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SOJOURN TIME OF SOME REFLECTED BROWNIAN MOTION IN THE UNIT DISK

BY

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Abstract. We study the heat diffusion in a domain with an obstacle inside. More precisely, we are interested in the quantity of heat in so far as a function of the position of the heat source at time 0. This quantity is also equal to the expectation of the sojourn time of the Brownian motion, reflected on the boundary of a small disk contained in the unit disk, and killed on the unit circle. We give the explicit expression of this expectation. This allows us to make some numerical estimates and thus to illustrate the behaviour of this expectation as a function of starting point of the Brownian motion.

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INTRODUCTION

Assume that a heat source is placed in a bounded domain D with a heat absorbing boundary ∂D. Also a small obstacle O with heat reflecting boundary ∂O is placed inside the domain. A natural question arises: given a position of the obstacle, what will the point $z \in D\backslash O$ be where we must place the heat source, so that the quantity of heat

$$Q(z) := \int_{D\backslash O} dw \int_0^\infty dt\, u(t, z, w)$$

will be a maximum? It is a classical optimisation problem for linear partial differential equations (see also [12] and references therein). Here $u(\cdot, z, \cdot)$ is the unique solution of the heat equation with mixed boundary conditions

$$\begin{cases} (\partial u/\partial t)(\cdot, z, \cdot) = (1/2)\Delta u(\cdot, z, \cdot) & \text{in } \mathbf{R}_+^* \times (D\backslash O), \\ u(0, z, \cdot) = \delta_z(\cdot) & \\ (\partial u/\partial \mathrm{n})(\cdot, z, w) = 0 & \text{for } w \in \partial O, \\ u(\cdot, z, w) = 0 & \text{for } w \in \partial D. \end{cases}$$

The purpose of this paper is to give, using some probabilistic remarks, an answer to this question for the simple case where D is the unit disk in the complex plan and O is a small disk inside D.

To modelise heat as a probabilistic object, we need to consider the Brownian motion starting from $z \in D$. The reflecting and the absorbing features are ensured by imposing that the stochastic process is reflected on ∂O and killed on ∂D. Then the quantity of heat is related to the sojourn time of this stochastic process in the domain $D \backslash O$. By a simple stochastic calculation we can see that

$$Q(z) = \text{const} \cdot E_z(\tau),$$

where τ is the first hitting time of ∂D by the stochastic process. Hence, finding the quantity of heat is equivalent to finding the preceding expectation (see also [4], Chapter II).

Let us note that we may follow the potential theory point of view: the quantity of heat is the integral on $D \backslash O$ of the fundamental solution (Green –Neumann function or 0-potential) associated with the stochastic process. If $D \backslash O$ is an annulus centered in the origin, the Green function for the Dirichlet problem was explicitly calculated by many authors: [13], p. 140; [3], p. 386; [11], p. 6.41 (see also [2], Section 11.7 and notes therein). In [6] a general Neumann problem is also described (see Sections 15.2 and 15.7). At our knowledge, there is no reference for a mixed boundary problem on the annulus.

It must be noticed that, since the expression of $Q(z)$ which we obtain is complicated, in general, computing its maximum is not easy. We illustrate the behaviour of this function using some numerical computations.

1. SETTING AND MAIN RESULT

1.1. Setting. Let us consider a complex Brownian motion B starting from a point z with $|z| < 1$. It is well known that the expectation of the exit time from the unit disk of B is $(1/2)(1-|z|^2)$, and it is a maximum when the starting point is $z = 0$.

Assume that a reflecting obstacle is placed in the unit disk. What will the expectation of the exit time be from the unit disk of the Brownian motion which is reflected when it hits the obstacle? Let us denote by $(x_t^0\colon t \geqslant 0)$ the process which is a reflected Brownian motion on an inner small circle γ_0, of radius R_0, killed at the first hitting time of the unit circle γ_1.

It is a simple calculation (see also [11], Chapter 6) to show that if the circles γ_0 and γ_1 are concentric, then the expectation of exit time from the unit disk of x_t^0 is $(1/2)(1-|z|^2+R_0^2 \log |z|^2)$. This expectation is a maximum when the starting point lies on the circle γ_0.

We shall also consider the process $(x_t^1\colon t \geqslant 0)$ which is the Brownian motion reflected on the unit circle γ_1 and killed at the first hitting time of γ_0 (see also Figure 1).

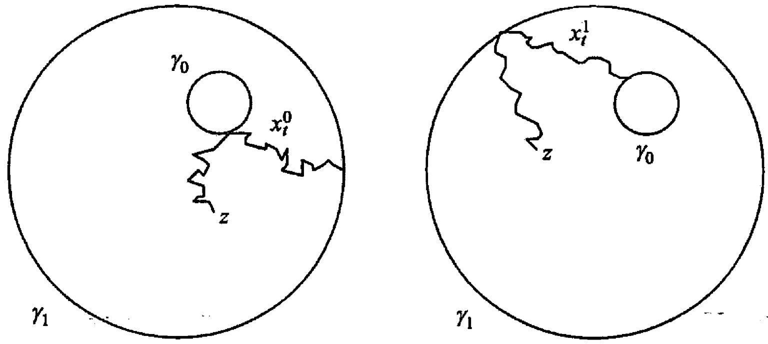

Figure 1. Reflected Brownian motion on γ_0 or on γ_1

We may assume, without loss of generality, that γ_0 is centered on the real axis. Take $c_0 \in]-1, 1[$, $R_0 \in]0, 1-|c_0|[$ and denote the domain between the circles γ_0 and γ_1 by

$$\Omega = \{z \in \boldsymbol{C}\colon\ |z| < 1,\ |z-c_0| > R_0\}.$$

For $j = 0$ or 1, we shall denote by τ_j the first hitting time of γ_{1-j} by the process x^j:

$$\tau_j = \inf\{t > 0\colon\ x_t^j \in \gamma_{1-j}\}. \tag{1.1}$$

It is known that, for any $z \in \Omega$, for $j = 0, 1$, τ_j is finite P_z-a.s.

1.2. Main result. We are interested in computing the expectation of the hitting time, $E_z(\tau_j)$, $j = 0, 1$, as a function of the starting point z. These functions are given in the following main result:

THEOREM 1.1. *The expectations of the hitting times are given by*

$$\begin{aligned} E_z(\tau_0) = {} & (1/2)(|z \sinh p - \cosh p|^2 - |z \cosh p - \sinh p|^2) \\ & - \log \frac{|z \cosh p - \sinh p|\,|z\, q \sinh 2p - r|}{|z \sinh p - \cosh p|\,|z\, r - q \sinh 2p|} \\ & + \frac{R^2}{r^2} \log \frac{|z \cosh p - \sinh p|\,|z\, r - q \sinh 2p|\,|z\, 2q(1-q)\sinh 2p - s|}{|z \sinh p - \cosh p|\,|z\, q \sinh 2p - r|\,|z\, s - 2q(1-q) \sinh 2p|} \\ & + \sum_{n=1}^{\infty} s_n^{(0)}(z, p, R) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} E_z(\tau_1) = {} & (1/2)(|z \sinh p - \cosh p|^2 - |z \cosh p - \sinh p|^2) \\ & - \log \frac{R\,|z \cosh p - \sinh p|}{|z \sinh p - \cosh p|\,|z\, r - q \sinh 2p|^2} \end{aligned} \tag{1.3}$$

$$+\frac{R^2}{r^2}\log\frac{|z\cosh p-\sinh p|\,|z\,q\sinh 2p-r|}{|z\sinh p-\cosh p|\,|z\,s-2q\,(1-q)\sinh 2p|}$$

$$-\frac{q}{r}\frac{|z\,(1+\cosh 2p)-\sinh 2p|^2-R^4\,|z\,(1-\cosh 2p)+\sinh 2p|^2}{|2z\,r-2q\sinh 2p|^2}$$

$$+\sum_{n=1}^{\infty} s_n^{(1)}(z,\,p,\,R),$$

where we have put

$$p=\frac{1}{4}\log\frac{(1+c_0)^2-R_0^2}{(1-c_0)^2-R_0^2},\qquad R=\tanh\left(\frac{1}{4}\log\frac{(1+R_0)^2-c_0^2}{(1-R_0)^2-c_0^2}\right), \tag{1.4}$$

$$q=(1-R^2)/2,\qquad r=\cosh^2 p-R^2\sinh^2 p,\qquad s=\cosh^2 p-R^4\sinh^2 p. \tag{1.5}$$

The sequences $(s_n^{(j)}(z,\,p,\,R))_{n\geqslant 1}$ *are explicit* (see (3.6)) *and converge to zero as* R^{4n}, *uniformly for* $z\in\Omega$.

