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LIMITING DISTRIBUTIONS OF DIFFERENCES AND QUOTIENTS OF SUCCESSIVE k-TH UPPER AND LOWER RECORD VALUES

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Abstract. Following the method of Gajek [3] we investigate the limiting properties of differences and quotients of successive k-th upper and lower record values.

1. Introduction. Let $\{X_i, i \geqslant 1\}$ be a sequence of i.i.d. random variables with common distribution function (df) F and probability density function (pdf) f. Let

$$X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$$

denote the *order statistics* of a sample $X_1, \ldots, X_n$.

For a fixed $k \geqslant 1$ we define (cf. [2]) the *k-th upper record times* $U_k(n)$, $n \geqslant 1$, of the sequence $\{X_i, i \geqslant 1\}$ as follows:

$$\begin{aligned} &U_k(1) = 1, \\ &U_k(n+1) = \min\{j > U_k(n)\colon\ X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \qquad n \geqslant 1, \end{aligned}$$

and the *k-th upper record values* as

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}, \qquad n \geqslant 1.$$

Note that for $k = 1$ we have $Y_n^{(1)} = X_{1:U_1(n)} := X_{U(n)}$, $n \geqslant 1$, (upper) record values of the sequence $\{X_i, i \geqslant 1\}$, and that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$.

Similarly, for a fixed $k \geqslant 1$ we define (cf. [6]) the *k-th lower record times* $L_k(n)$, $n \geqslant 1$, as

$$\begin{aligned} &L_k(1) = 1, \\ &L_k(n+1) = \min\{j > L_k(n)\colon\ X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, \qquad n \geqslant 1, \end{aligned}$$

and the *k-th lower record values* as

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \qquad n \geqslant 1.$$

Note that for $k = 1$ we have $Z_n^{(1)} = X_{1:L_1(n)} := X_{L(n)}$, $n \geqslant 1$, (lower) record values of the sequence $\{X_i, i \geqslant 1\}$, and that $Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k)$.

Let us put for $k \geqslant 1$, $n \geqslant 1$,

$$\Delta_n^{(k)} = Y_{n+1}^{(k)} - Y_n^{(k)}, \qquad D_n^{(k)} = Z_n^{(k)} - Z_{n+1}^{(k)},$$

$$U_n^{(k)} = Y_{n+1}^{(k)}/Y_n^{(k)}, \qquad T_n^{(k)} = Z_n^{(k)}/Z_{n+1}^{(k)},$$

and define

$$W_n^{(k)} = k\Delta_n^{(k)}, \qquad V_n^{(k)} = kD_n^{(k)},$$

$$R_n^{(k)} = n(U_n^{(k)} - 1), \qquad Q_n^{(k)} = n(T_n^{(k)} - 1).$$

Gajek [3] has shown that if a df F, concentrated on the interval $S \subset \boldsymbol{R}$ is absolutely continuous with a pdf f and if

$$r(x) = \frac{f(x)}{1-F(x)}$$

is a differentiable function with a bounded first derivative, then

$$W_n^{(k)} \xrightarrow{D} W_n \quad \text{as } k \to \infty$$

(D – in distribution), where W_n is exponentially distributed with a df

$$F_\lambda^*(x) = 1 - \exp(-\lambda x),$$

and where $\lambda = r(x_0^+)$ (the right limit of $r(x)$ at the point x_0), $x_0 = \inf S$, and F_0^*, F_∞^* denote the distribution concentrated at zero and the improper distribution concentrated at infinity, respectively.

In this paper we extend the class of sequences of df's described in [3] weakly convergent to an exponential distribution, using lower record values, to construct a sequence $\{V_n^{(k)}, k \geqslant 1\}$. Moreover, we study limiting properties of the sequences $\{R_n^{(k)}, n \geqslant 1\}$ and $\{Q_n^{(k)}, n \geqslant 1\}$. The results are illustrated by examples.

