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CONVERGENCE RATES IN THE STRONG LAW FOR ASSOCIATED RANDOM VARIABLES

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Abstract. We prove the Marcinkiewicz–Zygmund SLLN (MZ-SLLN) of order p, $p \in [1, 2[$, for associated sequences, not necessarily stationary. Our assumption on the moment of the random variables is minimal. We present an example of an associated and strongly mixing sequence, with infinite variance, to which our results apply. The conditions yielding such results for this example are discussed.

1. INTRODUCTION AND NOTATION

Let $(X_n)_{n \in N}$ be a sequence of *associated* random variables (r.v.'s) (as defined by Esary et al. [7]), i.e. for every finite subcollection $X_{i_1}, \ldots, X_{i_n}$ and every pair of coordinatewise non-decreasing functions $h, k: \boldsymbol{R}^n \to \boldsymbol{R}$

$$\operatorname{Cov}\big(h(X_{i_1}, \ldots, X_{i_n}), k(X_{i_1}, \ldots, X_{i_n})\big) \geqslant 0$$

whenever the covariance is defined. Define $S_n = \sum_{i=1}^{n} (X_i - EX_i)$ with the convention that $S_0 = 0$. Let p be a fixed real number in $[1, 2[$.

Our main purpose in the present note is to study the problem of the almost sure convergence of $n^{-1/p} S_n$. This problem is known as the Marcinkiewicz–Zygmund Strong Law of Large Numbers (MZ-SLLN). The case $p = 1$ is known as the Strong Law of Large Numbers (SLLN). An extreme case of association is the independence (cf. $(\mathscr{P}_4)$ of Esary et al. [7]). When the associated sequence $(X_n)_{n \in N}$ consists of i.i.d. r.v.'s, Baum and Katz [1] showed that the relations

$$(1) \qquad E|X_1|^p < \infty \quad \text{and} \quad \lim_{n \to \infty} n^{-1/p} S_n = 0 \text{ almost surely}$$

are equivalent.

As far as we know, for dependent sequences of associated r.v.'s there are two results that concern only the SLLN. Let us recall them briefly.

SLLN for associated sequences. The first result about this problem is in Newman [10]:

For a strictly stationary and associated sequence $(X_n)_{n\in\mathbf{N}}$ *with finite variance, the SLLN follows from the ergodic theorem if*

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\operatorname{Cov}(X_1, X_j) = 0.$$

Next Birkel [2] obtained an SLLN for associated (not necessarily stationary) sequences under a condition close to that of Kolmogorov's classical SLLN for independent r.v.'s. He assumed that *the associated sequence* $(X_n)_{n\in\mathbf{N}}$ *which has finite variance fulfills*

$$\sum_{n=1}^{\infty} n^{-2}\operatorname{Cov}(X_n, S_n) < \infty.$$

According to those results, the SLLN for associated sequences requires the existence of second moments, since the covariance structure is used to describe the asymptotic properties of the process. However, in some cases, the second moments do not exist, especially for the case of stable sequences.

A natural question in this context is then whether the (MZ-)SLLN holds for associated sequences for which *the second moment is not assumed to be finite* but rather the moment of order p ($p\in[1, 2[$). The main goal of this paper is to provide such a result for associated sequences. So, as in Dąbrowski and Jakubowski [5], a principal task is to find a suitable weak dependence coefficient defined for associated sequences with infinite variance.

There have been a great number of papers concerning the rates of convergence in the SLLN for weakly dependent r.v.'s (cf., for example, Chandra and Ghosal [4], Shao [16] and the references therein). For strongly mixing sequences, the above problem is completely solved (cf. Rio [13]). Strongly mixing coefficients (as defined by Rosenblatt [14]) refer more to σ-algebra than to random variables. Strongly mixing coefficients have explicit upper bounds for Markov chains or linear processes (cf. Doukhan [6] and the references therein). By contrast, the dependence structure for associated sequences appears only through the covariance quantities which are much easier to compute than the mixing coefficients.