The quantities $s_n^{(j)}$ are sums of logarithms and fractions of the same type as the other terms in (1.2) and (1.3).

Let us note that for $c_0=0$ we find the formulas of the concentric circles case. The expectation of the exit time of the Brownian motion from the unit disk (without obstacle) can be obtained by taking $c_0=R_0=0$.

1.3. Main ideas and plan. The reflected process that we study can be written, for $j=0,\,1,$ as

$$x_t^j=B_t-k_t^j \tag{1.6}$$

with

$$k_0^j=0,\qquad k_t^j=\int_0^t \mathrm{n}_j(x_s^j)\,d\,|k^j|_s,\qquad |k^j|_t=\int_0^t \mathbf{1}_{(x_s^j\in\gamma_j)}\,d\,|k^j|_s, \tag{1.7}$$

where n_j is the outward normal at γ_j to Ω, and, for $j=0,\,1$, $(k_t^j\colon\,t\geqslant 0)$ are adapted continuous and locally bounded variation processes (see also [10], p. 512).

The expectation of the hitting time τ_j is then expressed as a function of the starting point z:

$$E_z(\tau_j)=-2\,H_j(z),\qquad j=0,\,1. \tag{1.8}$$

Indeed, for $j=0,\,1,$ let H_j be a smooth function which satisfies the mixed boundary conditions of the (Dirichlet–Neumann) problem:

$$\Delta H_j=1\quad\text{in }\Omega, \tag{1.9}$$

$$\partial H_j/\partial \mathrm{n}_j=0\quad\text{on }\gamma_j, \tag{1.10}$$

$$H_j=0\quad\text{on }\gamma_{1-j}. \tag{1.11}$$

By Itô's formula we can write

$$H_j(x_t^j)-H_j(z)=\int_0^t \nabla H_j(x_s^j)\,dB_s-\int_0^t \nabla H_j(x_s^j)\,\mathrm{n}_j(x_s^j)\,d|k^j|_s+(1/2)\int_0^t \Delta H_j(x_s^j)\,ds.$$

To get (1.8) it suffices to put $t=\tau_j$ and to take the expectation.

Hence, it suffices to compute the function H_j. But, for $j=0, 1$,

$$H_j(z)=\int_\Omega G_j^{(\Omega)}(w, z)\,dw, \qquad z\in\Omega, \tag{1.12}$$

where $G_j^{(\Omega)}$ is the fundamental solution of the problem (1.9)–(1.11). For $j=0, 1$, $G_j^{(\Omega)}$ satisfies

$$\Delta G_j^{(\Omega)}(w, z)=\delta_w(z) \quad \text{for } z\in\Omega, \tag{1.13}$$

$$(\partial G_j^{(\Omega)}/\partial \mathrm{n}_j)(w, z)=0 \quad \text{if } z\in\gamma_j, \tag{1.14}$$

$$G_j^{(\Omega)}(w, z)=0 \quad \text{if } z\in\gamma_{1-j}. \tag{1.15}$$

In (1.13) and (1.14) the differentiation is with respect to z.

If the circles γ_0 and γ_1 are concentric, the expression of this fundamental solution can be obtained as for the reflected linear Brownian motion (see Section 2). By using a suitable linear fractional transformation we reduce the general case to the case of concentric circles (see Section 3.2).

The proof of Theorem 1.1 is given in Section 3.3, except for some technical lemmas postponed to the Appendix. Finally, Section 4 presents graphs of the expectation as a function of z.

2. CONCENTRIC CIRCLES CASE

2.1. Expectation of the hitting time. In this section we shall assume that $c_0=0$, that is, γ_0 is centered in the origin. For the sake of completeness, we recall some classical results (see also [11], Chapter 6).

PROPOSITION 2.1. *The expectations of the hitting times are given by*

$$E_z(\tau_0)=(1/2)(1-|z|^2+R_0^2\log|z|^2) \tag{2.1}$$

and

$$E_z(\tau_1)=(1/2)\big(R_0^2-|z|^2+\log(|z|^2/R_0^2)\big). \tag{2.2}$$

Note. We have already noted that (2.1) and (2.2) are nothing but (1.2) and (1.3) with $c_0=0$.

Proof of Proposition 2.1. It is classical: for $j=0, 1$, the function

$$H_j(z):=\ -(1/4)(a_j-|z|^2+b_j\log|z|^2), \qquad a_j,\, b_j\in\boldsymbol{R},$$

satisfies (1.9) in the annulus

$$A_{R_0} := \{w \in \mathbf{C}\colon\ R_0 < |w| < 1\}.$$

Then we find a_j, b_j such that the boundary conditions (1.10), (1.11) are fulfilled. Therefore (2.1) and (2.2) are obtained by (1.8). ∎

Remark 2.2. The maximum of the function $z \mapsto E_z(\tau_0)$ lies on the circle γ_0 and the one of the function $z \mapsto E_z(\tau_1)$ lies on γ_1. ∎

Note. It must be noticed that in this case we have the "intuition" of the solution of the problem (1.9)–(1.11) because the domain, the annulus A_{R_0}, is very particular.

2.2. Fundamental solution for the annulus. We shall point out the fundamental solution of the problem (1.9)–(1.11) for the annulus A_{R_0}. This function will be denoted, for $j = 0, 1$, by $G_j^{(R_0)}$, and it satisfies (1.13)–(1.15) with A_{R_0} instead of Ω.

The idea comes from the study of the linear Brownian motion on]0, 1[, reflected at 0 and absorbed at 1 (see also [4], pp. 77 and 79, or [9], p. 97). This process has the density

$$\sum_{n=-\infty}^{\infty} [q_t(y, -x+4n+2) + q_t(y, x+4n+2) - q_t(y, -x+4n+4) - q_t(y, x+4n)],$$

where $q_t(y, x) := (2\pi t)^{-1/2} \exp(-|x-y|^2/2t)$.

For the 2-dimensional case, we shall use the homothetic transformation $\mu(z) := z/R_0$ and the inversion $v(z) := R_0^2/\bar{z}$ instead of the translation and of the symmetry (see also [11], Section 4.3). The α-potential associated with the process x^j is obtained by integrating in t the product by $e^{-\alpha t}$ of the following density:

$$\sum_{n=-\infty}^{\infty} \big[p_t\big(w, (\mu^{4n+2} \circ v)(z)\big) + p_t\big(w, \mu^{4n+2}(z)\big) - p_t\big(w, (\mu^{4n+4} \circ v)(z)\big) - p_t\big(w, \mu^{4n}(z)\big)\big],$$

where $p_t(w, z) = (2\pi t)^{-1} \exp(-|w-z|^2/2t)$. It is known that

$$\int_0^\infty e^{-\alpha t} p_t(w, z)\, dt = (1/2\pi) K_0\big(\sqrt{2\alpha}\,|w-z|\big),$$

where K_0 denotes the modified Bessel function of index 0. Moreover, $K_0(\beta) \sim \log(1/\beta)$ as $\beta \downarrow 0$ (see, for instance, [8], p. 133).