2. Probability distributions of $U_n^{(k)}$, $D_n^{(k)}$ and $T_n^{(k)}$. It is well known that if a df F has pdf f, then $Y_n^{(k)}$ has pdf (cf. [2])

$$(1) \qquad f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\left[-\log(1-F(x))\right]^{n-1}[1-F(x)]^{k-1}f(x)$$

and the joint pdf of the vector $(Y_m^{(k)}, Y_n^{(k)})$ is of the form

$$^{(k)}f_{m,n}(x, y) = \frac{k^n}{(m-1)!\,(n-m-1)!}\left[-\log(1-F(x))\right]^{m-1}$$
$$\times \frac{f(x)}{1-F(x)}\left(\log\frac{1-F(x)}{1-F(y)}\right)^{n-m-1}[1-F(y)]^{k-1}f(y)$$

for $x < y$, and ${}^{(k)}f_{m,n}(x, y) = 0$ for $x \geqslant y$ (see [4]). The joint pdf of $(Y_n^{(k)}, Y_{n+1}^{(k)})$ is therefore

$$(2)\quad {}^{(k)}f_{n,n+1}(x, y) = \frac{k^{n+1}}{(n-1)!}[-\log(1-F(x))]^{n-1}\frac{f(x)}{1-F(x)}[1-F(y)]^{k-1}f(y).$$

Similarly, the random variable $Z_n^{(k)}$ has pdf

$$(3)\qquad f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}(-\log F(x))^{n-1}(F(x))^{k-1}f(x)$$

and the joint pdf of $(Z_m^{(k)}, Z_n^{(k)})$ is

$$(4)\qquad f_{m,n}^{(k)}(x, y) = \frac{k^n}{(m-1)!\,(n-m-1)!}(-\log F(x))^{m-1} \times\frac{f(x)}{F(x)}\left(\log\frac{F(x)}{F(y)}\right)^{n-m-1}(F(y))^{k-1}f(y)$$

for $x > y$, and $f_{m,n}^{(k)}(x, y) = 0$ for $x \leqslant y$ (see [6]), and the joint pdf of $(Z_n^{(k)}, Z_{n+1}^{(k)})$ is of the form

$$(5)\qquad f_{n,n+1}^{(k)}(x, y) = \frac{k^{n+1}}{(n-1)!}(-\log F(x))^{n-1}\frac{f(x)}{F(x)}(F(y))^{k-1}f(y)$$

for $x > y$.

In what follows we need the following statements concerning the distributions of the r.v.'s $D_n^{(k)}$, $T_n^{(k)}$ and $U_n^{(k)}$.

PROPOSITION 1. *The pdf of $D_n^{(k)}$ is of the form*

$$(6)\qquad f_{D_n^{(k)}}(u) = \frac{k^{n+1}}{(n-1)!}\int_{-\infty}^{\infty}(-\log F(u+v))^{n-1}\frac{f(u+v)}{F(u+v)}(F(v))^{k-1}f(v)\,dv$$

for $u > 0$ and $f_{D_n^{(k)}}(u) = 0$ for $u \leqslant 0$, and its df is

$$(7)\qquad F_{D_n^{(k)}}(z) = 1-\frac{k^n}{(n-1)!}\int_{-\infty}^{\infty}(-\log F(s))^{n-1}[F(s-z)]^k\frac{f(s)}{F(s)}ds$$

for $z \geqslant 0$.

Proof. Let us put $X = Z_n^{(k)}$ and $Y = Z_{n+1}^{(k)}$, where $n \geqslant 1$, $k \geqslant 1$. Let $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$g(x, y) = (x-y, y).$$

Then g maps the region $D = \{(x, y)\colon x > y\}$ onto $G = \{(u, v)\colon u > 0\}$. Moreover, for $(u, v) \in G$

$$g^{-1}(u, v) = (u+v, v)$$

and

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1.$$

Thus the pdf of the vector $(U, V) = g(X, Y) = (D_n^{(k)}, Z_{n+1}^{(k)})$ is

$$f_{UV}(u, v) = \begin{cases} f_{XY}(g^{-1}(u, v)), & u > 0,\ v \in \boldsymbol{R}, \\ 0, & u \leqslant 0,\ v \in \boldsymbol{R}, \end{cases}$$

where $f_{XY}(x, y) = f_{n,n+1}^{(k)}(x, y)$ is given by (5). Then

$$f_{D_n^{(k)}}(u) = f_U(u) = \int_{-\infty}^{+\infty} f_{UV}(u, v)\, dv.$$

Thus for $u > 0$, $v \in \boldsymbol{R}$,

$$f_{UV}(u, v) = \frac{k^{n+1}}{(n-1)!}(-\log(u+v))^{n-1}\frac{f(u+v)}{F(u+v)}(F(v))^{k-1} f(v)$$

