PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 20, Fasc. 1 (2000), pp. 75–91

MARTIN REPRESENTATION FOR α-HARMONIC FUNCTIONS

BY

KRZYSZTOF MICHALIK* AND KRZYSZTOF SAMOTIJ* (WROCŁAW)

Abstract. Let D be a nonempty open bounded subset of $\boldsymbol{R}^d$, $d \geqslant 2$, and let $0 < \alpha < 2$. For α-harmonic functions on D vanishing outside D an analogue of the Martin representation for harmonic functions is derived.

1. Introduction. The problem of the Martin boundary for harmonic functions is a part of potential theory. This theory can be expressed in a probabilistic way with the use of Markov processes. In the past the relationships between the classical Newtonian potential and the Brownian motion were investigated. A natural extension of the classical potential theory and harmonic functions is the theory of Riesz potentials and α-harmonic functions. This case has deep connections with the rotation invariant ('symmetric') stable processes with their index of stability $\alpha < 2$. Although in general these processes differ from the Brownian motion, they have some properties which are similar or analogous to the corresponding properties of the latter process. That is why the results concerning Riesz potentials and α-harmonic functions appear in many situations in probability theory, potential theory and in various analytical applications. They are often an interesting natural generalization of classical results.

For the rest of the paper let X_t be a symmetric stable process in $\boldsymbol{R}^d$ of index α for $d \geqslant 2$ and $0 < \alpha < 2$. For a Borel subset B of $\boldsymbol{R}^d$ let T_B and τ_B be the first entry time and the first exit time, respectively, i.e. $\tau_B = \inf\{t > 0\colon X_t \in B\}$ and $\tau_B = T_{B^c}$. D will stand for a nonempty open subset of $\boldsymbol{R}^d$. A nonnegative Borel function h on $\boldsymbol{R}^d$ is said to be *α-harmonic* on D if for each bounded open set B with $\bar{B} \subset D$ and for $x \in B$ we have

$$(1) \qquad h(x) = E^x h(X_{\tau_B}) < \infty.$$

This definition is equivalent to another one in which (1) is required to hold only for each ball $B = B(x, r) = \{y \in \boldsymbol{R}^d\colon |x-y| < r\}$ with $0 < r < \operatorname{dist}(x, D^c)$. In this

* Institute of Mathematics, Technical University of Wrocław.

case (1) can be written explicitly as

$$h(x) = \varepsilon_r * h(x) = \int_{R^d} \varepsilon_r(x-z)\,h(z)\,dz, \tag{2}$$

where

$$\varepsilon_r(y) = \begin{cases} \dfrac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{1+d/2}} \dfrac{1}{(|y|^2-r^2)^{\alpha/2}\,|y|^{\alpha}}, & |y| > r, \\ 0, & |y| \leqslant r. \end{cases} \tag{3}$$

The family of all α-harmonic functions on D will be denoted here by $\mathscr{H}^{\alpha}(D)$, and the family of those functions in $\mathscr{H}^{\alpha}(D)$ which vanish on D^c will be denoted by $\mathscr{H}^{\alpha}_0(D)$. Functions from $\mathscr{H}^{\alpha}(D)$ are continuous on D. Each nonnegative Borel function on D^c can be extended to D by the formula

$$h(x) = E^x h(X_{\tau_D}), \qquad x \in D.$$

Then h is either harmonic on D or infinite on D.

One of the classical results states that if D is a bounded Lipschitz domain in $\boldsymbol{R}^d$, then each nonnegative harmonic function on D admits a representation

$$h(x) = \int_{\partial D} P(x, y)\,d\mu(y), \qquad x \in D, \tag{4}$$

where $P(x, y)$ is the Poisson kernel for D, μ is some Borel measure, and ∂D denotes the Euclidean boundary of D. If D is a general bounded domain in $\boldsymbol{R}^d$, then a representation similar to (4), called the *Martin representation*, is valid, although the Euclidean boundary ∂D must be replaced by the so-called Martin boundary and the Poisson kernel for D is replaced by the Martin kernel $M(x, y)$. $M(x, y)$ is nonnegative and harmonic in D with respect to $x \in D$.

