a unique version of $\log\lambda$ ($\log g$) satisfying $\log\lambda(t) \to 1$ ($\log g(t) \to 1$) as $t \to \infty$ (see e.g. Feller [6], Chapter XV).

Application of Lemma 9 (note that $a_n \to \infty$, $a_{n+1}/a_n \to 1$ ($n \to \infty$) as in the proof of the Proposition) shows that the function R is regularly varying and $-\operatorname{Re}(\log g(t)) = |t|^{-\alpha}$ for $t \neq 0$.

Next we focus on (22). By setting $t = 1$ we get

$$\lim_{n\to\infty} nI(a_n) - b_n = \operatorname{Im}\log g(1),$$

and hence

$$\lim_{n\to\infty} n\,[I(a_n t) - t^{-1} I(a_n)] = \operatorname{Im}\log g(t) - t^{-1}\operatorname{Im}\log g(1).$$

Combining this with (21) for $t = 1$, we get for all real $t \neq 0$

$$\lim_{n\to\infty} \frac{a_n tI(a_n t) - a_n I(a_n)}{a_n R(a_n)} = \frac{t\operatorname{Im}\log g(t) - \operatorname{Im}\log g(1)}{-\operatorname{Re}\log g(1)} =: \tau(t). \tag{23}$$

In a similar way, by Lemma 9 this implies

$$\frac{txI(tx) - xI(x)}{xR(x)} \to c\,\frac{t^\gamma - 1}{\gamma}, \qquad x \to \infty,\ t > 0, \tag{24}$$

where $c \in \boldsymbol{R}$ is a constant. Since $R \in RV_{-\alpha}$, it follows from Lemma 1 that $c = 0$ or, if $c \neq 0$, then $\gamma = 1 - \alpha$. Using the fact that I is an odd function we now have

$$\frac{txI(tx) - xI(x)}{xR(x)} \to c\,\frac{|t|^\gamma - 1}{\gamma}, \qquad x \to \infty,\ t \in \boldsymbol{R}\backslash\{0\}. \tag{25}$$

We have now (iii) with $1-U$ replaced with R and V replaced with I.

Since for complex z, $|z| < 1/2$,

$$|e^z - 1 - z| \leqslant |z|^2,$$

we have

$$|\lambda(a_n)-1-\log\lambda(a_n)| \leqslant |\log\lambda(a_n)|^2$$

for n sufficiently large. From (21) and (22) we obtain

$$n|\log\lambda(a_n)|^2 = n\big(R(a_n)+I(a_n)\big)^2 \leqslant 2n\big(R(a_n)^2+I(a_n)^2\big) \to 0 \qquad (n\to\infty). \tag{26}$$

It follows that

$$\lim_{n\to\infty} n|-\log\lambda(a_n)-1+\lambda(a_n)| = 0; \tag{27}$$

hence we may replace $-\log\lambda$ in (20) with $1-\lambda$. Consequently, we can repeat the above argument with $-\operatorname{Re}\log\lambda$ replaced with $1-U$ and $\operatorname{Im}\log\lambda$ replaced with V to obtain (iii).

(ii) $\Rightarrow$ (iii). Define the functions H and K by

$$H(t) := 1-F(t)+F(-t) \quad \text{and} \quad K(t) := 1-F(t)-F(-t).$$

First we prove that

$$\lim_{t\to\infty} \frac{\lambda(t)-1-it^{-1}\int_0^t K(s)\,ds}{H(t)} = -s_\alpha + i(2p-1)c_\alpha. \tag{28}$$

Now for any $A>0$

$$\frac{\lambda(t)-1-it^{-1}\int_0^t K(s)\,ds}{H(t)}$$

$$= -\int_0^A \sin x\frac{H(tx)}{H(t)}dx - i\int_0^A (1-\cos x)\frac{K(tx)}{H(t)}dx + i\int_1^A \frac{K(tx)}{H(t)}dx$$

$$- \int_A^\infty \sin x\frac{H(tx)}{H(t)}dx + i\int_A^\infty \cos x\frac{K(tx)}{H(t)}dx.$$