and $f_{UV}(u, v) = 0$, $u < 0$, which implies (6). Furthermore,

$$\begin{aligned}
F_{D_n^{(k)}}(z) &= \int_0^z f_{D_n^{(k)}}(u)\, du \\
&= \frac{k^{n+1}}{(n-1)!}\int_0^z du \int_{-\infty}^{+\infty} (-\log F(u+v))^{n-1}\frac{f(u+v)}{F(u+v)}(F(v))^{k-1} f(v)\, dv \\
&= \frac{k^{n+1}}{(n-1)!}\int_0^z du \int_{-\infty}^{+\infty} (-\log F(s))^{n-1}\frac{f(s)}{F(s)}(F(s-u))^{k-1} f(s-u)\, ds \\
&= \frac{k^{n+1}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}\frac{f(s)}{F(s)} ds \int_0^z (F(s-u))^{k-1} f(s-u)\, ds \\
&= \frac{k^{n+1}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}\frac{f(s)}{F(s)} ds \int_{F(s-z)}^{F(s)} t^{k-1}\, dt \\
&= \frac{k^{n}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}\frac{f(s)}{F(s)}\{(F(s))^k - (F(s-z))^k\}\, ds \\
&= \frac{k^{n}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}(F(s))^{k-1} f(s)\, ds \\
&\quad - \frac{k^{n}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}(F(s-z))^{k}\frac{f(s)}{F(s)} ds \\
&= 1 - \frac{k^{n}}{(n-1)!}\int_{-\infty}^{+\infty} (-\log F(s))^{n-1}(F(s-z))^{k}\frac{f(s)}{F(s)} ds,
\end{aligned}$$

which completes the proof. ∎

The next two propositions can be proved similarly.

PROPOSITION 2. *Assume that $F(x) = 0$ for $x \leqslant 0$. Then the pdf of the r.v. $U_n^{(k)}$ is*

$$(8)\qquad f_{U_n^{(k)}}(u) = \frac{k^{n+1}}{(n-1)!}\int_0^\infty \frac{v}{u^2}\left[-\log\left(1-F(v/u)\right)\right]^{n-1} \times \frac{f(v/u)}{1-F(v/u)}\left[1-F(v)\right]^{k-1} f(v)\,dv$$

for $u \geqslant 1$ and $f_{U_n^{(k)}}(u) = 0$ for $u < 1$, and its df is of the form

$$(9)\qquad F_{U_n^{(k)}}(z) = 1-\frac{k^{n+1}}{n!}\int_0^\infty \left[-\log\left(1-F(v/z)\right)\right]^n \left(1-F(v)\right)^{k-1} f(v)\,dv$$

for $z \geqslant 1$ and $F_{U_n^{(k)}} = 0$ for $z < 1$.

PROPOSITION 3. *Assume that $F(x) = 0$ for $x \leqslant 0$. Then the pdf of the r.v. $T_n^{(k)}$ is*

$$(10)\qquad f_{T_n^{(k)}}(u) = \frac{k^{n+1}}{(n-1)!}\int_0^\infty v\left(-\log F(uv)\right)^{n-1}\frac{f(uv)}{F(uv)}\left(F(v)\right)^{k-1} f(v)\,dv$$

for $u \geqslant 1$ and $f_{T_n^{(k)}} = 0$ for $u < 1$, and its df is of the form

$$(11)\qquad F_{T_n^{(k)}}(z) = 1-\frac{k^{n+1}}{n!}\int_0^\infty \left(-\log F(vz)\right)^n \left(F(v)\right)^{k-1} f(v)\,dv$$

for $z \geqslant 1$ and $F_{T_n^{(k)}} = 0$ for $z < 1$.

Remark. Analogous propositions may be proved in the case when F has the support $S \subset \boldsymbol{R}$.

3. Examples. We now give some examples illustrating the limit theorems formulated in Sections 4, 5 and 6.

EXAMPLE 1. *Generalized extreme value distributions* [1].