The main purpose of this paper is to establish a version of Martin representation for functions in $\mathscr{H}^{\alpha}_0(D)$ (Theorem 5.12). The recent result shows that for a bounded Lipschitz domain the Martin boundary coincides, as for $\alpha = 2$, with the topological boundary (cf. [5]). Our paper refers to the general case of bounded domains in $\boldsymbol{R}^d$. Contrary to what may seem at a first glance, Theorem 5.12 is not a theorem about the zero function. Examples of nontrivial functions in $\mathscr{H}^{\alpha}_0(D)$ are given in Section 2. In Section 3 we prove a decomposition theorem for functions in $\mathscr{H}^{\alpha}(D)$, which states that each $h \in \mathscr{H}^{\alpha}(D)$ can be represented uniquely as the sum of an $\mathscr{H}^{\alpha}_0(D)$ function and a function from $\mathscr{H}^{\alpha}(D)$ "orthogonal" to $\mathscr{H}^{\alpha}_0(D)$. This second term has a natural representation (5).

In general, as long as possible, we follow the classical schemes. But, unlike in the case of $\alpha = 2$, α-harmonicity is a global property. For this reason the kernels $M(\cdot, y)$ are not always α-harmonic on D. We show this at the end of this paper. Hence, to prove the uniqueness part, we use different arguments. However, the boundary which we construct can be characterized as for $\alpha = 2$. Finally, we must remark that this boundary coincides with the boundary S_1' introduced in [8] in a more general setting. Also, as in the classical case, the kernel M coincides on D with the kernel κ defined in [8], except for the definition for $x = y = x_0$.

In the sequel an open bounded nonempty subset D of $\mathbf{R}^d$ will be fixed. We will also use the notation (D_n) for a fixed sequence of nonempty bounded open subsets of D such that $\overline{D_n} \subset D_{n+1}$, $n = 1, 2, \ldots$, and $\bigcup_n D_n = D$.

2. Examples of functions in $\mathscr{H}_0^\alpha(D)$

EXAMPLE 2.1. The function

$$h(x) = \begin{cases} |x-y|^{\alpha-d}, & x \neq y, \\ 0, & x = y \end{cases}$$

is in $\mathscr{H}_0^\alpha(\mathbf{R}^d \backslash \{y\})$ ([9], I.6.19). More generally, if $g_D(x, y)$ is (discussed later) the Green function for D and $y \in D$, then the function

$$h(x) = \begin{cases} g_D(x, y), & x \in D\backslash\{y\}, \\ 0, & x \notin D\backslash\{y\} \end{cases}$$

is in $\mathscr{H}_0^\alpha(D\backslash\{y\})$.

EXAMPLE 2.2. Let K be a compact subset of D of nonzero α-capacity and zero Lebesgue measure. Let us define the function h_1 by the following formula:

$$h_1(x) = \begin{cases} E^x(\mathbf{1}_K(X_{\tau_{D\backslash K}})), & x \in D\backslash K, \\ \mathbf{1}_K(x), & x \notin D\backslash K, \end{cases}$$

and let $h = \mathbf{1}_{D\backslash K} \cdot h_1$. The function h_1 is α-harmonic on $D\backslash K$, equal to h on $D\backslash K$, and equal to h a.e. on $\mathbf{R}^d$. Therefore for $x \in D\backslash K$ and r, $0 < r < \operatorname{dist}(x, (D\backslash K)^c)$, we have

$$h(x) = h_1(x) = \varepsilon_r * h_1(x) = \varepsilon_r * h(x).$$

Hence h is α-harmonic on $D\backslash K$. Since $h = 0$ on $D\backslash K$, we have $h \in \mathscr{H}_0^\alpha(D\backslash K)$. Since K is of nonzero α-capacity, we have

$$h(x) = P^x\{X_{\tau_{D\backslash K}} \in K\} > 0, \quad x \in D\backslash K,$$

so h is also nontrivial.