Examples in Wood [17] and in Louhichi [9] prove that mixing and association define two different classes of processes. Thus one may expect that the technical difficulties that arise during manipulating association are different from those that concern strongly mixing sequences.

Let us now describe the methods and the contents of the paper. Recall that our main task is to find a suitable covariance quantity for associated sequences, with infinite variance, for which the MZ-SLLN holds. For this, let

$$\Delta_{i,i+r}(x, y) := \mathbf{P}(X_i \geqslant x,\ X_{i+r} \geqslant y) - \mathbf{P}(X_i \geqslant x)\,\mathbf{P}(X_{i+r} \geqslant y). \tag{2}$$

It follows from the association property that $\Delta_{i,i+r}(x, y) \geqslant 0$ for all x, y in $\boldsymbol{R}$. For fixed v in $\boldsymbol{R}^+$, we define the non-decreasing real-valued function g_v by

$$g_v(u) := (u \wedge v) \vee (-v). \tag{3}$$

Finally, let $G_{i,i+r}(v)$ denote the following covariance quantity:

$$G_{i,i+r}(v) := \operatorname{Cov}\big(g_v(X_i),\, g_v(X_{i+r})\big).$$

Let us note that $G_{i,i+r}(v)$ is well defined, bounded by v^2 and positive (this follows from association). Moreover, if the r.v.'s $(X_i)_i$ have finite variance, then

$$G_{i,i+r}(\infty) = \operatorname{Cov}(X_i, X_{i+r}),$$

as can be seen from the equality

$$G_{i,i+r}(v) = \int_{|x| \leqslant v} \int_{|y| \leqslant v} \Delta_{i,i+r}(x, y)\, dx dy.$$

In Section 2, we prove (cf. Theorem 1) that (1) holds for associated sequences having a finite moment of order p under the summability condition

$$\sum_{1 \leqslant i < j < \infty} \int_{j^{1/p}}^{+\infty} v^{-3} G_{i,j}(v)\, dv < \infty. \tag{4}$$

Note that the above integral expression is finite as soon as the r.v.'s have a finite moment of order p (cf. Lemma 5).

All the involved assumptions are stated in terms of the covariance quantity $G_{i,j}(v)$. Those coefficients $G_{i,j}(v)$ are explicitly evaluated for moving average with innovations having a stable distribution. We check the mixing property for those linear processes (cf. Appendix). In particular, we compare the conditions yielding the MZ-SLLN for those linear processes using first their association properties and next their mixing ones (cf. Section 3). The proofs are given in Section 4.

2. RESULTS: MARCINKIEWICZ–ZYGMUND STRONG LAW OF LARGE NUMBERS

The following theorem gives rate of convergence in the SLLN for associated sequences. The moments of order two of the r.v.'s are not assumed to be finite.

THEOREM 1. *Let p be a fixed real number in $[1, 2[$. Let $(X_n)_{n \in N}$ be a sequence of associated r.v.'s. Suppose that there exists a positive r.v. X such that $EX^p < \infty$ and that $\sup_i P(|X_i| > x) \leqslant P(X > x)$ for any positive x. If*

$$\sum_{1 \leqslant i < j < \infty} \int_{j^{1/p}}^{+\infty} v^{-3} G_{i,j}(v)\, dv < \infty, \tag{5}$$

then $\lim_{n \to \infty} n^{-1/p} S_n = 0$ almost surely.

Remark. For pairwisely identically distributed and associated sequences, condition (5) holds as soon as

$$\sum_{r=1}^{\infty} \int_{(r+1)^{1/p}}^{+\infty} v^{p-3} G_{0,r}(v)\, dv < \infty. \tag{6}$$

For random variables with finite variance, Theorem 1 yields the following result, providing rate of convergence in the SLLN of Birkel [2].