Generalized extreme value distributions are defined by

$$(12)\qquad F(x) = \begin{cases} \exp\{-(1-\gamma x)^{1/\gamma}\}, & x < 1/\gamma,\ \gamma > 0,\\ \exp\{-(1-\gamma x)^{1/\gamma}\}, & x > 1/\gamma,\ \gamma < 0,\\ \exp(-e^{-x}), & x \in \boldsymbol{R},\ \gamma = 0,\end{cases}$$

and for such distributions

$$f(x) = (1-\gamma x)^{(1/\gamma)-1}\exp\{-(1-\gamma x)^{1/\gamma}\},$$

$$f(x)/F(x) = (1-\gamma x)^{(1/\gamma)-1}, \qquad -\log F(x) = (1-\gamma x)^{1/\gamma}.$$

If $\gamma > 0$, then $x < 1/\gamma$, and changing the variables in (6) to $t = 1-\gamma(u+v)$ we obtain

$$f_{D_n^{(k)}}(u) = \frac{k^{n+1}}{(n-1)!}\frac{1}{\gamma}\int_0^\infty t^{(n/\gamma)-1}\exp\{-k(t+\gamma u)^{1/\gamma}\}(t+\gamma u)^{(1/\gamma)-1}\,dt.$$

Putting $\gamma = 1$ we easily obtain

$$f_{D_n^{(k)}}(u) = ke^{-ku}, \quad u \geqslant 0.$$

Thus $D_n^{(k)}$ is exponentially distributed with df

$$F_n^{(k)}(u) = 1-e^{-ku},$$

and $V_n^{(k)} = kD_n^{(k)}$ has the df

$$F(x) = 1-e^{-x}, \quad x \geqslant 0.$$

Then of course the limiting distribution of $V_n^{(k)}$, as $k \to \infty$, is exponential.

Now consider the distribution given by (12) with $\gamma = 0$. This is the so-called *Gumbel distribution.* Now we have

$$f(x) = \exp(-e^{-x})e^{-x}, \quad x \in \boldsymbol{R},$$

$$f(x)/F(x) = e^{-x}, \quad -\log F(x) = e^{-x}.$$

Thus from (6) we obtain (after changing the variables to $t = e^{-v}$)

$$f_{D_n^{(k)}}(u) = ne^{-nu},$$

and $D_n^{(k)}$ has the df

$$F_{D_n^{(k)}}(z) = 1-e^{-nz}, \quad z \geqslant 0,$$

while $V_n^{(k)}$ has the df

$$F_k(z) = 1-e^{-nz/k} \to 0, \quad k \to \infty.$$

Therefore in this case the limit of the sequence $\{F_k, k \geqslant 1\}$ is not a proper probability distribution; it may be considered as an improper distribution concentrated at infinity.

EXAMPLE 2. *The Fréchet distribution.*

Let us consider now the *Fréchet distribution* with df

$$F(x) = \begin{cases} \exp(-1/x^\alpha), & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

where $\alpha > 0$. Then

$$f(x) = \frac{\alpha}{x^{\alpha+1}}\exp(-1/x^\alpha),$$

$$f(x)/F(x) = \alpha/x^{\alpha+1}, \quad -\log F(x) = 1/x^\alpha.$$

Thus, using (11) and making the substitution $t = 1/x^{\alpha}$ we obtain for $x \geqslant 1$

$$F_{T_n^{(k)}}(x) = 1 - x^{-n\alpha}.$$

It follows that $Q_n^{(k)} = n(T_n^{(k)} - 1)$ has the df

$$F_{R_n^{(k)}}(x) = F_{T_n^{(k)}}(1 + x/n) = 1 - \frac{1}{(1 + x/n)^{n\alpha}} \to 1 - e^{-\alpha x}, \qquad n \to \infty,$$

and the limit distribution of $Q_n^{(k)}$, $n \to \infty$, is again exponential.