EXAMPLE 2.3. Let $\hat{P}(x, y)$, $y \in \mathbf{R}^d$, $|x| < 1$, be the Poisson kernel for the unit ball, i.e. the density function of $P^x(X_{\tau_{B(0,1)}} \in dy)$. This function is explicitly given by the formula ([9], I.6.23)

$$\hat{P}(x, y) = \begin{cases} \dfrac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{1+d/2}} \dfrac{(1-|x|^2)^{\alpha/2}}{(|y|^2-1)^{\alpha/2}|x-y|^d}, & |y| > 1, \\ 0, & |y| \leqslant 1. \end{cases}$$

Let $|y| > 1$ and $0 < \varrho < |y|-1$. The function

$$h_{y,\varrho}(x) = \frac{1}{|B(y, \varrho)|} E^x\big(\mathbb{1}_{B(y,\varrho)}(X_{\tau_{B(0,1)}})\big)$$

is α-harmonic in $B(0, 1)$. Therefore for each x, $|x| < 1$, and each r, $0 < r < 1-|x|$, we have

$$h_{y,\varrho}(x) = \varepsilon_r * h_{y,\varrho}(x).$$

When we express $h_{y,\varrho}$ in the above equality in terms of $\hat{P}$ and we let ϱ tend to zero, then we obtain

$$\hat{P}(x, y) = \varepsilon_r * \hat{P}(\cdot, y)(x) + \varepsilon_r(x-y), \qquad 0 < r < 1-|x|.$$

Multiplying both sides of the above equality by $(|y|^2-1)^{\alpha/2}$ and letting y, $|y| > 1$, tend to a fixed point y_0, $|y_0| = 1$, we obtain

$$h_{y_0}(x) = \varepsilon_r * h_{y_0}(x), \qquad 0 < r < 1-|x|,$$

where

$$h_{y_0}(x) = \begin{cases} \dfrac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{1+d/2}} \dfrac{(1-|x|^2)^{\alpha/2}}{|x-y_0|^d}, & |x| < 1, \\ 0, & |x| \geqslant 1. \end{cases}$$

Hence the function h_{y_0} belongs to $\mathscr{H}_0^\alpha(B(0, 1))$. As can be easily verified by Fubini's theorem, a more general example of a nontrivial function from $\mathscr{H}_0^\alpha(B(0, 1))$ is given by the formula

$$h(x) = \int_{|y|=1} h_y(x)\,\mu(dy),$$

with μ being any nonzero Borel measure on the unit sphere.

3. A decomposition theorem for α-harmonic functions

THEOREM 3.1. *Every α-harmonic function h in $\mathscr{H}^\alpha(D)$ can be uniquely represented in the form $h = h_1 + h_2$, where $h_1 \in \mathscr{H}_0^\alpha(D)$, $h_2 \in \mathscr{H}^\alpha(D)$ and h_2 is such that the condition: $h_2 \geqslant v$ with $v \in \mathscr{H}_0^\alpha(D)$ implies $v \equiv 0$. Moreover, h_2 is of the form*

$$h_2(x) = \begin{cases} E^x\big(h(X_{\tau_D}) \colon X_{\tau_D -} \in D\big), & x \in D, \\ h(x), & x \notin D. \end{cases} \tag{5}$$

Proof. For $h \in \mathscr{H}^\alpha(D)$ and for a positive integer n define

$$T_n h(x) = \begin{cases} E^x\big(h(X_{\tau_{D_n}}) \colon X_{\tau_{D_n}} \in D^c\big), & x \in D_n, \\ h(x), & x \notin D_n. \end{cases}$$

Note that we have either $\tau_{D_n} < \tau_D$ for all n or $\tau_{D_n} = \tau_D$ for all but finite numbers of n's. By quasi-continuity of X_t we have

$$X_{\tau_D -} = \lim_n X_{\tau_{D_n}} = X_{\tau_D} \in D^c \text{ a.s.} \quad \text{on } \{\tau_{D_n} < \tau_D\colon n = 1, 2, \ldots\}$$

and

$$X_{\tau_D -} \in \bigcup_n \bar{D}_n = D \quad \text{on } \bigcup_n \{\tau_{D_n} = \tau_D\}.$$

Hence

$$\{X_{\tau_D -} \in D\} = \bigcup_n \{X_{\tau_{D_n}} \in D^c\} \text{ a.s.} \quad \text{with } \{X_{\tau_{D_n}} \notin D\} \subset \{X_{\tau_{D_{n+1}}} \notin D\}.$$