Take $\varepsilon>0$. By Lemma 4 the last two integrals are less than ε for $t>t_0$ and $A>A_0$. For fixed $A>0$ the first three integrals converge by Lemma 3 to

$$-\int_0^A \sin x\frac{dx}{x^\alpha} - i(2p-1)\int_0^A (1-\cos x)\frac{dx}{x^\alpha} + i(2p-1)\int_1^A \frac{dx}{x^\alpha}.$$

Now (28) follows if we take $A\to\infty$. By separating the real and imaginary parts in (28) we get the limiting behavior as $t\to+\infty$ in (9) and (10). The limiting behavior as $t\to-\infty$ follows since U is an even function and V an odd one. Obviously, (9) implies that $1-U\in RV_{-\alpha}$ (since $H\in RV_{-\alpha}$). Note that $s_\alpha\neq 0$ for $0<\alpha<2$. Now (10) implies that for $x>0$

$$\lim_{t\to\infty} \frac{txV(tx)-\int_0^{tx}K(s)\,ds}{tH(t)} = (2p-1)c_\alpha x^{1-\alpha}$$

(use $H \in RV_{-\alpha}$). Combination with (10) gives

$$\lim_{t\to\infty} \frac{txV(tx)-tV(t)}{tH(t)} - \int_1^x \frac{K(ts)}{H(t)}\,ds = (2p-1)\,c_\alpha(x^{1-\alpha}-1). \tag{29}$$

Note that the integral on the left-hand side converges to

$$(2p-1)\frac{x^{1-\alpha}-1}{1-\alpha} \quad \text{as } t\to\infty$$

by Lemma 3. Now (8) follows since $1-U$ satisfies (9).

(iii)$\Rightarrow$(ii). In this part of the proof c denotes a constant which may take different values at each occurrence. In order to prove the results in this part we make use of Lévy's inversion relation

$$F(x+h)-F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\zeta)\frac{1-e^{-i\zeta h}}{i\zeta}e^{-i\zeta x}\,d\zeta, \tag{30}$$

valid for all x, $x+h$ for which F is continuous. See e.g. Chow and Teicher [3]. Note that the above integral is to be understood as the limit as $A\to\infty$ of the integral over $(-A, +A)$. A similar remark holds for the other inversion integrals below. Using the relation (30), the equality $\phi(t) = \lambda(1/t) = U(1/t)+ +iV(1/t)$ and the fact that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2},$$

we obtain the following inversion formula for H:

$$H(x) = \frac{2}{\pi}\int_0^\infty \frac{1-U(1/t)}{t}\sin tx\,dt, \qquad x>0. \tag{31}$$

First we prove that H is regularly varying with order $-\alpha$. For $t \geqslant 0$ define

$$H_2(t) = \int_0^t H_1(x)\,dx,$$

where $H_1(t) = \int_0^t xH(x)\,dx = \frac{1}{2}\int_0^{t^2} H(\sqrt{u})\,du$ as in Theorem 2 (ii). By (31) it follows that

$$H_1(t) = \frac{2}{\pi}\int_0^t\int_0^\infty \frac{1-U(1/s)}{s}x\sin sx\,ds\,dx.$$

From Lemma 8 it follows that we may reverse the order of integration, and so

$$H_1(t) = \frac{2}{\pi}\int_0^\infty \{1-U(1/s)\}\frac{\sin ts - ts\cos ts}{s^3}\,ds.$$

Since this integral is absolutely convergent, by Fubini's theorem we have

$$H_2(t) = \frac{2}{\pi}\int_0^\infty \int_0^t \{1-U(1/s)\}\frac{\sin xs - xs\cos xs}{s^3}\,dxds \tag{32}$$

$$= \frac{2}{\pi}\int_0^\infty \{1-U(1/s)\}\frac{2(1-\cos ts)-ts\sin ts}{s^4}\,ds.$$