COROLLARY 1. *Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s that fulfills the requirement of Theorem 1. Suppose moreover that $(X_n)_{n\in N}$ has finite variance. If*

$$\sum_{j=1}^{\infty} j^{-2/p} \operatorname{Cov}(X_j, S_{j-1}) < \infty \quad \textit{for some } p \in [1, 2[,$$

then $\lim_{n\to\infty} n^{-1/p} S_n = 0$ almost surely.

3. EXAMPLE: LINEAR SEQUENCES WITH STABLE INNOVATIONS

We shall derive here the almost sure limiting behavior of $n^{-1/p} S_n$ of a moving average process, with infinite variance, defined by

$$X_n = \sum_{i\geq 0} a_i \varepsilon_{n-i}, \tag{7}$$

where the ε_n's are independent and stable $(\sigma, \alpha, 0)$ with $\alpha \in]1, 2[$. The constants a_i defined by (7) are assumed to satisfy

$$\gamma_n^{\alpha} := \sum_{i\geq n} |a_i|^{\alpha} \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Recall that a random variable is *stable* $(\sigma, \alpha, 0)$ if it has the characteristic function $\phi(u) = \exp\{-\sigma^{\alpha}|u|^{\alpha}\}$ with $\sigma \geq 0$, $\alpha \in]0, 2[$. Recall also that a stable $(\sigma, \alpha, 0)$ r.v. X fulfills $E|X|^p < \infty$ for any $p \in]0, \alpha[$ and $E|X|^p = \infty$ if $p \geq \alpha$ (cf. Property 1.2.16 in [15]). The stationary sequence $(X_n)_{n\in Z}$ (cf. (7)) is defined if and only if (8) holds (cf. Leadbetter et al. [8]). Moreover, X_0 is stable $\big(\sigma(\sum_{i\geq 0}|a_i|^{\alpha})^{1/\alpha}, \alpha, 0\big)$. Hence X_0 has a finite moment of order p if and only if $p < \alpha$. We note, in particular, that the moment of order two of X_0 does not exist.

Under some additional conditions on the weights $(a_i)_{i\in N}$ the requirements of Theorem 1 are fulfilled.

COROLLARY 2. *Let $\sigma > 0$, $\alpha \in]1, 2[$ and $p \in [1, \alpha[$ be fixed. Let $(\varepsilon_n)_{n\in Z}$ be a sequence of i.i.d. random variables with a stable $(\sigma, \alpha, 0)$ marginal distribution.*

Let $(a_i)_{i\in N}$ *be a sequence of real numbers fulfilling* (8). *If*

$$(9)\qquad \sum_{n>0} n^{1-2/p+\beta/p}\gamma_n < \infty \quad \textit{for some } \beta\in\,]2-\alpha,\, 2-p[,$$

then the assumptions of Theorem 1 *hold and, in particular, MZ-SLLN holds for the sequence* $(X_n)_{n\in Z}$ *defined by* (7).

Remark. Let $(X_n)_{n\in Z}$ be the linear process defined as in (7). Suppose that

$$(10)\qquad |a_i| = \mathcal{O}(i^{-a}) \text{ for some } a>1$$

$$\text{and} \quad \sum_{k=0}^{\infty} a_k z^k \neq 0 \text{ for all } z\in C \text{ with } |z|\leqslant 1.$$

Then the following properties hold:

1. According to Corollary 2, the sequence $(X_n)_{n\in Z}$ satisfies the MZ-SLLN as soon as (9) holds, i.e., if

$$(11)\qquad a > (2+1/\alpha-\alpha/p)\vee 1.$$

2. The sequence $(X_n)_{n\in Z}$ is *strongly mixing* (s.m.). In order to prove this property it suffices to check the conditions of Pham and Tran [12]: provided that (10) holds, the sequence $(X_n)_{n\in Z}$ is s.m. if the density g of ε_1 satisfies

$$(12)\qquad \int_{-\infty}^{\infty} |g(x+u)-g(x)|\,dx \leqslant Cu$$

for some positive constant C and for all $u\in R^+$ (we check (12) in the Appendix).