Therefore $T_n h(x) \uparrow Th(x)$ as $n \to \infty$, where $Th(x)$ denotes the right-hand side of (5). Moreover, $h(x) = E^x\big(h(X_{\tau_{D_n}})\big) \geqslant T_n h(x)$, $x \in D$. The monotone convergence theorem implies that the function Th is α-harmonic on D as a finite limit of a nondecreasing sequence $(T_n h)$ of functions α-harmonic on D. It is clear that $T(Th) = Th$ and $Tv = 0$ if $v \in \mathscr{H}_0^\alpha(D)$. So, if $h \in \mathscr{H}_0^\alpha(D)$, we set $h_2 = Th$ and $h_1 = h - Th$. The function h_1 is α-harmonic on D as the nonnegative difference of functions α-harmonic on D, and $h_1 = 0$ on D^c. To complete this proof assume that $v \in \mathscr{H}_0^\alpha(D)$ and $0 \leqslant v \leqslant h_2$. Then $h_2 - v \geqslant T(h_2 - v) = Th_2 - Tv = Th_2 = T(Th) = Th = h_2$. Consequently, we have $v = 0$.

4. Auxiliary results. Together with the process X_t we will consider the process $\hat{X}_t$ which is "X_t killed on exiting D". This is a Markov process on the state space $D_\Delta = D \cup \{\Delta\}$, where $\Delta \notin D$, defined by the formula

$$\hat{X}_t(\omega) = \begin{cases} X_t(\omega), & t < \tau_D(\omega), \\ \Delta, & t \geqslant \tau_D(\omega), \end{cases}$$

if $\hat{X}_0(\omega) \in D$, and $\hat{X}_t(\omega) = \Delta$, $t > 0$, if $\hat{X}_0(\omega) = \Delta$. As is customary, all functions on D_Δ considered here will be assumed to vanish at Δ. Therefore, we may identify each function on D_Δ with its restriction to D. The potential operator for the process $\hat{X}_t$ is denoted by G_D, i.e.

$$G_D f(x) = E^x \int_0^\infty f(\hat{X}_t)\, dt = E^x \int_0^{\tau_D} f(X_t)\, dt, \quad x \in D,$$

for each nonnegative Borel function f on D. The operator G_D has a kernel which will be denoted by $g_D(x, y)$, i.e.

$$G_D f(x) = \int_D g_D(x, y) f(y)\, dy, \quad x \in D.$$

The kernel g_D is positive on $D \times D$ and symmetric. Also, for each fixed $x \in D$, the function $g_D(x, \cdot)$ is finite and continuous on $D\backslash\{x\}$ while near x we have $\lim_{y\to x} g_D(x, y) = +\infty$ (see [7]). In the case $D = \mathbf{R}^d$ we have

$$g_D(x, y) = u(x-y) = A_{d,\alpha}|x-y|^{\alpha-d}, \quad \text{where } A_{d,\alpha} = \frac{\pi^{\alpha-d/2}\,\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}.$$

LEMMA 4.1. *If $h \in \mathcal{H}^\alpha(D)$, then $h|_D$ is excessive with respect to $\hat{X}_t$.*

Proof. Since h is continuous on D, it is enough to show that $h(x) \geq E^x h(\hat{X}_t)$, $x \in D$, $t > 0$. To see this note that for each $x \in D$, each $t > 0$ and each n with $x \in D_n$, we have

$$\begin{aligned} h(x) &= E^x h(X_{\tau_{D_n}}) \geq E^x \{h(X_{\tau_{D_n}}): t < \tau_{D_n}\} \\ &= E^x \{E^{X_t} h(X_{\tau_{D_n}}): t < \tau_{D_n}\} = E^x \{h(X_t): t < \tau_{D_n}\}. \end{aligned}$$

Since $\{t < \tau_{D_n}\} \uparrow \{\tau < \tau_D\}$ a.s., letting n tend to infinity we obtain

$$h(x) \geq \lim_n E^x \{h(X_t): t < \tau_n\} = E^x \{h(X_t): t < \tau_D\} = E^x h(\hat{X}_t).$$

PROPOSITION 4.2. *Let $h \in \mathcal{H}^\alpha(D)$ and let $h_n(x) = E^x h(\hat{X}_{T_{D_n}})$, $x \in D$. Then there is a Borel measure ν_n concentrated on $\overline{D_n}$ such that*

$$h_n(x) = G_D \nu_n(x), \qquad x \in D.$$

In particular, we have $G_D \nu_n(x) = h(x)$, $x \in D_n$.