EXAMPLE 3. *Exponential distribution.*

Assume that X_i are i.i.d. r.v.'s with df

$$F(u) = \begin{cases} 1 - \exp\{-(u-\mu)/\lambda\}, & u \geqslant \mu, \\ 0, & u < \mu. \end{cases}$$

Then

$$-\log(1 - F(u)) = (u-\mu)/\lambda, \qquad f(u) = [1 - F(u)]/\lambda.$$

Then Proposition 2 implies that for $z \geqslant 1$

$$F_{U_n^{(k)}}(z) = 1 - \frac{k^{n+1}}{n!} \int_{\mu}^{\infty} \left(\frac{v/z - \mu}{\lambda}\right)^n \exp(-k(v-\mu)/\lambda) \frac{1}{\lambda} dv,$$

which after the change of variables $t = \lambda^{-1}(v/z - \mu)$ gives

$$F_{U_n^{(k)}}(z) = 1 - z^{-n} \exp(-k\mu(z-1)/\lambda).$$

Thus, for $z \geqslant 0$,

$$\begin{aligned} F_{R_n^{(k)}}(z) &= F_{U_n^{(k)}}(1 + z/n) \\ &= 1 - \frac{1}{(1+z/n)^n} \exp\left(-\frac{k\mu z}{n\lambda}\right) \to 1 - e^{-z}, \qquad n \to \infty. \end{aligned}$$

4. Limiting distributions of the random variables $V_n^{(k)}$, $k \to \infty$. The theorems in this section, concerning the k-th lower record values $Z_n^{(k)}$, are counterparts of theorems formulated in [3].

THEOREM 1. *Suppose that X_i have df F and pdf f, with the interval $S \subset \boldsymbol{R}$ as the support, and that $q(x) = f(x)/F(x)$ is a differentiable function with bounded first derivative. Moreover, assume that $\{F_k, k \geqslant 1\}$ is a sequence of distribution functions of the form*

$$F_k(z) = \begin{cases} 1 - \int_S \left(\frac{F(v - z/k)}{F(x)}\right)^k dG_k(v) & \text{for } z \geqslant 0, \\ 0 & \text{for } z < 0, \end{cases} \tag{13}$$

where $\{G_k, k \geqslant 1\}$ *is a sequence of distribution functions such that*

$$G_k \to G, \quad k \to \infty, \tag{14}$$

and G *is a distribution concentrated at a point* $x_0 \in \partial S$. *Then*

$$F_k \to F_\mu, \quad k \to \infty,$$

where

$$F_\mu(z) = 1 - \exp(-\mu z) \quad \text{for } z > 0,$$

and

$$\mu = \begin{cases} q(x_0^-) & \text{if } x_0 = \sup S, \\ q(x_0^+) & \text{if } x_0 = \inf S. \end{cases} \tag{15}$$

Proof. Put $g(x) = \log F(x)$. Then using Taylor's formula and the differentiability of q we obtain for $z \geqslant 0$

$$\log \frac{F(v-z/k)}{F(v)} = \log F(v-z/k) - \log F(v) = g(v-z/k) - g(v)$$

$$= g'(v)\left(-\frac{z}{k}\right) + \frac{1}{2} g''\left(v - \frac{\theta z}{k}\right)\left(-\frac{z}{k}\right)^2 = -q(v)\frac{z}{k} + q'\left(v - \frac{\theta z}{k}\right)\frac{z^2}{2k^2},$$

where $0 < \theta < 1$. Thus

$$1 - F_k(z) = \int_S \exp\left\{q'\left(v - \frac{\theta z}{k}\right)\frac{z^2}{2k}\right\} \exp(-q(x)z)\, dG_k(v). \tag{16}$$

Define

$$H_k(z) = \int_S \exp(-q(v)z)\, dG_k(v). \tag{17}$$

Now the assumption $|q'(x)| \leqslant M$ for $x \in S$ and (16) together imply that

$$H_k(z) \exp\left\{-\frac{Mz^2}{2k}\right\} \leqslant 1 - F_k(z) \leqslant \exp\left\{\frac{Mz^2}{2k}\right\} H_k(z). \tag{18}$$

From (14) and (17) we get

$$H_k(z) = E \exp(-q(Y_k)z) \to \exp(-q(x_0)z), \quad k \to \infty, \tag{19}$$

where the random variable Y_k has the df G_k. Thus (18) and (19) imply that

$$F_k(z) \to 1 - \exp(-\mu z), \quad k \to \infty,$$

where μ is given by (15). From (13) it follows that $F_k(z) \to 0$ if $z \leqslant 0$, which completes the proof. ∎

THEOREM 2. *Suppose that F, f and q are as in Theorem 1. Then*

$$V_n^{(k)} \xrightarrow{D} V_n, \quad k \to \infty, \tag{20}$$

where V_n has the exponential distribution with df

$$F_\mu(x) = 1-\exp(-\mu x),$$

and $\mu = q(x_0^-)$ and $x_0 = \sup S$.