Hence Theorem 2.1 in Pham and Tran [12] bounds the strong mixing coefficients α_n of this linear sequence as follows:

$$\alpha_n \leqslant C\sum_{j\geqslant n}\Big(\sum_{k\geqslant j} a_k\Big)^{\delta/(1+\delta)} \leqslant Cn^{1+(1-a)\delta/(1+\delta)},$$

where δ is some real number in $]0, \alpha[$.

For s.m. sequences Rio [13] gives an optimal condition yielding the MZ-SLLN of order $p\in[1, 2[$:

$$(13)\qquad \sum_{i\geqslant 0}(i+1)^{p-2}\int_0^{\alpha_i} Q^p(u)\,du < \infty,$$

where Q denotes the inverse of the tail function $t\to P(X\geqslant t)$ (the r.v. X is defined in Theorem 1). We recall that the stationary sequence defined as in (7) has a stable $\big(\sigma(\sum_{i\geqslant 0}|a_i|^{\alpha})^{1/\alpha}, \alpha, 0\big)$ marginal distribution. It follows then from Property 1.2.15 in [15] that $P(|X_1|\geqslant t) = \mathcal{O}(t^{-\alpha})$. Hence

$$Q(u)\leqslant Cu^{-1/\alpha},$$

and condition (13) is satisfied as soon as

(14) $a > 1+p(\alpha-1)/(\alpha-p)+p(\alpha-1)/\delta(\alpha-p)$ for some $\delta\in]0, \alpha[$.

Clearly, condition (11) improves on (14).

4. PROOFS

4.1. Main tools. Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s. Let $S_n^* = \sup_{k\in[0,n]} S_k$. The following lemma is the main tool for the proofs:

LEMMA 1. *Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s. Suppose that there exist a positive r.v. X such that for any positive x*

$$\sup_{i>0} P(|X_i| > x) \leqslant P(X > x).$$

Then for any positive real numbers x and M

$$P(S_n^* > x) \leqslant \frac{4n}{x^2} EX^2 \mathbf{1}_{X\leqslant M} + \frac{4n}{x} EX\mathbf{1}_{X>M} + \frac{4nM^2}{x^2} P(X>M) + \frac{8}{x^2} \sum_{1\leqslant i<j\leqslant n} G_{i,j}(M).$$

The proof of Lemma 1 is based on the following lemmas well known in the literature.

LEMMA 2 (Newman and Wright [11]). *If (X_n) is a sequence of associated, mean zero, finite variance real-valued r.v.'s, then*

$$E(S_n^*)^2 \leqslant \operatorname{Var} S_n. \tag{15}$$

LEMMA 3 (Yu [18]). *Let (X_1, X_2) be a random vector. Let f and g be two absolutely continuous real-valued functions satisfying $Ef^2(X_1)+Eg^2(X_2) < +\infty$. We have*

$$\operatorname{Cov}\big(f(X_1), g(X_2)\big) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f'(x)g'(y)\operatorname{Cov}(\mathbf{1}_{X_1\geqslant x}, \mathbf{1}_{X_2\geqslant y})\,dxdy.$$

Proof of Lemma 1. Define, for fixed $M > 0$, the random variables

$$\bar{X}_i := \bar{X}_{i,M} = (X_i \wedge M)\vee(-M) \quad \text{and} \quad \tilde{X}_i := \tilde{X}_{i,M} = X_i - \bar{X}_i.$$