Proof. Note that since $g_D(x, y) \leq A_{d,\alpha}|x-y|^{d-\alpha}$, and since the function $g_D(\cdot, y)$ is continuous, from the dominated convergence theorem we may conclude that $G_D f$ is continuous whenever f is a bounded function on D vanishing outside a compact subset of D. By Lemma 4.1, the function h is excessive with respect to $\hat{X}_t$. Therefore our proposition follows by Theorem VI.2.8. of [3].

PROPOSITION 4.3. *Let $h \in \mathcal{H}_0^\alpha(D)$. Let h_n and ν_n be such as in Proposition 4.2. Let K be an arbitrary compact subset of D and let $\nu_n' = \nu_n|_K$. Then $\sup_{x\in K} G_D \nu_n'(x)$ converges to zero as n tends to infinity. In particular, $\lim_n \nu_n(K) = 0$.*

Proof. Let $x \in K$ and let $0 < 2r < \operatorname{dist}(K, D^c)$. Then for each n such that $K \subset D_n$ we have

$$\begin{aligned} (6)\qquad 0 &\leq G_D \nu_n'(x) - \varepsilon_r * G_D \nu_n'(x) \leq G_D \nu_n(x) - \varepsilon_r * G_D \nu_n(x) \\ &= h(x) - \varepsilon_r * h_n(x) = \varepsilon_r * (h - h_n)(x) \leq \varepsilon_r * (1_{D\backslash D_n}(x)\, h(x)). \end{aligned}$$

By the dominated convergence theorem, the last expression converges to zero uniformly with respect to $x \in K$, when n tends to infinity. On the other hand, if we assume additionally that $r < \operatorname{dist}(K, D_n^c)$, then we have

$$\begin{aligned} (7)\qquad & G_D \nu_n'(x) - \varepsilon_r * G_D \nu_n'(x) = G_{\mathbf{R}^d} \nu_n'(x) - \varepsilon_r * G_{\mathbf{R}^d} \nu_n'(x) \\ &= \int_K [u(x-y) - (\varepsilon_r * u)(x-y)]\, \nu_n'(dy) \geq \int_{|x-y|<r/3} (1-2^{\alpha-d})\, u(x-y)\, \nu_n'(dy). \end{aligned}$$

Since $u(x-y) \geqslant A_{d,\alpha}(r/3)^{\alpha-d}$, $|x-y| \leqslant r/3$, by compactness of K, the inequalities (6) and (7) imply that

$$\lim_n \nu'_n(K) = 0. \tag{8}$$

To complete this proof note that

$$G_D \nu'_n(K) \leqslant \int u(x-y)\,\nu'_n(dy) \leqslant \int_{|x-y|<r/3} u(x-y)\,\nu'_n(dy) + \int_K A_{d,\alpha}(r/3)^{\alpha-d}\,\nu'_n(dy).$$

But, by (6)–(8), each of the last two integrals converges to zero as n tends to infinity.

5. The Martin representation. Our construction of the Martin boundary follows a classical scheme (cf. e.g. [6], Chapter XIV). For a fixed point $x_0 \in D$ let us define a function M on $R^d \times D$ with values in $[0, \infty]$ by the formula

$$M(x, y) = \begin{cases} g_D(x, y)/g_D(x_0, y), & x \neq x_0,\ x \in D, \\ 1, & x = x_0, \\ 0, & x \in D^c. \end{cases}$$

Let K be an arbitrary compact subset of D containing x_0 and let B be an open set with $K \subset B \subset D$. For a fixed $y \in D \backslash B$ the function $M(\cdot, y)$ is α-harmonic on B and assumes value 1 at x_0. Hence, by the generalized Harnack inequality there is a constant C depending only on K, B and D such that

$$M(x, y) \leqslant C, \quad x \in K,\ y \in D \backslash B. \tag{9}$$

Let D^* be the Constantinescu–Cornea compactification ([6], Chapter XIII) of D with respect to the family of functions $M(x, \cdot)$, $x \in D$. The set D is a dense subset of D^*, and functions of the family $M(x, \cdot)$, $x \in D$, extend uniquely to continuous functions on D^*. The space D^* is metrizable, since the family $M(x, \cdot)$, $x \in D$, contains a countable subfamily separating points of D.