Proof. By Proposition 1, the distribution function of $D_n^{(k)}$ may be rewritten as

$$F_{D_n^{(k)}}(z) = 1-\int_S \left(\frac{F(v-z)}{F(v)}\right)^k dG_k(v),$$

where

$$G_k(x) = \frac{k^n}{(n-1)!} \int_{-\infty}^{x} (-\log F(y))^{n-1} (F(y))^{k-1} f(y)\, dy$$

is the df of $Z_n^{(k)}$. Therefore $V_n^{(k)}$ has the df

$$F_{V_n^{(k)}}(z) = 1-\int_S \left(\frac{F(v-z/k)}{F(v)}\right)^k dG_k(v).$$

Since

$$G_k(x) = \frac{1}{(n-1)!} \int_{-k\log F(x)}^{\infty} u^{n-1} e^{-u}\, du,$$

condition (14) is valid with $x_0 = \sup S$. Thus Theorem 1 implies that

$$F_{V_n^{(k)}} \to F_n, \quad k \to \infty,$$

where

$$F_n(x) = 1-\exp(-\mu x), \quad x \geqslant 0,$$

which is equivalent to (20). ∎

5. Limiting distributions of the random variables $R_n^{(k)}$, $n \to \infty$.

THEOREM 3. *Suppose that X_i have df F and pdf f, with the interval $S \subset [0, \infty)$ as the support, and that*

$$q(x) = \frac{1}{-\log(1-F(x))} \frac{f(x)}{1-F(x)}$$

is a differentiable function satisfying the condition

$$|x^2 q'(x)| \leqslant M \quad \textit{for } x \in S. \tag{21}$$

Moreover, assume that $\{F_n, n \geqslant 1\}$ *is a sequence of distribution functions of the form*

$$F_n(z) = \begin{cases} 1 - \int\limits_S \left(\dfrac{\log\left(1 - F\left(v/(1+z/n)\right)\right)}{\log\left(1 - F(v)\right)} \right)^n dG_n(v), & z \geqslant 1, \\ 0, & z < 1, \end{cases} \tag{22}$$

where $\{G_n, n \geqslant 1\}$ *is a sequence of distribution functions such that*

$$G_n \to G, \quad n \to \infty, \tag{23}$$

and G *is a distribution concentrated at a point* $x_0 \in \partial S$. *Then*

$$F_n \to F_\mu, \quad n \to \infty,$$

where

$$F_\mu(z) = 1 - \exp(-\mu z) \quad \textit{for } z > 0$$

and

$$\mu = \begin{cases} \lim_{x \to x_0^-} xq(x) & \textit{if } x_0 = \sup S, \\ \lim_{x \to x_0^+} xq(x) & \textit{if } x_0 = \inf S. \end{cases} \tag{24}$$

Proof. Let $g(x) = \log\left[-\log\left(1 - F(x)\right)\right]$. Then using Taylor's formula and the differentiability of q we obtain for $z \geqslant 0$

$$\log \frac{\log\left(1 - F\left(v/(1+z/n)\right)\right)}{\log\left(1 - F(v)\right)} = -q(v)\frac{vz}{n+z} + q'\left(\frac{v}{1+\theta z/n}\right)\left(\frac{vz}{n+z}\right)^2,$$

where $0 < \theta < 1$. Thus

$$1 - F_n(z) = \int\limits_S \exp\left\{ q'\left(\frac{v}{1+\theta z/n}\right)\frac{n(vz)^2}{(n+z)^2} \right\} \exp\left(-q(v)\frac{nvz}{n+z}\right) dG_n(v). \tag{25}$$

Define

$$H_n(z) = \int\limits_S \exp\left(-q(v)\frac{nvz}{n+z}\right) dG_n(v). \tag{26}$$