Let $\bar{S}_n = \sum_{i=1}^n (\bar{X}_i - E\bar{X}_i)$ and $\bar{S}_n^* = \sup_{k\in[0,n]} \bar{S}_k$. Clearly,

$$S_n^* \leqslant \bar{S}_n^* + \sum_{i=1}^n (|\tilde{X}_i| + E|\tilde{X}_i|).$$

Hence Markov's inequality yields

$$P(S_n^* > x) \leqslant P(\bar{S}_n^* > x/2) + \frac{4}{x}\sum_{i=1}^{n} E|\tilde{X}_i| \leqslant \frac{4}{x^2} E\bar{S}_n^{*2} + \frac{4}{x}\sum_{i=1}^{n} E|\tilde{X}_i|. \tag{16}$$

In order to bound $E\bar{S}_n^{*2}$ note that the sequence $(\bar{X}_i)_i$ is associated, since it is a non-decreasing function of the original associated sequence $(X_i)_i$ (cf. $(\mathscr{P}_4)$ in [7]). Hence the maximal inequality stated in (15) applied to the sequence $(\bar{X}_i - E\bar{X}_i)_i$, together with (16), yields

$$\begin{aligned} P(S_n^* \geqslant x) &\leqslant \frac{4}{x^2} E\bar{S}_n^2 + \frac{4}{x}\sum_{i=1}^{n} E|\tilde{X}_i| \\ &\leqslant \frac{4}{x^2}\sum_{i=1}^{n} E\bar{X}_i^2 + \frac{8}{x^2}\sum_{1 \leqslant i<j \leqslant n} \operatorname{Cov}(\bar{X}_i, \bar{X}_j) + \frac{4}{x}\sum_{i=1}^{n} E|\tilde{X}_i|. \end{aligned}$$

The above inequality yields Lemma 1, since

$$E\bar{X}_i^2 \leqslant EX^2 \mathbf{1}_{X \leqslant M} + M^2 P(X > M),$$

$$E|\tilde{X}_i| \leqslant E(|X_i| - M)\mathbf{1}_{|X_i| > M} \leqslant EX\mathbf{1}_{X > M}.$$

In the following, we give a technical lemma that will be useful in the sequel. Define, for some $p \in [1, 2[$, $i, j \geqslant 1$ and $s \geqslant 0$, the quantity

$$A_s(i, j, p) := \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x| \vee |y| \vee s^{1/p})^{-2} \Delta_{i,j}(x, y)\, dx dy. \tag{17}$$

The forthcoming lemma establishes the relation between $A_s(i, j, p)$ and the covariance quantity $G_{i,j}(v)$ whenever $s \geqslant 1$.

LEMMA 4. *Let $p \in [1, 2[$, $i, j \geqslant 1$ and $s \geqslant 1$ be fixed. Then*

$$A_s(i, j, p) = 2\int_{s^{1/p}}^{+\infty} v^{-3} G_{i,j}(v)\, dv.$$

Proof. Let $p \in [1, 2[$, $s \geqslant 1$ be fixed. Clearly,

$$(|x| \vee |y| \vee s^{1/p})^{-2} = \int_0^1 \mathbf{1}_{|x| \leqslant u^{-1/2}} \mathbf{1}_{|y| \leqslant u^{-1/2}} \mathbf{1}_{u \leqslant s^{-2/p}}\, du.$$

The last equality, together with Fubini's theorem, yields

$$A_s(i, j, p) = \int_0^1 \mathbf{1}_{u \leqslant s^{-2/p}} G_{i,j}(u^{-1/2})\, du,$$

and proves Lemma 4.

From Lemma 4 we deduce that the integral expression in (5) is well defined as soon as the r.v.'s have finite moment of order p, as it is shown by the forthcoming lemma.