Let h be a fixed function in $\mathscr{H}_0^\alpha(D)$. For each positive integer n we define a Borel measure μ_n on D by the formula

$$\mu_n(dy) = g_D(x_0, y)\,\nu_n(dy),$$

where ν_n is the measure from Proposition 4.3. Hence we have

$$\mu_n(D) = \int_D g_D(x_0, y)\,\nu_n(dy) \leqslant h(x_0) < \infty. \tag{10}$$

Since D is an open subset of D^*, each measure μ_n may be considered to be a measure on D^* which is concentrated on D. The space D^* is metrizable and compact, therefore (10) implies that the sequence (μ_n) is relatively weakly compact. Let μ be a weak accumulation point of (μ_n). By Proposition 4.3 we have $\mu(D) = 0$. Moreover, by Proposition 4.2, for each $x \in D$ and each n with $x \in D_n$ we have

$$h(x) = G_D \nu_n(x) = \int_D g_D(x, y)\, \nu_n(dy) = \int_{D^*} M(x, y)\, \mu_n(dy).$$

Hence

$$h(x) = \int_{D^* \setminus D} M(x, y)\, \mu(dy).$$

The set $\partial_M D = D^* \setminus D$ may be called the *Martin boundary* of D. Note that ∂_M depends on α. Thus we have proved the following theorem.

THEOREM 5.1. *For each $h \in \mathscr{H}_0^\alpha(D)$ there exists a Borel measure μ on $\partial_M D$ such that*

$$h(x) = \int_{\partial_M D} M(x, y)\, \mu(dy), \qquad x \in D. \tag{11}$$

It was shown by K. Bogdan (private communication) that the converse of this theorem is not true in general. More precisely, he constructed a region D in $\mathbf{R}^d$ such that for some $y \in \partial_M D$ the function

$$h(x) = \begin{cases} M(x, y), & x \in D, \\ 0, & x \notin D, \end{cases} \tag{12}$$

is not α-harmonic in D. However, if we denote by $\partial_{\tilde{M}} D$ the set of those points $y \in \partial_M D$ for which the function h defined by (12) is α-harmonic in D, then we have the following theorem.

THEOREM 5.2. *Let h be a nonnegative Borel function on $\mathbf{R}^d$ vanishing off D. Then $h \in \mathscr{H}_0^\alpha(D)$ iff there is a Borel measure μ on $\partial_{\tilde{M}} D$ such that*

$$h(x) = \int_{\partial_{\tilde{M}} D} M(x, y)\, \mu(dy), \qquad x \in D.$$

For the proof of this theorem the following lemma is needed.

LEMMA 5.3. *Let us assume that $x_1, x_2 \in D$ and $y_0 \in \partial_M D$ are given. Let r_1, r_2 be real numbers with $0 < r_1 < \operatorname{dist}(x_1, D^c)$ and $0 < r_2 < \operatorname{dist}(x_2, D^c)$. Then*

$$M(\cdot, y_0) * \varepsilon_{r_1}(x_1) = M(x_1, y_0) \tag{13}$$

if and only if

$$M(\cdot, y_0) * \varepsilon_{r_2}(x_2) = M(x_2, y_0). \tag{14}$$

Proof of Lemma 5.3. Let us choose any sequence (y_n) of points of D convergent to y_0 in D^*. Let us consider the following formal equality:

$$\int_K \big(\lim_n M(z, y_n)\big)\, \varepsilon_{r_1}(x_j - z)\, dz = \lim_n \int_K \big(M(z, y_n)\big)\, \varepsilon_{r_2}(x_j - z)\, dz. \tag{K, j}$$

Assume that (13) holds. This is equivalent to $(D, 1)$. Since M is nonnegative, $(D, 1)$, together with the pointwise convergence of $M(\cdot, y_n)$ to $M(\cdot, y_0)$, implies that $M(\cdot, y_n)$ converges in $L^1(D\backslash B(x_1, r_1), \varepsilon_{r_1}(x_1-z)\,dz)$ to $M(\cdot, y_0)$. Since

$$\varepsilon_{r_2}(x_2-z) \leqslant C\varepsilon_{r_1}(x_1-z) \quad \text{for } z\in D\backslash(B(x_1, r_1)\cup B(x_2, \varrho)),$$

with ϱ being a real number such that $r_2 < \varrho < \operatorname{dist}(x_2, D^c)$, also $M(\cdot, y_n)$ converges to $M(\cdot, y_0)$ in $L^1(D\backslash(B(x_1, r_1)\cup B(x_2, \varrho)), \varepsilon_{r_2}(x_2-z)\,dz)$. Hence $(D\backslash(B(x_1, r_2)\cup B(x_2, \varrho)), 2)$ is true.