Letting $\alpha \downarrow 0$ in the expression of the α-potential for the process x^j, we get the 0-potential or the Green function

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \log \frac{|w-\mu^{4n}(z)|\,|w-(\mu^{4n+4} \circ v)(z)|}{|w-(\mu^{4n+2} \circ v)(z)|\,|w-\mu^{4n+2}(z)|}.$$

To obtain the solution of (1.13)–(1.15) on A_{R_0}, we need to add a harmonic function (see also [13], p. 140, [3], p. 386, [11], p. 6.41, for the Dirichlet problem):

PROPOSITION 2.3. *For* $j = 0, 1$ *and for* $w \neq z$, *we introduce*

$$(2.3)\quad G_j^{(R_0)}(w, z) := \frac{1}{4\pi}\log\frac{|z|^2}{R_0^{2j}} + \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\log\frac{|w-z/R_0^{4n}|^2\,|w-1/\bar{z}R_0^{4n+2-2j}|^2}{|w-1/\bar{z}R_0^{4n-2j}|^2\,|w-z/R_0^{4n+2}|^2}.$$

Then $G_j^{(R_0)}$ *satisfies* (1.13)–(1.15) *on the annulus* A_{R_0}.

Proof. We prove the result for $j=0$, the case $j=1$ being similar. The function $G_0^{(R_0)}$ is well defined. Indeed, the series in (2.3) is convergent. For instance, if $n \in N$, the series behaves as $k\sum_{n=0}^{\infty} R_0^{4n}$, which is convergent since $0 < R_0 < 1$ (here $k = -w/z - w\bar{z}R_0^2 + w\bar{z} + wR_0^2/z$).

Since, for $z \in \gamma_1$, $z\bar{z} = 1$, the general term of the series in (2.3) equals 0 and (1.15) is obvious.

For the proof of (1.14) let us put $z = \varrho e^{i\theta}$. To compute the normal derivative it suffices to differentiate with respect to ϱ, since the circles are centered at the origin:

$$(*)\quad (\partial G_0^{(R_0)}/\partial\varrho)(w, \varrho e^{i\theta}) = 1/(2\pi\varrho)$$
$$+\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{-e^{i\theta}/R_0^{4n}}{w-\varrho e^{i\theta}/R_0^{4n}} + \frac{e^{i\theta}/\varrho^2 R_0^{4n+2}}{w-e^{i\theta}/\varrho R_0^{4n+2}} - \frac{e^{i\theta}/\varrho^2 R_0^{4n}}{w-e^{i\theta}/\varrho R_0^{4n}} + \frac{e^{i\theta}/R_0^{4n+2}}{w-\varrho e^{i\theta}/R_0^{4n+2}}\right).$$

Clearly, the series is uniformly convergent. For instance, if $n \in N$, the series behaves as $k\sum_{n=0}^{\infty} R_0^{4n}$, where $k = w(1+\varrho^2)(1-R_0^2)/(\varrho^2 e^{i\theta})$. For $z \in A_{R_0}$, $\varrho^{-2}+1 < R_0^{-2}+1$, so the series converges.

Then we can verify (1.14), since, by $(*)$, we get

$$\frac{\partial G_0^{(R_0)}}{\partial\varrho}(w, \varrho e^{i\theta})|_{\varrho=R_0} = \frac{1}{2\pi R_0} + \frac{1}{2\pi}\lim_{N\uparrow\infty}\sum_{n=-N}^{n=N}\left(\frac{-e^{i\theta}/R_0^{4n}}{w-e^{i\theta}/R_0^{4n-1}} + \frac{e^{i\theta}/R_0^{4n+4}}{w-e^{i\theta}/R_0^{4n+3}}\right)$$
$$= \frac{1}{2\pi}\frac{1}{R_0} + \frac{1}{2\pi}\lim_{N\uparrow\infty}\left(\frac{-e^{i\theta}/R_0^{-4N}}{w-e^{i\theta}/R_0^{-4N-1}} + \frac{e^{i\theta}/R_0^{4N+4}}{w-e^{i\theta}/R_0^{4N+3}}\right) = 0.$$

Finally, to prove (1.13) we note that, for $n \in Z$, the points λ_n of the form

$$wR_0^{4n},\ 1/wR_0^{4n+2},\ 1/wR_0^{4n},\ wR_0^{4n+2}$$

lie in the complementary of the annulus A_{R_0}, excepting w. Hence the functions $\log|z-\lambda_n|^2$ and $\log|\bar{z}-\lambda_n|^2$ are harmonic in A_{R_0}, and

$$\Delta_z G_0^{(R_0)}(w, z) = (1/2\pi)\,\Delta_z(\log|z-w| + \log|z|) = \delta_w(z). \quad ∎$$

Remark 2.4. The function $G_j^{(R_0)}$ is not symmetric in (w, z). However, we can prove that, for $j = 0, 1$,

$$\Delta_w G_j^{(R_0)}(w, z) = \delta_z(w) \quad \text{for } w \in A_{R_0}. \tag{2.4}$$

Indeed, we note that, for $n \in \boldsymbol{Z}$, the points of the form

$$z/R_0^{4n}, \; 1/\bar{z}R_0^{4n+2-2j}, \; 1/\bar{z}R_0^{4n-2j}, \; z/R_0^{4n+2}$$

lie in the complementary of the annulus A_{R_0}, excepting z. Then we can proceed as for the proof of (1.13). ∎

2.3. Find the expectation using the fundamental solution. By integrating (2.3) on A_{R_0} we find the expectation $E_z(\tau_j)$, $j = 0, 1$, that is (2.1) and (2.2) up to the multiplication by -2 (see (1.8) and (1.12)).

Let us remark first that the integrals of the terms of the series are zero except for the term corresponding to $n = 0$. Indeed, this is a consequence of the facts that, for $j = 0, 1$,

$$|z/R_0^{4n}|, \; |1/\bar{z}R_0^{4n+2-2j}|, \; |1/\bar{z}R_0^{4n-2j}|, \; |z/R_0^{4n+2}| \text{ are } \begin{cases} > 1 & \text{if } n \in \boldsymbol{Z}_+^*, \\ < R_0 & \text{if } n \in \boldsymbol{Z}_-^*, \end{cases}$$

and of the following

LEMMA 2.5. *We have*

$$\int_{A_{R_0}} \log|\zeta - \lambda|^2 \, d\zeta = \begin{cases} 2\pi(1 - R_0^2)\log|\lambda| & \text{if } |\lambda| > 1, \\ -2\pi R_0^2 \log|\lambda| - \pi(1 - |\lambda|^2) & \text{if } R_0 < |\lambda| < 1, \\ -2\pi R_0^2 \log R_0 - \pi(1 - R_0^2) & \text{if } |\lambda| < R_0. \end{cases} \tag{2.5}$$

The proof of this lemma is postponed to the Appendix.

On the other hand,

$$\int_{A_{R_0}} G_j^{(R_0)}(w, z)\, dw = \frac{1}{4\pi}\int_{A_{R_0}} \log\frac{|z|^2}{R_0^{2j}}\, dw + \frac{1}{4\pi}\int_{A_{R_0}} \log\frac{|w-z|^2\,|w - 1/\bar{z}R_0^{2-2j}|^2}{|w - 1/\bar{z}R_0^{-2j}|^2\,|w - z/R_0^2|^2}\, dw.$$

Since, for $j = 0$, $R_0 < |z| < 1$, $|z|/R_0^2 > 1$, $1/|\bar{z}|R_0^2 > 1$ and $1/|\bar{z}| > 1$, using again (2.5) we get

$$\begin{aligned} \int_{A_{R_0}} G_0^{(R_0)}(w, z)\, dw &= (1/2\pi)\log|z| \int_{A_{R_0}} dw + (1/4\pi)\big(-2\pi R_0^2 \log|z| - \pi(1 - |z|^2) \\ &\quad + 2\pi(1 - R_0^2)\log(1/|\bar{z}|R_0^2) - 2\pi(1 - R_0^2)\log(1/|\bar{z}|) - 2\pi(1 - R_0^2)\log(|\bar{z}|/R_0^2)\big) \\ &= (1/2)(1 - R_0^2)\log|z| + (1/4)\big(|z|^2 - 1 - 2\log|z|\big) = (-1/2)\, E_z(\tau_0). \end{aligned}$$

The calculation is similar for $j = 1$, by noting that $R_0^2/|\bar{z}| < R_0$. ∎

3. FRACTIONAL LINEAR TRANSFORMATION AND GENERAL CASE

3.1. Fractional linear transformation. Let us consider, for $t \in \boldsymbol{R}$, the transformation

$$a_t(\zeta) := \frac{\zeta \cosh t + \sinh t}{\zeta \sinh t + \cosh t}, \qquad \zeta \in \boldsymbol{C}. \tag{3.1}$$

This is a one-parameter transformation group: $a_t a_s = a_{t+s}$. We recall below some well-known properties of these transformations.