LEMMA 5. *Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s. If $\sup_i E|X_i|^p < \infty$ for some p in $[1, 2[$, then for all positive integers i, j*

$$\int_1^{+\infty} v^{-3} G_{i,j}(v)\,dv \leqslant \frac{2}{p^2} \sup_i E|X_i|^p.$$

Proof. Define $\mathrm{Sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$. Lemmas 4 and 3 and some elementary bounds yield

$$2\int_1^{+\infty} v^{-3} G_{i,j}(v)\,dv \leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x|^{-2} \wedge |y|^{-2} \wedge 1)\,\Delta_{i,j}(x, y)\,dx dy$$
$$\leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x|^{p/2-1}|y|^{p/2-1}\Delta_{i,j}(x, y)\,dx dy$$
$$= \frac{4}{p^2}\mathrm{Cov}(\mathrm{Sgn}(X_i)|X_i|^{p/2}, \mathrm{Sgn}(X_j)|X_j|^{p/2}) \leqslant \frac{4}{p^2}\sup_{i>0} E|X_i|^p.$$

4.2. Strong Law of Large Numbers

PROPOSITION 1. *Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s. Suppose that all the assumptions of Theorem 1 are fulfilled for some $p\in[1, 2[$. Then for any $\varepsilon > 0$*

$$\sum_{n>0} n^{-1} P(S_n^* > \varepsilon n^{1/p}) < \infty.$$

Proof. (i) First suppose that $1 < p < 2$. Lemma 1, applied with $x = \varepsilon n^{1/p}$ and $M = n^{1/p}$, yields

$$n^{-1} P(S_n^* > \varepsilon n^{1/p}) \leqslant \frac{4}{\varepsilon n^{1/p}} EX\mathbf{1}_{X^p>n} + \frac{4}{\varepsilon^2 n^{2/p}} EX^2\mathbf{1}_{X^p\leqslant n} + \frac{1}{\varepsilon^2}P(X > n^{1/p}) + \frac{8n^{-1-2/p}}{\varepsilon^2}\sum_{1\leqslant i<j\leqslant n} G_{i,j}(n^{1/p}). \tag{18}$$

Using Fubini's theorem we deduce the existence of some positive constant c_p such that

$$\sum_{n>0} n^{-1/p} E(X\mathbf{1}_{X^p>n}) = E(X\sum_{n>0} n^{-1/p}\mathbf{1}_{X^p>n}) \leqslant c_p EX^p. \tag{19}$$

Again Fubini's theorem yields

$$\sum_{n>0} n^{-2/p} E(X^2\mathbf{1}_{X^p\leqslant n}) + \sum_{n>0} P(X > n^{1/p}) \leqslant c_p EX^p. \tag{20}$$

Now Lemma 4, together with Fubini's theorem, yields

$$(21)\quad \sum_{n>0} n^{-1-2/p} \sum_{1\leqslant i<j\leqslant n} G_{i,j}(n^{1/p})$$
$$= \sum_{1\leqslant i<j\leqslant \infty} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sum_{n>0} n^{-1-2/p} \mathbb{1}_{n\geqslant |x|^p\vee|y|^p\vee j} \Delta_{i,j}(x, y)\,dx\,dy$$
$$\leqslant c_p \sum_{1\leqslant i<j\leqslant \infty} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x|^p\vee|y|^p\vee j)^{-2/p}\Delta_{i,j}(x, y)\,dx\,dy$$
$$= 2c_p \sum_{1\leqslant i<j\leqslant \infty} \int_{j^{1/p}}^{+\infty} v^{-3}\, G_{i,j}(v)\,dv,$$

which is finite if the requirement of Theorem 1 holds. Now, taking the sum over $n > 0$ in (18) and using inequalities (19)–(21), we complete the proof of Proposition 1 for $p \in]1, 2[$.