From assumption (21) we obtain

$$\left| v^2 q'\left(\frac{v}{1+\theta z/n}\right) \right| \leqslant M\left(1 + \frac{\theta z}{n}\right)^2,$$

which together with (25) implies

$$(27)\quad H_n(z)\exp\left\{-\frac{Mz}{2n}\left(\frac{n+\theta z}{n+z}\right)\right\} \leqslant 1-F_n(u) \leqslant \exp\left\{\frac{Mz}{2n}\left(\frac{n+\theta z}{n+z}\right)\right\} H_n(z).$$

Note that

$$H_n(z) = E\exp(-Z_n), \quad \text{where } Z_n = Y_n q(Y_n)\frac{nz}{n+z},$$

and $\{Y_n, n \geqslant 1\}$ is a sequence of random variables such that Y_n has the df G_n. The convergence $Y_n \xrightarrow{D} x_0$, as $n \to \infty$, and (one-sided) continuity of the function $v \mapsto vq(v)$ at x_0 together imply that $Z_n \xrightarrow{D} \mu z$, $n \to \infty$, and we obtain

$$(28)\quad H_n(z) \to \exp(-\mu z), \quad n \to \infty.$$

Thus (27) and (28) imply that

$$F_n(z) \to 1-\exp(-\mu z), \quad n \to \infty,$$

where μ is given by (24). It follows that $F_k(z) \to 0$ for $z \leqslant 0$, which completes the proof of the theorem. ■

THEOREM 4. *Suppose that F, f and q are as in Theorem 3. Then*

$$(29)\quad R_n^{(k)} \xrightarrow{D} R^{(k)}, \quad n \to \infty,$$

where $R^{(k)}$ has an exponential distribution

$$F_\mu(x) = 1-\exp(-\mu x)$$

and $\mu = \lim_{x \to x_0^-} xq(x)$ and $x_0 = \sup S$.

Proof. Using Proposition 2 we may write the df of $U_n^{(k)}$ as

$$F_{U_n^{(k)}}(z) = 1-\int_S \left(\frac{\log(1-F(v/z))}{\log(1-F(v))}\right)^n dG_n(v),$$

where

$$G_n(x) = \frac{k^{n+1}}{n!}\int_0^x [-\log(1-F(y))]^n (1-F(y))^{k-1} f(y)\,dy$$

is the df of $Y_{n+1}^{(k)}$. Therefore $R_n^{(k)}$ has the df

$$F_{R_n^{(k)}}(z) = 1-\int_S \left(\frac{\log(1-F(v/(1+z/n)))}{\log(1-F(v))}\right)^n dG_n(v).$$

To prove that (23) is satisfied, we use the formula (cf. [5])

$$(30)\quad F_{Y_n^{(k)}}(x) = 1-(1-F(x))^k \sum_{i=0}^{n-1} \frac{k^i}{i!}[-\ln(1-F(x))]^i.$$

If $x_0 = \sup S$, then for $x < x_0$

$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} F_{Y_{n+1}^{(k)}}(x) = 1-(1-F(x))^k \sum_{i=0}^{\infty} \frac{k^i}{i!}[-\ln(1-F(x))]^i$$
$$= 1-(1-F(x))^k \exp\{-k\ln(1-F(x))\} = 0,$$

and $G_n(x) = 1$ for $x \geqslant x_0$. Theorem 3 implies now that

$$F_{R_n^{(k)}} \to F_k, \quad n \to \infty,$$

where

$$F_k(x) = 1-\exp(-\mu x), \quad x \geqslant 0,$$

which is equivalent to (29). ∎

6. Limiting distributions of the random variables $Q_n^{(k)}$, $n \to \infty$.

THEOREM 5. *Suppose that X_i have df F and pdf f, with the interval $S \subset [0, \infty)$ as the support, and that*

$$p(x) = -\frac{f(x)}{F(x)\log F(x)}$$

is a differentiable function such that

$$|x^2 p'(x)| \leqslant M \tag{31}$$

for $x \in S$. Moreover, assume that $\{F_k, k \geqslant 1\}$ is a sequence of distribution functions of the form

$$F_n(z) = \begin{cases} 1-\int_0^{\infty} \left(\frac{\log F(v(1+z/n))}{\log F(v)}\right) dG_n(v), & z \geqslant 0, \\ 0, & z < 0, \end{cases} \tag{32}$$