Remark 3.1. The image of the circle centered in the origin, with radius $\tanh r$, by the fractional linear transformation a_t is a circle centered on the real axis. Moreover, this transformation leaves the unit circle invariant.

Indeed, it is no difficult to see that $|a_t(\zeta)| = 1$, provided $|\zeta| = 1$. On the other hand, by classical properties of the fractional linear transformations, the image of an orthogonal circle to the real axis is a circle orthogonal to the real axis. Therefore, the center of the image circle lies on the real axis.

Moreover, the images of the points $\tanh r$ and $-\tanh r$ are $\tanh(t+r)$ and $\tanh(t-r)$, respectively. Hence the image circle has the center $(1/2)\big(\tanh(t+r)+\tanh(t-r)\big)$ and the radius $(1/2)\big(\tanh(t+r)-\tanh(t-r)\big)$.

In particular, the circle γ_0, centered in c_0, with radius R_0, is the image of a circle centered in 0 having the radius R, by the transformation of parameter p, where R and p are given by (1.4). ∎

Remark 3.2. Let α and α^{-1} be the solutions of the equation

$$\alpha + \alpha^{-1} = (1 + c_0^2 - R_0^2)/c_0.$$

By a geometric reasoning we can see that, for $j = 0, 1$,

$$|(\zeta-\alpha)/(\zeta-\alpha^{-1})| = \text{const} \quad \text{provided } \zeta \in \gamma_j. \tag{3.2}$$

Therefore, γ_0 and γ_1 are in the family of Apollonius circles with respect to α and α^{-1} (see [1], p. 84). There exists an orthogonal family of circles which pass through the points α and α^{-1}, and satisfy

$$\arg(\zeta-\alpha)/(\zeta-\alpha^{-1}) = \text{const} \tag{3.2'}$$

(see also [7], p. 53). ∎

3.2. Fundamental solution for the general domain. We shall point out the fundamental solution of the problem (1.9)–(1.11) using the one on the annulus A_{R_0} and the linear transformation defined above (see also [11], p. 6.19):

PROPOSITION 3.3. *For $j = 0, 1$ and for $w \neq z$ we introduce*

$$G_j^{(\Omega)}(w, z) := G_j^{(R)}\big(a_{-p}(w), a_{-p}(z)\big), \tag{3.3}$$

where $G_j^{(R)}$ is given by (2.3) *with the parameter R given in* (1.4). *Then $G_j^{(\Omega)}$ satisfies* (1.13)–(1.15).

Proof. Let us first notice that

$$(*) \qquad a_{-p}(z) \in A_R = \{\zeta \in C \colon R < |\zeta| < 1\} \quad \text{provided } z \in \Omega$$

(see also [11], p. 4.29). It follows, by a reasoning similar to that for the proof of Proposition 2.3, that

$$\begin{aligned} \Delta_z G_j^{(\Omega)}(w, z) &= \Delta_z G_j^{(R)}(a_{-p}(w), a_{-p}(z)) \\ &= (1/2\pi)\Delta_z\big(\log|a_{-p}(z) - a_{-p}(w)| + \log(|a_{-p}(z)|/R^j)\big). \end{aligned}$$

Since a_t is a holomorphic function, the second term in the above sum is a harmonic function. Then, by (3.1), we get

$$|a_{-p}(z) - a_{-p}(w)| = |z - w|/(|-z\sinh p + \cosh p|\,|-w\sinh p + \cosh p|).$$

Therefore,

$$\Delta_z G_j(w, z) = (1/2\pi)\Delta_z\big(\log|z - w| - \log|(1/\tanh p) - z|\big)$$

and we get (1.13), since $\tanh p < 1$.

In order to obtain (1.14) we use the corresponding property of $G_j^{(R)}$, $(*)$ and Remark 3.2.

Indeed, to calculate the derivative of $G_j^{(\Omega)}$ in the normal direction to the circle γ_j we can use an infinitesimal shift along an arc of circle from the orthogonal family given by (3.2′). We have already observed that a_{-p} send the Apollonius circles (3.2) in the concentric circles in 0 and the orthogonal family (3.2′) in straight lines through 0 (that is, into two other orthogonal families of circles in wider sense; see [1], p. 79). Therefore, using an infinitesimal shift along a segment on a line through 0, the upper derivative is equal to the derivative of $G_j^{(R)}$ in the normal direction to the image circle centered in 0. But this derivative is zero and (1.14) follows.

Finally, (1.15) is a consequence of $(*)$ and the corresponding property of $G_j^{(R)}$. ∎

3.3. Expectation of the hitting time: the proof of Theorem 1.1. As in Section 2.3 we shall find the expectation of the hitting time using (1.8), (1.12) and the fundamental solution obtained in Proposition 3.3.

By (3.3) and $(*)$ from the proof of Proposition 3.3, we can write, for $j = 0, 1$,

$$H_j(z) = \int_\Omega G_j^{(R)}(a_{-p}(w), a_{-p}(z))\,dw = \int_{A_R} G_j^{(R)}(\zeta, a_{-p}(z))\,|\mathrm{Jac}(\zeta)|\,d\zeta,$$

where $\mathrm{Jac}(\zeta)$ is the Jacobian of the transformation a_p. Let us write this Jacobian as

$$\mathrm{Jac}(\zeta) = \frac{1}{|\zeta\sinh p + \cosh p|^4} = \frac{1}{\sinh^4 p}\frac{1}{|\zeta - k|^4} = \frac{1}{4\sinh^4 p}\cdot\Delta_\zeta\left(\frac{1}{|\zeta - k|^2}\right),$$

where $k = -1/\tanh p < -1$. Therefore,

$$H_j(z) = \frac{1}{4\sinh^4 p} \int_{A_R} G_j^{(R)}(\zeta, a_{-p}(z))\, \Delta_\zeta \left(\frac{1}{|\zeta-k|^2}\right) d\zeta.$$

Using Green's formula, we obtain

$$\text{(i)} \qquad H_j(z) = \frac{1}{4\sinh^4 p} \int_{A_R} \Delta_\zeta G_j^{(R)}(\zeta, a_{-p}(z)) \cdot \frac{1}{|\zeta-k|^2} d\zeta + \frac{1}{4\sinh^4 p} \int_{\partial A_R} \left(G_j^{(R)}(\zeta, a_{-p}(z)) \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2} \frac{\partial}{\partial \mathrm{n}} G_j^{(R)}(\zeta, a_{-p}(z)) \right) d\sigma := \Lambda_1^{(j)} + \Lambda_2^{(j)},$$

where the normal derivatives and the surface integral are with respect to ζ.