Hence

$$\frac{H_2(t)}{t^3(1-U(t))} = \frac{2}{\pi}\int_0^\infty \frac{1-U(t/s)}{1-U(t)}\,\frac{2(1-\cos s)-s\sin s}{s^4}\,ds.$$

Since $1-U$ is regularly varying with index $-\alpha$, in view of Lemma 3 (substitute $s = x^{-1}$) the right-hand side converges to

$$\frac{2}{\pi}\int_0^\infty \frac{2(1-\cos s)-s\sin s}{s^{4-\alpha}}\,ds \qquad \text{as } t\to\infty.$$

As a consequence, $H_2 \in RV_{3-\alpha}$. By the monotone density theorem (Lemma 5) it follows that $H_1 \in RV_{2-\alpha}$; then $H \in RV_{-\alpha}$. In order to prove the tail balance condition we need an inversion relation for K. As for the inversion relation for H we obtain

$$K(x)-K(y) = \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}(\cos tx - \cos ty)\,dt, \qquad x,\, y > 0; \tag{33}$$

hence by Lemma 8 the function

$$K(x) - \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}\cos tx\,dt$$

is constant for $x > 0$. The constant is necessarily 0. This follows by taking the limit as $x \to \infty$ and applying the Riemann–Lebesgue lemma in (18). See e.g. Feller [6], Chapter XV.4. For $y > 0$ we have

$$K_1(y) := \int_0^y K(x)\,dx = \frac{2}{\pi}\int_0^y\int_0^\infty \frac{V(1/t)}{t}\cos tx\,dtdx \tag{34}$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^y \frac{V(1/t)}{t}\cos tx\,dxdt = \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}\,\frac{\sin ty}{t}\,dt.$$

Interchanging the order of integration is justified by Lemma 8. Now we integrate once more, use (34) and Lemma 8 to find for $t > 0$

$$\bar{K}_1(t) := \frac{1}{t}\int_0^t K_1(y)\,dy = \frac{2}{\pi t}\int_0^\infty \frac{V(1/s)(1-\cos st)}{s^3}\,ds.$$

It follows that for $b, t > 0$

$$\frac{\bar{K}_1(bt)-\bar{K}_1(t)}{a(t)} = \frac{2}{\pi}\int_0^\infty \frac{btsV(bts)-tsV(ts)}{a(ts)}\frac{a(ts)}{a(t)}\{1-\cos(s^{-1})\}\,ds, \tag{35}$$

where $a(t) := t(1-U(t))$.

Taking the limit as $t \to \infty$, using (8) and Lemmas 1, 2 and 3 we find for $x > 0$

$$\lim_{t\to\infty} \frac{\bar{K}_1(tx)-\bar{K}_1(t)}{a(t)} = c\frac{x^{1-\alpha}-1}{1-\alpha}. \tag{36}$$

Application of Lemma 6 then shows that

$$\lim_{t\to\infty} \frac{K_1(tx)-K_1(t)}{a(t)} = c\frac{x^{1-\alpha}-1}{1-\alpha}, \qquad x > 0. \tag{37}$$

It follows from (9), since $H \in RV_{-\alpha}$, that as $t \to \infty$ for $x > 0$

$$\frac{\int_0^{tx} H(s)\,ds - \int_0^t H(s)\,ds}{a(t)} = \frac{tH(t)}{t(1-U(t))}\int_1^x \frac{H(ts)}{H(t)}\,ds \to c\frac{x^{1-\alpha}-1}{1-\alpha}. \tag{38}$$