On the other hand, by (9) and the bounded convergence theorem, we have $(B(x_1, r_2)\cup B(x_2, \varrho), 2)$. Combining

$$(D\backslash(B(x_1, r_1)\cup B(x_2, \varrho)), 2) \quad \text{and} \quad (B(x_1, r_1)\cup B(x_2, \varrho), 2),$$

we obtain $(D, 2)$ which is equivalent to (14).

Proof of Theorem 5.2. Let μ be any Borel measure on $\partial_{\tilde{M}} D$. Let us define

$$h(x) = \begin{cases} \int\limits_{\partial_{\tilde{M}} D} M(x, y)\,\mu(dy), & x\in D,\\ 0, & x\in \mathbf{R}^d\backslash D.\end{cases}$$

Then for $0 < r < \operatorname{dist}(x, D^c)$ we have by Fubini's theorem

$$\varepsilon_r * h(x) = \int\limits_D h(z)\,\varepsilon_r(x-z)\,dz = \int\limits_D\Big(\int\limits_{\partial_{\tilde{M}} D} M(z, y)\,\mu(dy)\Big)\varepsilon_r(x-z)\,dz$$

$$= \int\limits_{\partial_{\tilde{M}} D} M(x, y)\,\mu(dy) = h(x).$$

Hence $h\in \mathscr{H}_0^\alpha(D)$, and the "if" part of the theorem follows.

To prove the "only if" part of the theorem note that Lemma 5.3 and the Fatou lemma imply that $y\in\partial_M D\backslash\partial_{\tilde{M}} D$ if and only if for each $x\in D$ and each r, $0 < r < \operatorname{dist}(x, D^c)$, we have $M(\cdot, y) * \varepsilon_r(x) < M(x, y)$. Now, let h be any element of $\mathscr{H}_0^\alpha(D)$. By Theorem 5.1 there exists a Borel measure μ on $\partial_M D$ such that we have (11). Let us fix $x\in D$ and r, $0 < r < \operatorname{dist}(x, D^c)$. Then we have

$$0 = h(x)-(h*\varepsilon_r)(x) = \Big(\int\limits_{\partial_{\tilde{M}} D} + \int\limits_{\partial_M D\backslash\partial_{\tilde{M}} D}\Big)\Big[M(x, y)-\int\limits_D M(z, y)\,\varepsilon_r(x-z)\,dz\Big]\,\mu(dy)$$

$$= \int\limits_{\partial_M D\backslash\partial_{\tilde{M}} D} [M(x, y)-M(\cdot, y)*\varepsilon_r(x)]\,\mu(dy).$$

Since the last integrand is positive, we have $\mu(\partial_M D\backslash\partial_{\tilde{M}} D) = 0$. Therefore, by (11) we have

$$h(x) = \int\limits_{\partial_{\tilde{M}} D} M(x, y)\,\mu(dy), \qquad x\in D.$$

Now we are going to obtain, as for $\alpha = 2$, the uniqueness in the Martin representation. The main obstacle is the fact that a pointwise limit of α-harmonic functions does not need to be α-harmonic. To overcome this we introduce the following definition.

DEFINITION 5.4. Let μ be a Borel measure on R^d such that $\mu|_D$ is absolutely continuous with respect to the Lebesgue measure and its density u is locally integrable. We say that μ is *α-harmonic* on D if for every $x \in D$ and for every positive number δ such that $\overline{B(x, \delta)} \subset D$ the following equation holds:

$$u(x) = E^x \mu(X_{\tau_{B(x,\delta)}}),$$

where

$$E^x \mu(X_{\tau_{B(x,\delta)}}) = \int P^x(X_{\tau_{B(x,\delta)}} \in dy)\, d\mu(y) = \int_D \varepsilon_\delta(x-y) u(y)\, dy + \int_{D^c} \varepsilon_\delta(x-y)\, d\mu(y). \tag{15}$$

Note that we may put $u \equiv 0$ on D^c. Then every function $f \in \mathscr{H}_0^\alpha(D)$ may be considered to be an α-harmonic measure μ such that $\mu|_{D^c} \equiv 0$. It is easy to verify that the basic properties such as the Poisson integral representation and the Harnack principle remain true also for α-harmonic measures. Since the notion of α-harmonic measure is a natural extension of the notion of an α-harmonic function, as for functions we will denote the set of these measures by $\mathscr{H}^\alpha(D)$. We will also denote by $\mathscr{H}_0^\alpha(D)$ the set of the measures $\mu \in \mathscr{H}^\alpha(D)$ such that $\mu|_{\bar{D}^c} \equiv 0$. Moreover, from now on we will denote the densities of some measures $\mu, \nu \in \mathscr{H}^\alpha(D)$ by u and v, respectively.