By (2.4), the first integral in (i) is the same for $j = 0, 1$, and equals

$$\text{(ii)} \qquad \Lambda_1 = \frac{1}{4\sinh^4 p} \cdot \frac{1}{|a_{-p}(z)-k|^2} = \frac{1}{4\sinh^4 p} \cdot \frac{\tanh^2 p}{|1+a_{-p}(z)\tanh p|^2}.$$

By (2.3), the second integral in (i) can be written as

$$\text{(iii)} \qquad \Lambda_2^{(j)} = \frac{1}{16\pi \sinh^4 p} \int_{\partial A_R} \left(\log\frac{|a_{-p}(z)|^2}{R^{2j}} \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2} \frac{\partial}{\partial \mathrm{n}} \log\frac{|a_{-p}(z)|^2}{R^{2j}} \right) d\sigma + \frac{1}{16\pi \sinh^4 p} \sum_{n=-\infty}^{\infty} \int_{\partial A_R} \left(t_{j,n}(\zeta, a_{-p}(z), R) \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2} \frac{\partial}{\partial \mathrm{n}} t_{j,n}(\zeta, a_{-p}(z), R) \right) d\sigma := \Lambda_{21}^{(j)} + \Lambda_{22}^{(j)},$$

where

$$t_{j,n}(\zeta, a_{-p}(z), R) = \log \frac{|\zeta - a_{-p}(z)/R^{4n}|^2\, |\zeta - 1/\bar{a}_{-p}(z)\, R^{4n+2-2j}|^2}{|\zeta - 1/\bar{a}_{-p}(z)\, R^{4n-2j}|^2\, |\zeta - a_{-p}(z)/R^{4n+2}|^2}.$$

Since the derivatives are in ζ, the first integral in (iii) equals

$$\Lambda_{21}^{(j)} = \frac{1}{16\pi \sinh^4 p} \left(\int_{\{|\zeta|=1\}} \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} d\sigma - \int_{\{|\zeta|=R\}} \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} d\sigma \right) \log\frac{|a_{-p}(z)|^2}{R^{2j}}.$$

To compute the preceding quantity we shall use the following

LEMMA 3.4. *For* $0 < \varrho \leqslant 1$,

$$\text{(3.4)} \qquad \frac{1}{4\pi} \int_{\{|\zeta|=\varrho\}} \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} d\sigma = \frac{\varrho^2}{(k^2-\varrho^2)^2}.$$

We postpone the proof of this lemma to the Appendix, and we use its result to obtain

$$\text{(iv)}\quad \Lambda_{21}^{(j)} = \frac{1}{4}\left(1-\frac{R^2}{(k^2-R^2)^2/(k^2-1)^2}\right)\log\frac{|a_{-p}(z)|^2}{R^{2j}} = \frac{1}{4}\left(1-\frac{R^2}{r^2}\right)\log\frac{|a_{-p}(z)|^2}{R^{2j}},$$

since $k^2-1 = 1/\sinh^2 p$ and here r is given in (1.5).

The second term in (iii) can be written as

$$\begin{aligned}\text{(v)}\quad \Lambda_{22}^{(j)} = \frac{1}{16\pi\sinh^4 p}\sum_{n=-\infty}^{\infty} &\{(J(1, a_{-p}(z)/R^{4n}) - J(R, a_{-p}(z)/R^{4n}))\\ &+(J(1, 1/\bar{a}_{-p}(z)R^{4n+2-2j}) - J(R, 1/\bar{a}_{-p}(z)R^{4n+2-2j}))\\ &-(J(1, 1/\bar{a}_{-p}(z)R^{4n-2j}) - J(R, 1/\bar{a}_{-p}(z)R^{4n-2j}))\\ &-(J(1, a_{-p}(z)R^{4n+2}) - J(R, a_{-p}(z)R^{4n+2}))\} := \sum_{n=-\infty}^{\infty}\Lambda_{22n}^{(j)},\end{aligned}$$

where

$$\text{(vi)}\quad J(\varrho, \lambda) := \int_{\{|\zeta|=\varrho\}}\left(\log|\zeta-\lambda|^2\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}}\log|\zeta-\lambda|^2\right)d\sigma.$$

The expression of $J(\varrho, \lambda)$ is contained in the following result, the proof of which will be given in the Appendix:

LEMMA 3.5. *For* $0 < \varrho \leqslant 1$,

$$\text{(3.5)}\quad J(\varrho, \lambda) = \begin{cases} \dfrac{8\pi\varrho^2}{(k^2-\varrho^2)^2}\log|\lambda-\varrho^2/k| & \textit{if } |\lambda| > \varrho, \\[2ex] \dfrac{8\pi\varrho^2}{(k^2-\varrho^2)^2}\log\varrho|1-\lambda/k| - \dfrac{4\pi}{k^2-\varrho^2}\dfrac{1-|\lambda/k|^2}{|1-\lambda/k|^2} & \textit{if } |\lambda| < \varrho. \end{cases}$$

Recalling that $1/k = -\tanh p$, by (vi) and (3.5) we can compute the general term of the series in (v):

$$\text{(vii)}\quad \frac{J(1, \lambda) - J(R, \lambda)}{16\pi\sinh^4 p}$$

$$= \begin{cases} \frac{1}{2}\log|\lambda+\tanh p| - \dfrac{R^2}{2r^2}\log|\lambda+R^2\tanh p| & \text{if } |\lambda| > 1, \\[2ex] \frac{1}{2}\log|1+\lambda\tanh p| - \dfrac{R^2}{2r^2}\log|\lambda+R^2\tanh p| - \dfrac{1}{4\sinh^2 p}\cdot\dfrac{1-|\lambda|^2\tanh^2 p}{|1+\lambda\tanh p|^2} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } R < |\lambda| < 1, \\[2ex] \frac{1}{2}\log|1+\lambda\tanh p| - \dfrac{R^2}{2r^2}\log R|1+\lambda\tanh p| - \dfrac{q}{2r}\cdot\dfrac{1-|\lambda|^2\tanh^2 p}{|1+\lambda\tanh p|^2} & \text{if } |\lambda| < R. \end{cases}$$

Here q and r are given in (1.5), and

$$\lambda \in \{a_{-p}(z)/R^{4n},\ 1/\bar{a}_{-p}(z)R^{4n+2-2j},\ 1/\bar{a}_{-p}(z)R^{4n-2j},\ a_{-p}(z)/R^{4n+2}:\ n\in \boldsymbol{Z}, j=0, 1\}.$$

In order to apply (vii) for the calculation of $\Lambda^{(j)}_{22n}$, we need to study the modulus of the complex in the preceding set for $n\in\boldsymbol{Z}$ and for $j=0, 1$.

For $n=0$,

$$R<|a_{-p}(z)|<1,\quad |1/\bar{a}_{-p}(z)R^2|>1,\quad |1/\bar{a}_{-p}(z)|>1,$$
$$|1/\bar{a}_{-p}(z)R^{-2}|<R,\quad |a_{-p}(z)/R^2|>1.$$

Therefore,

$$\text{(viii)}\quad \Lambda^{(0)}_{220}=\frac{1}{2}\log\frac{|1+a_{-p}(z)R^2\tanh p|}{|a_{-p}(z)+R^2\tanh p|}-\frac{1}{4\sinh^2 p|}\cdot\frac{1-|a_{-p}(z)|^2\tanh^2 p}{|1+a_{-p}(z)\tanh p|^2}$$
$$-\frac{R^2}{2r^2}\log\frac{|a_{-p}(z)+R^2\tanh p|\,|1+a_{-p}(z)R^4\tanh p|}{|1+a_{-p}(z)R^2\tanh p|\,|a_{-p}(z)+R^4\tanh p|}$$

and

$$\text{(ix)}\quad \Lambda^{(1)}_{220}=\frac{1}{2}\log\frac{R^2|1+a_{-p}(z)\tanh p|^2}{|a_{-p}(z)+R^2\tanh p|^2}-\frac{1}{4\sinh^2 p}\cdot\frac{1-|a_{-p}(z)|^2\tanh^2 p}{|1+a_{-p}(z)\tanh p|^2}$$
$$-\frac{R^2}{2r^2}\log\frac{R|1+a_{-p}(z)R^2\tanh p|}{|a_{-p}(z)+R^4\tanh p|}+\frac{q}{2r}\cdot\frac{|a_{-p}(z)|^2-R^4\tanh^2 p}{|a_{-p}(z)+R^2\tanh p|^2}.$$