Adding both sides of (36) and (38) we obtain for $x > 0$, as $t \to \infty$,

$$t\int_1^x \frac{1-F(ts)}{a(t)}\,ds \to c\frac{x^{1-\alpha}-1}{1-\alpha}.$$

In view of (9) this implies

$$\int_1^x \frac{1-F(ts)}{1-F(t)+F(-t)}\,ds \to c\frac{x^{1-\alpha}-1}{1-\alpha}, \qquad t \to \infty,\ x > 0. \tag{39}$$

For $x > 1$ the left-hand side is at most

$$(x-1)(1-F(t))/(1-F(t)+F(-t)).$$

Hence

$$\liminf_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} \geqslant c\frac{x^{1-\alpha}-1}{(1-\alpha)(x-1)}.$$

Letting $x \downarrow 1$ then gives

$$\liminf_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} \geqslant c. \tag{40}$$

Starting with $0 < x < 1$ in (39) and applying similar inequalities we obtain

$$\limsup_{t\to\infty} (1-F(t))/(1-F(t)+F(-t)) \leqslant c,$$

where c equals the constant in (40).

(iii) $\Rightarrow$ (i). Define the sequence a_n, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} s_\alpha n\left(1-F(a_n)+F(-a_n)\right) = 1. \tag{41}$$

Note that this is possible since $1-F(t)+F(-t)$ is regularly varying. Moreover, $a_n \to \infty$ as $n \to \infty$. By (9) we have, since $1-U \in RV_{-\alpha}$,

$$\lim_{n\to\infty} n\left(1-U(a_n t)\right) = |t|^{-\alpha} \quad \text{for all } t \in \boldsymbol{R},\ t \neq 0. \tag{42}$$

Define the sequence b_n, $n = 1, 2, \ldots$, by

$$b_n = \frac{n}{a_n}\int_0^{a_n} K(s)\,ds + \frac{2p-1}{s_\alpha} c_\alpha.$$

Then as $n \to \infty$ for all $t \in \boldsymbol{R}$, $t \neq 0$, by (9) and (42)

$$\begin{aligned} nV(a_n t) - \frac{b_n}{t} &= \frac{n}{a_n t}\left(a_n t V(a_n t) - a_n V(a_n)\right) + \frac{n}{t}\left(V(a_n) - \frac{b_n}{n}\right) - \frac{2p-1}{t s_\alpha} c_\alpha \\ &\sim \frac{1}{t}\,\frac{a_n t V(a_n t) - a_n V(a_n)}{a_n\left(1-U(a_n)\right)} + \frac{1}{t s_\alpha}\,\frac{V(a_n) - a_n^{-1}\int_0^{a_n} K(s)\,ds}{H(a_n)} - \frac{2p-1}{t s_\alpha} c_\alpha. \end{aligned}$$

Substituting relations (8) and (10) on the right-hand side we find

$$\lim_{n\to\infty}\left\{nV(a_n t) - \frac{b_n}{t}\right\} = \frac{2p-1}{t s_\alpha}\left\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\right\}. \tag{43}$$

Combining (42) and (43) we get

$$\lim_{n\to\infty} n\left(1-\lambda(a_n t)\right) + i b_n/t = |t|^{-\alpha} - \frac{i(2p-1)}{t s_\alpha}\left\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\right\}. \tag{44}$$

We want to prove that

$$\begin{aligned} &\lim_{n\to\infty} \lambda^n(a_n t)\exp(-i b_n/t) \\ &= \exp\left(-\left[|t|^{-\alpha} - \frac{i(2p-1)}{t s_\alpha}\left\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\right\}\right]\right). \end{aligned} \tag{45}$$

Now for $|z| < 1/2$ we have $|e^z - 1 - z| \leqslant |z|^2$. In particular, for fixed $t \in \boldsymbol{R}$, $t \neq 0$, there exists n_0 such that for $n > n_0$

$$\left|\exp\left(-1+\lambda(a_n t)\right) - \lambda(a_n t)\right| \leqslant |1-\lambda(a_n t)|^2,$$

and hence

$$\exp\left\{-n\left(1-\lambda(a_n t)\right)\right\}\exp(-i b_n/t) = \lambda^n(a_n t)\exp(-i b_n/t)\left\{1 + O\left(\frac{|1-\lambda(a_n t)|^2}{\lambda(a_n t)}\right)\right\}^n.$$

So it is sufficient to prove that $n|1-\lambda(a_n t)|^2 \to 0$ as $n \to \infty$. This follows from (42) and (43).