where $\{G_n, n \geqslant 1\}$ is a sequence of distribution functions such that

$$G_n \to G, \quad n \to \infty, \tag{33}$$

and G is a distribution concentrated at a point $x_0 \in \partial S$. Then

$$F_n \to F_\mu, \quad n \to \infty, \tag{34}$$

where

$$F_\mu(z) = 1-\exp(-\mu z) \quad \textit{for } z > 0$$

and

$$\mu = \begin{cases} \lim_{x\to x_0^-} xp(x) & \textit{if } x_0 = \sup S, \\ \lim_{x\to x_0^+} xp(x) & \textit{if } x_0 = \inf S. \end{cases} \tag{35}$$

Proof. Put $g(v) = \log(-\log F(v))$. Then

$$g'(x) = \frac{f(x)}{F(x)\log F(x)} = -p(x)$$

and, by Taylor's formula,

$$\log\frac{\log F(v(1+z/n))}{\log F(v)} = -p(v)\frac{vz}{n} - \frac{1}{2}v^2 p'(v(1+\theta z/n))(z/n)^2.$$

Therefore, using (32) we obtain

$$(36) \qquad 1-F_n(z) = \int_0^{\infty} \exp\{-vp(v)z\}\exp\left\{-v^2 p'(v(1+\theta z/n))\frac{z^2}{2n}\right\} dG_n(v)$$

and, by (31),

$$(37) \qquad |v^2 p'(v(1+\theta z/n))| = \left|\frac{1}{(1+\theta z/n)^2} v^2(1+\theta z/n)^2 p'(v(1+\theta z/n))\right| \leqslant M\left(\frac{n}{n+\theta z}\right)^2.$$

Define

$$H_n(z) = \int_0^{\infty} \exp\{-vp(v)z\}\, dG_n(v).$$

Then from (36) and (37) it follows that

$$(38) \qquad \exp\left\{-\frac{Mnz^2}{2(n+\theta z)^2}\right\} H_n(z) \leqslant 1-F_n(z) \leqslant \exp\left\{\frac{Mnz^2}{2(n+\theta z)^2}\right\} H_n(z).$$

Since from (33) and (35) we get

$$H_n(z) \to \exp(-\mu z), \qquad n \to \infty,$$

inequality (38) implies (34). ∎

THEOREM 6. *Suppose that F, f and p are as in Theorem 5. Then*

$$(39) \qquad Q_n^{(k)} \xrightarrow{D} Q^{(k)}, \qquad k \to \infty,$$

where Q_n has an exponential distribution

$$F_\mu(x) = 1-\exp(-\mu x), \qquad x \geqslant 0,$$

and $\mu = \lim_{x\to x_0^+} xp(x)$ and $x_0 = \inf S$.

Proof. Using Proposition 3 we may write the df of $T_n^{(k)}$ as

$$F_{T_n^{(k)}}(z) = 1 - \int_S \left(\frac{\log F(vz)}{\log F(v)}\right)^n dG_n(v),$$

where

$$G_n(x) = \frac{k^{n+1}}{n!} \int_0^x (-\log F(y))^n (F(y))^{k-1} f(y)\, dy$$

is the df of $Z_{n+1}^{(k)}$. Therefore $Q_n^{(k)}$ has the df

$$F_{Q_n^{(k)}}(z) = 1 - \int_S \left(\frac{\log F(v(1+z/n))}{\log F(v)} \right) dG_n(v).$$

Analogously to (30) we show that

$$F_{Z_n^{(k)}}(x) = (F(x))^k \sum_{i=0}^{n-1} \frac{k^i}{i!} (-\ln F(x))^i.$$

Thus for $x \geqslant x_0$

$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} F_{Z_{n+1}^{(k)}}(x) = (F(x))^k \sum_{i=0}^{\infty} \frac{k^i}{i!} (-\ln F(x))^i$$
$$= (F(x))^k \exp\{-k \ln F(x)\} = 1$$

and condition (33) is valid with $x_0 = \inf S$. Theorem 5 implies that

$$F_{R_n^{(k)}} \to F_k, \qquad n \to \infty,$$

where

$$F_k(x) = 1 - \exp(-\mu x), \qquad x \geqslant 0,$$

which is equivalent to (39). ∎

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