For $n\in\boldsymbol{Z}^*_+$ and $j=0, 1$,

$$|a_{-p}(z)/R^{4n}|,\ |1/\bar{a}_{-p}(z)R^{4n+2-2j}|,\ |1/\bar{a}_{-p}(z)R^{4n-2j}|,\ |a_{-p}(z)/R^{4n+2}|>1,$$

while, for $n\in\boldsymbol{Z}^*_-$ and $j=0, 1$,

$$|a_{-p}(z)/R^{4n}|,\ |1/\bar{a}_{-p}(z)R^{4n+2-2j}|,\ |1/\bar{a}_{-p}(z)R^{4n-2j}|,\ |a_{-p}(z)/R^{4n+2}|<R.$$

Thus, for $n\in\boldsymbol{Z}^*_+$ and $j=0, 1$, the computations give

$$\text{(x)}\quad \Lambda^{(j)}_{22n}+\Lambda^{(j)}_{22(-n)}=\frac{1}{2}\left(\log\frac{|a_{-p}(z)+R^{4n}\tanh p|\,|1+a_{-p}(z)R^{4n+2-2j}\tanh p|}{|1+a_{-p}(z)R^{4n-2j}\tanh p|\,|a_{-p}(z)+R^{4n+2}\tanh p|}\right.$$
$$\left.+\log\frac{|1+a_{-p}(z)R^{4n}\tanh p|\,|a_{-p}(z)+R^{4n-2+2j}\tanh p|}{|a_{-p}(z)+R^{4n+2j}\tanh p|\,|1+a_{-p}(z)R^{4n-2}\tanh p|}\right)$$
$$-\frac{R^2}{2r^2}\left(\log\frac{|a_{-p}(z)+R^{4n+2}\tanh p|\,|1+a_{-p}(z)R^{4n+4-2j}\tanh p|}{|1+a_{-p}(z)R^{4n+2-2j}\tanh p|\,|a_{-p}(z)+R^{4n+4}\tanh p|}\right.$$
$$\left.+\log\frac{|1+a_{-p}(z)R^{4n}\tanh p|\,|a_{-p}(z)+R^{4n-2+2j}\tanh p|}{|a_{-p}(z)+R^{4n+2j}\tanh p|\,|1+a_{-p}(z)R^{4n-2}\tanh p|}\right)$$

$$-\frac{q}{2r}\left(\frac{1-|a_{-p}(z)|^2 R^{8n}\tanh^2 p}{|1+a_{-p}(z)R^{4n}\tanh p|^2}+\frac{|a_{-p}(z)|^2-R^{8n-4+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n-2+2j}\tanh p|^2}\right.$$

$$\left.-\frac{|a_{-p}(z)|^2-R^{8n+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n+2j}\tanh p|^2}-\frac{1-|a_{-p}(z)|^2 R^{8n-4}\tanh^2 p}{|1+a_{-p}(z)R^{4n-2}\tanh p|^2}\right),$$

where q and r are given in (1.5), and $a_{-p}(z)=(z\cosh p-\sinh p)/(-z\sinh p+\cosh p)$.

Combining (i), (ii), (iv), (vii) and (x) we get

(xi) $$H_0(z)=A_1+A_{21}^{(0)}+A_{220}^{(0)}+\sum_{n=1}^{\infty}(A_{22n}^{(0)}+A_{22(-n)}^{(0)}),$$

while from (i), (ii), (iv), (ix) and (3.6) we get

(xii) $$H_1(z)=A_1+A_{21}^{(1)}+A_{220}^{(1)}+\sum_{n=1}^{\infty}(A_{22n}^{(1)}+A_{22(-n)}^{(1)}).$$

Thus (1.2)–(1.3) are obtained by using (1.8) and (x), where we have put

(3.6) $$s_n^{(j)}(z,p,R):=-2(A_{22n}^{(j)}+A_{22(-n)}^{(j)}),\qquad n\in N^*,\ j=0,1.$$

In order to end the proof, let us show that, for $j=0,1$, the series with general term given by (3.6) are convergent. For instance,

$$\sum_{n=1}^{\infty}\log\frac{|a_{-p}(z)+R^{4n}\tanh p|\,|1+a_{-p}(z)R^{4n+2-2j}\tanh p|}{|1+a_{-p}(z)R^{4n-2j}\tanh p|\,|a_{-p}(z)+R^{4n+2}\tanh p|}$$

has the same behaviour as $k\sum_{n=1}^{\infty}R^{4n}$, where

$$k=\left(1/a_{-p}(z)+a_{-p}(z)R^{2-2j}-a_{-p}(z)R^{-2j}-R^2/a_{-p}(z)\right)\cdot\tanh p.$$

The other series with logarithm can be treated in the same manner. Similarly,

$$\sum_{n=1}^{\infty}\left(\frac{1-|a_{-p}(z)|^2R^{8n}\tanh^2 p}{|1+a_{-p}(z)R^{4n}\tanh p|^2}+\frac{|a_{-p}(z)|^2-R^{8n-4+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n-2+2j}\tanh p|^2}\right.$$

$$\left.-\frac{|a_{-p}(z)|^2-R^{8n+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n+2j}\tanh p|^2}-\frac{1-|a_{-p}(z)|^2R^{8n-4}\tanh p}{|1+a_{-p}(z)R^{4n-2}\tanh p|^2}\right)$$

behaves as $k\sum_{n=1}^{\infty}R^{4n}$, where

$$k=(-2\tanh p)(1-R^{-2})\left(|a_{-p}(z)|-R^{2j}/|a_{-p}(z)|\right).$$

This ends the proof of the theorem except for the proofs of the lemmas. ■

4. NUMERICAL RESULTS

In this section we present numerical approximations of the analytical representation of the expectations given in (1.2) or (1.3). We plot two views of the expectation as a function of the starting point: three-dimensional graph and its vertical section.

Figures 2 and 3 illustrate the case $j = 0$. Recall that for the concentric circles case the maximum lies on γ_0. Suppose now that $c_0 \neq 0$. When R_0 is small, the position of the maximum is close to zero as we can see in Figure 2. The increase of R_0 gives a displacement of this position towards γ_1. This is quite different with respect to a deterministic motion in which it is quite natural to think that z must be far from this circle.

Figures 5 and 6 correspond to the case $j = 1$. The maximum lies on γ_1 and its value is a decreasing function with respect to R_0.

In parallel to the semi-analytical method present in this paper we consider a classical finite element method to solve the partial differential equations (1.9), (1.10) and (1.11). A solution of this problem is in $H^2(\Omega)$, the Sobolev space of function u such that u, ∇u and Δu belong to $L^2(\Omega)$. By the finite element we compute an approximation of H_j belonging to a subspace V_h of continuous functions piecewise linear in Ω. The results obtained by the finite element method are similar to those obtained by the semi-analytical method considered in this paper. One difference between the two methods is that the maximum obtained by the finite element method is an approximation with an error of order h^2, h being the discretisation parameter. In Figures 4 and 7 we represent results computed by the finite element technique, which correspond to Figures 2b and 6b, respectively.