Proof of Theorem 2

(i) $\Rightarrow$ (iii). Following the reasoning of the proof of Theorem 1, part (i) $\Rightarrow$ (iii), we find that $1-U \in RV_{-2}$. Since (24) now holds with $\gamma = -1$, application of Lemma 1 (or its extended form from Remark 2 following the lemma) shows that $\lim_{t\to\infty} tI(t) =: c_0$ exists. Hence (25) holds with c possibly 0 and the right-hand side equals

$$\tau(t) = t \operatorname{Im} \log g(t) - \operatorname{Im} \log g(1) = -c(|t|^{-1}-1).$$

Since $-\operatorname{Re} \log g(t) = t^{-2}$, $t \neq 0$, we have

$$g(t) = \exp\{-t^{-2} + it^{-1}(c_3 + c_4|t|^{-1})\},$$

where c_3 and c_4 are constants. Since any bounded continuous function ω with $\omega(0) = 1$ is a characteristic function only if for all x and $\varepsilon > 0$

$$\int_{-\infty}^{\infty} e^{-i\zeta x} \omega(\zeta) \exp(-\varepsilon\zeta^2)\, d\zeta \geqslant 0$$

(see Feller [6], Chapter XIX.2), we must have $c_4 = 0$ (see Steutel [18]). Hence $\psi(t) = g(t^{-1}) = \exp(-t^2)$ and (24) holds with $c = 0$. Remark 2 following Lemma 1 now shows that $\lim_{t\to\infty} tV(t) =: \mu$ exists and (12) holds.

(ii) $\Rightarrow$ (iii). By Lemma 7 (take $F_0(t) = 1-H(t)$, $t \geqslant 0$), $t^2 H(t)/H_1(t) \to 0$ as $t \to \infty$. Note that

$$\frac{\lambda(t) - 1 - i\mu/t + H_1(t)/t^2}{H_1(t)/t^2}$$
$$= -\int_0^1 \frac{(tx)^2 H(tx)}{H_1(t)} \frac{\sin x - x}{x^2} dx - \int_1^\infty \frac{(tx)^2 H(tx)}{H_1(t)} \frac{\sin x}{x^2} dx$$
$$+ i\int_0^1 \frac{(tx)^2 K(tx)}{H_1(t)} \frac{\cos x - 1}{x^2} dx + i\int_1^\infty \frac{(tx)^2 K(tx)}{H_1(t)} \frac{\cos x - 1}{x^2} dx.$$

Application of Lemma 3 shows that the integrals on the right-hand side all tend to zero as $t \to \infty$. Now (13) and (14) follow by taking the real and imaginary part and (iii) follows from (ii), (13) and (14).

(iii) $\Rightarrow$ (ii). Compared to the corresponding part in the proof of Theorem 1 we have to integrate once more in order to get an absolutely convergent integral. For the function H_3 defined by $H_3(t) = \int_0^t H_2(s)\, ds$ an expression similar to (32) can be given. A similar calculation shows that H_1 is slowly varying.

(iii) $\Rightarrow$ (i). With the sequences a_n and b_n, $n = 1, 2, \ldots$, defined by

$$\frac{nH_1(a_n)}{a_n^2} \to 1 \text{ as } n \to \infty \quad \text{and} \quad b_n = n\mu/a_n,$$

the proof is similar to the proof of the corresponding part of Theorem 1. We omit the details.

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Econometric Institute
Erasmus University Rotterdam
P.O. Box 1738
NL-3000 DR Rotterdam, The Netherlands

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