(ii) Suppose now that $p = 1$. For fixed positive integer n, define the sequences $(\overline{Y}_{i,n})_i$ and $(\widetilde{Y}_{i,n})_i$ by

$$Y_i := Y_{i,n} = (X_i \wedge n)\vee(-n) \quad \text{and} \quad \widetilde{Y}_i := \widetilde{Y}_{i,n} = X_i - Y_i.$$

Let $T_n^* = \sup_{k\in[0,n]} \sum_{i=1}^k (Y_i - EY_i)$. Clearly,

$$E|X_i - Y_i| \leqslant E(|X_i| - n)\mathbb{1}_{|X_i|\geqslant n} \leqslant E(X-n)\mathbb{1}_{X\geqslant n} \leqslant \varepsilon/2 \quad \text{for } n \text{ large enough.}$$

Hence inequality (3.12) of Rio [13] yields

$$(22)\quad \sum_{n>0} n^{-1} P(S_n^* > \varepsilon n) \leqslant \sum_{n>0} P(X \geqslant n) + \sum_{n>0} n^{-1} P(T_n^* > \varepsilon n/2).$$

The first term on the right-hand side of the last inequality is finite since $EX < \infty$. The second term is bounded, arguing exactly as in the proof of the first part of Proposition 1, by some constant times

$$\Big(EX + \sum_{1\leqslant i<j<\infty} \int_j^{+\infty} v^{-3}\, G_{i,j}(v)\,dv\Big),$$

which is finite, by the assumption of Proposition 1. Hence Proposition 1 is proved.

End of the proof of Theorem 1. We first apply Proposition 1 to the sequences $(X_i)_{i\in N}$ and $(-X_i)_{i\in N}$ (recall that if the sequence $(X_i)_{i\in N}$ is associated, then this property holds also for the sequence $(-X_i)_{i\in N}$). Then we obtain

$$(23)\quad \sum_{n>0} n^{-1} P(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon n^{1/p}) < \infty.$$

Let k be such that $2^k \leqslant n < 2^{k+1}$. Then

$$2^{-k-1} P(\max_{1\leqslant j\leqslant 2^k} |S_j| > \varepsilon 2^{(k+1)/p}) \leqslant n^{-1} P(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon n^{1/p}).$$

Next we take the sums first over n: $2^k \leqslant n < 2^{k+1}$, and next over $k \in N$ in the last inequality and we use (23) to deduce that

$$\sum_{k>0} P\big(\max_{1 \leqslant j \leqslant 2^k} |S_j| > \varepsilon 2^{(k+1)/p}\big) < +\infty.$$

The last summation together with the Borel–Cantelli lemma proves Theorem 1.

4.3. Proof of Corollary 2. The linear sequence $(X_n)_{n\in Z}$ can be written as a *difference of two associated linear processes.* This fact follows by writing a_i as the difference $a_i^+ - a_i^-$ (where $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$). Hence we may suppose without loss of generality that the coefficients (a_i) are *non-negative* for each $i \in N$.

Let p be fixed in $[1, \alpha[$, where α is a fixed real number in $]1, 2[$.

For any $p \in [1, \alpha[$, $E|X_0|^p < \infty$. This follows from the fact that X_0 is a stable $\big(\sigma(\sum_{i\geqslant 0} |a_i|^\alpha)^{1/\alpha}, \alpha, 0\big)$ r.v.

The sequence (X_n) *is associated.* This property follows from $(\mathscr{P}_4)$, $(\mathscr{P}_5)$, $(\mathscr{P}_6)$ of Esary et al. [7] (recall that we have supposed: $a_i \geqslant 0$ for each i).

Condition (6) *is fulfilled.* In order to check this property, write for $n \in N^*$

$$X_n = \sum_{i=0}^{n-1} a_i \varepsilon_{n-i} + \sum_{i=n}^{\infty} a_i \varepsilon_{n-i} =: Z_n + Y_n.$$

Clearly, the previously defined r.v. Z_n is independent of (X_0, Y_n). This property, together with the fact that the function g_v defined as in (3) is 1-Lipschitz, yields:

$$\begin{aligned} G_n(v) &= \mathrm{Cov}\big(g_v(X_0), g_v(Z_n + Y_n) - g_v(Z_n)\big) \\ &\leqslant E|g_v(X_0)|\,|Y_n| + E|g_v(X_0)|\,E|Y_n|; \end{aligned} \tag{24}$$

here and in the sequel we put $G_n(v) := G_{0,n}(v)$.