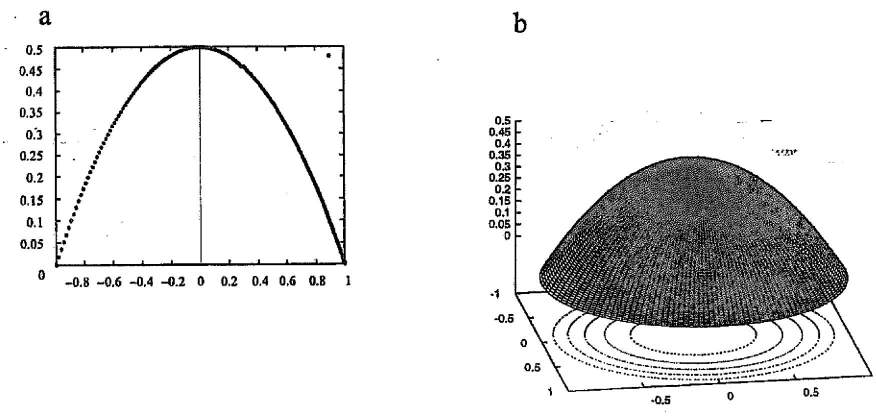

Figure 2. Case $j = 0$ with $c_0 = 0.1$ and $R_0 = 0.01$

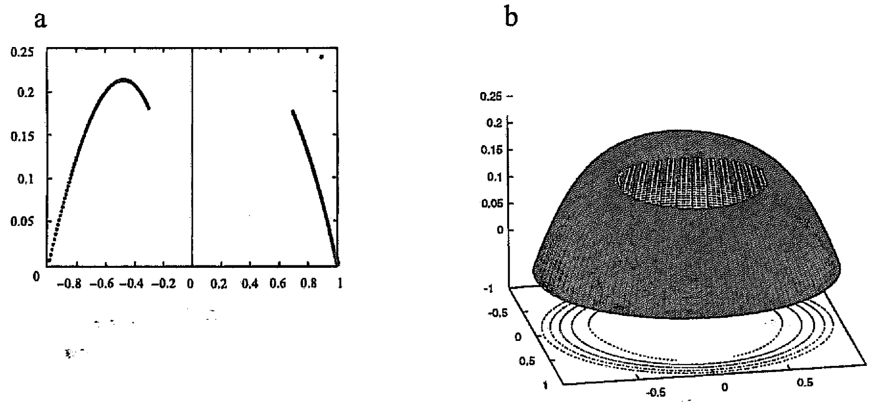

Figure 3. Case $j = 0$ with $c_0 = 0.2$ and $R_0 = 0.4$

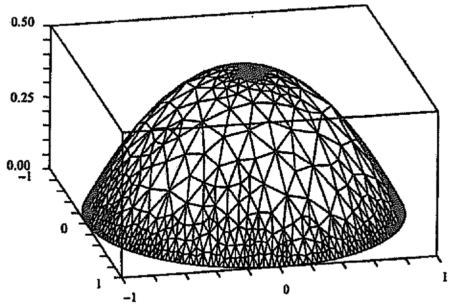

Figure 4. Case $j = 0$ with $c_0 = 0.1$ and $R_0 = 0.01$, by the finite element technique

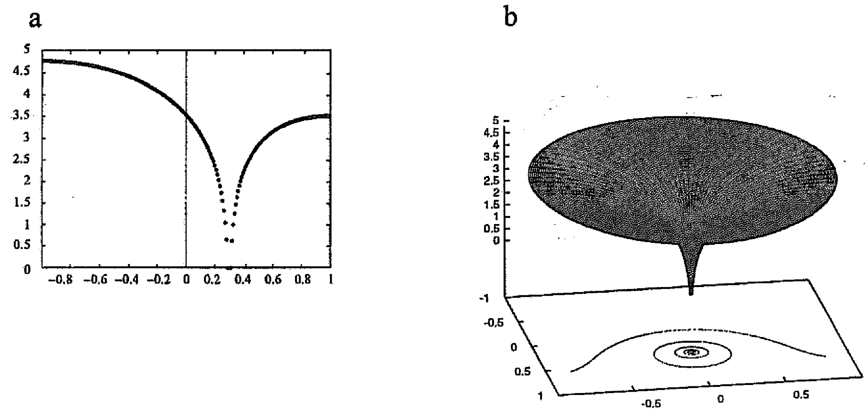

Figure 5. Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.01$

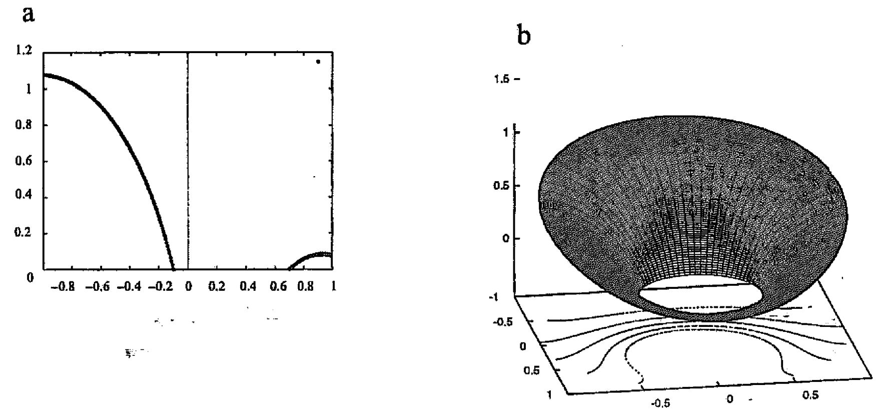

Figure 6. Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.4$

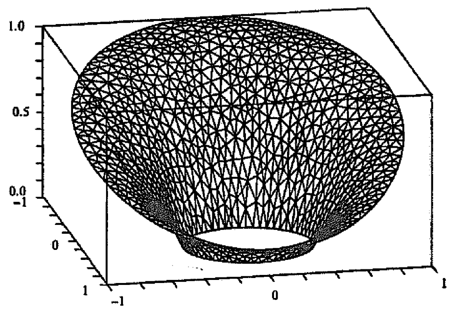

Figure 7. Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.4$, by the finite element technique

APPENDIX

We give here the proofs of lemmas which we used.

Proof of Lemma 2.5. Let us put $\zeta := \varrho e^{i\theta}$, $\lambda := \sigma e^{i\tau}$ and, for $\sigma \neq 0$, $a := (\varrho/\sigma)\sin\tau$ and $b := (\varrho/\sigma)\cos\tau$. We can write

$$I := \int_{A_{R_0}} \log|\zeta - \lambda|^2 \, d\zeta = \int_{R_0}^{1} \varrho \, d\varrho \int_0^{2\pi} d\theta \log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau)$$

$$= \int_{R_0}^{1} \varrho \, d\varrho \Big(2\pi \log\sigma^2 + \int_0^{2\pi} d\theta \log(1 + a^2 + b^2 - 2a\cos\theta - 2b\sin\theta)\Big).$$

Since

$$\int_0^{2\pi} d\theta \log(1 + a^2 + b^2 - 2a\cos\theta - 2b\sin\theta) = 2\pi \mathbb{1}_{\{a^2+b^2 \geq 1\}} \log(a^2 + b^2)$$

(see also [5], p. 528, 4.225.4), we obtain

$$I = 2\pi \int_{R_0}^{1} \varrho d\varrho \left(\log\sigma^2 + \mathbf{1}_{\{\varrho \geq \sigma\}} \log(\varrho^2/\sigma^2)\right)$$

$$= \begin{cases} 2\pi\left(\int_{R_0}^1 \varrho d\varrho\right)\log\sigma^2 & \text{if } \sigma > 1, \\ 2\pi\left(\int_{R_0}^{\sigma} \varrho d\varrho\right)\log\sigma^2 + 2\pi\int_{\sigma}^1 \varrho\log\varrho^2\, d\varrho & \text{if } R_0 < \sigma < 1, \\ 2\pi\int_{R_0}^1 \varrho\log\varrho^2\, d\varrho & \text{if } \sigma < R_0. \end{cases}$$