Let β be a fixed real number in $]2-\alpha, 2-p[$ and p' be fixed in $]\alpha/(\alpha+\beta-1), \alpha[$. We deduce from (24) that

$$\begin{aligned} G_n(v) &\leqslant E(v \wedge |X_0|)\,|Y_n| + E(v \wedge |X_0|)\,E|Y_n| \\ &\leqslant v^\beta E|X_0|^{1-\beta}|Y_n| + v^\beta E|X_0|^{1-\beta} E|Y_n| \\ &\leqslant 2v^\beta \|Y_n\|_{p'} (E|X_0|^{p'(1-\beta)/(p'-1)})^{1-1/p'}. \end{aligned} \tag{25}$$

Let us now bound $\|Y_n\|_{p'}$. Recall that $Y_n = \sum_{i=n}^{\infty} a_i \varepsilon_{n-i}$ and that $(\varepsilon_n)_{n\in Z}$ is a sequence of i.i.d. r.v.'s with a stable $(\sigma, \alpha, 0)$ marginal distribution. Hence Y_n has the same law as $\varepsilon_0\big(\sum_{i\geqslant n} a_i^\alpha\big)^{1/\alpha}$. This yields

$$\|Y_n\|_{p'} = \|\varepsilon_0\|_{p'} \big(\sum_{i\geqslant n} a_i^\alpha\big)^{1/\alpha}. \tag{26}$$

Inequality (25), together with (26), yields

$$\sum_{n=1}^{\infty} \int_{n^{1/p}}^{+\infty} v^{p-3} G_n(v)\,dv \leqslant C(E|X_0|^{p'(1-\beta)/(p'-1)})^{1-1/p'} \|\varepsilon_0\|_{p'} \sum_{n>0} n^{1-2/p+\beta/p} \big(\sum_{i\geqslant n} a_i^\alpha\big)^{1/\alpha},$$

where C is some positive constant. Since $p'(1-\beta)/(p'-1) \in]0, \alpha[$ and $p' \in]1, \alpha[$, we deduce that all the terms on the right-hand side of the last inequality are finite as soon as (9) holds. The proof of Corollary 2 is then complete.

APPENDIX

In this Appendix we check condition (12). When $\alpha \in]0, 1[$, (12) is proved (cf. Chanda and Ruymgaart [3]). Here we suppose that $\alpha \in]1, 2[$.

Let $b > 1$ be an arbitrary real number. It follows from Jensen's inequality and Parseval's identity (exactly as (4.22) and (4.23) in Chanda and Ruymgaart [3]) that

$$\int_0^\infty |g(x+u)-g(x)|\,dx \leqslant Cu + \int_b^\infty |g(x+u)-g(x)|\,dx; \tag{27}$$

here and in the sequel C denotes a generic positive constant that may be different from line to line. Since

$$g(x) = (1/\pi)\int_0^\infty \cos(tx)\exp(-t^\alpha)\,dt,$$

we deduce, using integration by parts twice, that

$$g'(x) = \mathcal{O}(1/x^2) \quad \text{for all } x > 0.$$

Hence the mean value theorem yields

$$\int_b^\infty |g(x+u)-g(x)|\,dx \leqslant Cu.$$

The last inequality, together with (27), yields

$$\int_0^\infty |g(x+u)-g(x)|\,dx \leqslant Cu. \tag{28}$$

The facts that g is a bounded and an even density yield

$$\int_{-\infty}^0 |g(x+u)-g(x)|\,dx = \int_0^\infty |g(x-u)-g(x)|\,dx = \int_{-u}^\infty |g(x+u)-g(x)|\,dx$$

$$\leqslant 2u\,\|g\|_\infty + \int_0^\infty |g(x+u)-g(x)|\,dx.$$

Thus (12) follows from (28).

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