The preceding equality lies also for $\sigma = 0$, and we get (2.5). ■

Proof of Lemma 3.4. Let us put $\zeta := \varrho e^{i\theta}$ and $a := \varrho/k < 1$. We can write

$$\int_{\{|\zeta| = \varrho\}} \frac{\partial}{\partial n} \frac{1}{|\zeta - k|^2} d\sigma = \varrho \int_0^{2\pi} d\theta \frac{\partial}{\partial\varrho} \frac{1}{\varrho^2 + k^2 - 2k\varrho\cos\theta}$$

$$= \frac{a}{k^2} \int_0^{2\pi} \frac{(-2a + 2\cos\theta)\, d\theta}{(1 + a^2 - 2a\cos\theta)^2} = \frac{a}{k^2} \frac{d}{da} \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{a}{k^2} \frac{d}{da} \frac{2\pi}{1 - a^2}.$$

From this we get (3.4). ■

For the proof of Lemma 3.5 we need the following result:

Lemma A.1. *For* $\lambda \in \boldsymbol{C}$,

$$\text{(A.1)} \qquad \int_{\{|\zeta| = \varrho\}} \frac{\log|\zeta - \lambda|^2}{|\zeta - k|^2} d\sigma = \frac{2\pi\varrho}{k^2 - \varrho^2} \cdot \begin{cases} \log|\lambda - \varrho^2/k|^2 & \textit{if } |\lambda| > \varrho, \\ \log\varrho^2 |1 - \lambda/k|^2 & \textit{if } |\lambda| < \varrho. \end{cases}$$

Proof. Let us put $\zeta := \varrho e^{i\theta}$, $\lambda := \sigma e^{i\tau}$, $a := \varrho/k < 1$ and, for $\sigma \neq 0$, $b := \varrho/\sigma$. We can write

$$I := \int_{\{|\zeta| = \varrho\}} \frac{\log|\zeta - \lambda|^2}{|\zeta - k|^2} d\sigma = \frac{a}{k} \int_0^{2\pi} d\theta \frac{\log\sigma^2}{1 + a^2 - 2a\cos\theta}$$

$$+ \frac{a}{k} \int_0^{2\pi} d\theta \frac{\log(1 + b^2 - 2b\cos\theta\cos\tau - 2b\sin\theta\sin\tau)}{1 + a^2 - 2a\cos\theta}.$$

The first integral is equal to $(2\pi\log\sigma^2)/(1 - a^2)$. Then (A.1) is easily obtained for real λ (that is, for $\tau = 0$ or $\tau = \pi$) since

$$\int_0^{2\pi} d\theta \frac{\log(1 + b^2 - 2b\cos\theta)}{1 + a^2 - 2a\cos\theta} = \frac{4\pi}{1 - a^2} \cdot \begin{cases} \log(1 - ab) & \text{if } b^2 \leq 1, \\ \log(b - a) & \text{if } b^2 > 1 \end{cases}$$

(see also [5], p. 594, 4.397.16).

Then, by a classical argument of analytic continuation, we obtain

$$\int_{\{|\zeta| = \varrho\}} \frac{\log(\zeta - \lambda)^2}{|\zeta - k|^2} d\sigma = \frac{2\pi\varrho}{k^2 - \varrho^2} \cdot \begin{cases} \log(\lambda - \varrho^2/k)^2 & \text{if } |\lambda| > \varrho, \\ \log\varrho^2(1 - \lambda/k)^2 & \text{if } |\lambda| < \varrho. \end{cases}$$

The preceding equality lies also for $\sigma = |\lambda| = 0$ and its real part is nothing but (A.1). ■

Proof of Lemma 3.5. We shall use the same notation as in the proof of Lemma A.1. We write $J(\varrho, \lambda) = I_1 - I_2$, where

$$\begin{aligned}
I_1 :&= \int_{\{|\zeta| = \varrho\}} \log|\zeta - \lambda|^2 \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta - k|^2} d\sigma \\
&= \varrho \int_0^{2\pi} d\theta \log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau) \frac{\partial}{\partial\varrho} \frac{1}{\varrho^2 + k^2 - 2k\varrho\cos\theta} \\
&= \frac{a}{k^2} \int_0^{2\pi} d\theta \log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau) \frac{-2a + 2\cos\theta}{(1 + a^2 - 2a\cos\theta)^2} \\
&= \frac{a}{k^2} \frac{\partial}{\partial a} \int_0^{2\pi} d\theta \frac{\log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau)}{1 + a^2 - 2a\cos\theta},
\end{aligned}$$

$$\begin{aligned}
I_2 :&= \int_{\{|\zeta| = \varrho\}} \frac{1}{|\zeta - k|^2} \frac{\partial}{\partial \mathrm{n}} \log|\zeta - \lambda|^2 d\sigma \\
&= \varrho \int_0^{2\pi} d\theta \frac{1}{\varrho^2 + k^2 - 2k\varrho\cos\theta} \frac{\partial}{\partial\varrho} \log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau) \\
&= b \int_0^{2\pi} d\theta \frac{1}{\varrho^2 + k^2 - 2k\varrho\cos\theta} \cdot \frac{2b - 2\cos\theta\cos\tau - 2\sin\theta\sin\tau}{1 + b^2 - 2b\cos\theta\cos\tau - 2b\sin\theta\sin\tau} \\
&= b \frac{\partial}{\partial b} \int_0^{2\pi} d\theta \frac{\log(1 + b^2 - 2b\cos\theta\cos\tau - 2b\sin\theta\sin\tau)}{\varrho^2 + k^2 - 2k\varrho\cos\theta}.
\end{aligned}$$

But (A.1) can be written as

$$\text{(A.2)} \quad \int_0^{2\pi} d\theta \frac{\log(\varrho^2 + \sigma^2 - 2\varrho\sigma\cos\theta\cos\tau - 2\varrho\sigma\sin\theta\sin\tau)}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2} \cdot \begin{cases} \log\varrho^2(1/b^2 + a^2 - 2(a/b)\cos\tau) & \text{if } |\lambda| > \varrho, \\ \log\varrho^2(1 + a^2/b^2 - 2(a/b)\cos\tau) & \text{if } |\lambda| < \varrho \end{cases}$$

or as

$$\text{(A.3)} \quad \int_0^{2\pi} d\theta \frac{\log(1 + b^2 - 2b\cos\theta\cos\tau - 2b\sin\theta\sin\tau)}{\varrho^2 + k^2 - 2k\varrho\cos\theta} = \frac{2\pi}{k^2 - \varrho^2} \cdot \begin{cases} \log(1 + a^2 b^2 - 2ab\cos\tau) & \text{if } |\lambda| > \varrho, \\ \log(b^2 + a^2 - 2ab\cos\tau) & \text{if } |\lambda| < \varrho. \end{cases}$$

By the derivation of (A.2) and (A.3) with respect to a and b, respectively, we get

$$I_1 = \begin{cases} \dfrac{4\pi\varrho^2}{k^2(k^2 - \varrho^2)} \cdot \dfrac{\varrho^2 - |\lambda| k\cos\tau}{|\lambda - \varrho^2/k|^2} + \dfrac{8\pi\varrho^2}{(k^2 - \varrho^2)^2} \log|\lambda - \varrho^2/k| & \text{if } |\lambda| > \varrho, \\ \dfrac{4\pi}{k^2(k^2 - \varrho^2)} \cdot \dfrac{|\lambda|(|\lambda| - k\cos\tau)}{|1 - \lambda/k|^2} + \dfrac{8\pi\varrho^2}{(k^2 - \varrho^2)^2} \log\varrho|1 - \lambda/k| & \text{if } |\lambda| < \varrho, \end{cases}$$

and

$$I_2 = \begin{cases} \dfrac{4\pi\varrho^2}{k^2(k^2-\varrho^2)} \cdot \dfrac{\varrho^2 - |\lambda| k \cos\tau}{|\lambda - \varrho^2/k|^2} & \text{if } |\lambda| > \varrho, \\ \dfrac{4\pi}{k^2-\varrho^2} \cdot \dfrac{1 - |\lambda/k| \cos\tau}{|1-\lambda/k|^2} & \text{if } |\lambda| < \varrho, \end{cases}$$

and the proof of Lemma 3.5 is done. ∎

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