Now, let $\mathscr{C} = \mathscr{H}_0^\alpha(D)$ and let $\mathscr{K} = \{\mu \in \mathscr{H}_0^\alpha(D)\colon u(x_0) = 1\}$. Both $\mathscr{C}$ and $\mathscr{K}$ are convex. Moreover, $\mathscr{K}$ is the intersection of $\mathscr{C}$ with the hyperplane $\{\mu\colon L\mu = 1\}$, where L is a linear functional defined as $L(\mu) = u(x_0)$.

To start with the next theorem we need some technical lemmas.

LEMMA 5.5. *Let A be an open subset of R^d. Let μ be a measure on R^d. Let ν be a measure which is absolutely continuous on A with respect to the Lebesgue measure and let its density be given by the formula*

$$v(x) = E^x \mu(X_{\tau_A}) = \int_{A^c} P^x(X_{\tau_A} \in dy)\, d\mu(y), \qquad x \in A. \tag{16}$$

Moreover, let $\nu \equiv \mu$ on A^c. Then $\nu \in \mathscr{H}^\alpha(A)$.

Proof. Let B be an open subset of A such that $\bar{B} \in A$. Let $x \in B$. Hence, by the strong Markov property and the Fubini theorem, we have

$$E^x \nu(X_{\tau_B}) = E^x\big(v(X_{\tau_B});\ \tau_B < \tau_A\big) + \int_{A^c} P^x(X_{\tau_B} \in dy)\, d\mu(y)$$

$$= E^x\big(E^{X_{\tau_B}} \mu(X_{\tau_A});\ \tau_B < \tau_A\big) + \int_{A^c} P^x(X_{\tau_B} \in dy;\ \tau_B = \tau_A)\, d\mu(y)$$

$$= E^x \int_{A^c} P^{X_{\tau_B}}(X_{\tau_A} \in dy;\ \tau_B < \tau_A)\, d\mu(y) + \int_{A^c} P^x(X_{\tau_A} \in dy;\ \tau_B = \tau_A)\, d\mu(y)$$

$$= \int_{A^c} ([P^x(X_{\tau_A} \in dy;\ \tau_B < \tau_A)] \circ \theta_{\tau_B})\, d\mu(y) + \int_{A^c} P^x(X_{\tau_A} \in dy;\ \tau_B = \tau_A)\, d\mu(y)$$

$$= \int_{A^c} P^x(X_{\tau_A} \in dy;\ \tau_B < \tau_A)\, d\nu(y) + \int_{A^c} P^x(X_{\tau_A} \in dy;\ \tau_B = \tau_A)\, d\nu(y)$$

$$= \int_{A^c} P^x(X_{\tau_A} \in dy)\, d\nu(y) = E^x \nu(X_{\tau_A}) = v(x).$$

This completes the proof.

We will call the measure ν defined in Lemma 5.5 an *α-harmonic extension* of μ onto A.

LEMMA 5.6. *Let δ be a positive number. Let $\phi \in C^\infty(\mathbf{R})$ be a nonnegative function for which $\int_{\mathbf{R}} \phi(x)\, dx = \int_{\delta/2}^{\delta} \phi(x)\, dx = 1$. Define the kernel ε_δ^ϕ as*

$$\varepsilon_\delta^\phi(y) = \int_{\delta/2}^{\delta} \phi(r)\, \varepsilon_r(y)\, dr.$$

Then $\varepsilon_\delta^\phi \in C^\infty(\mathbf{R}^d)$. Moreover, $\mu \in \mathscr{H}^\alpha(D)$ iff for each $x \in D$, for every $\delta < \operatorname{dist}(x, \partial D)$ and for each function ϕ defined as above we have

$$u(x) = \int \varepsilon_\delta^\phi(x-y)\, d\mu(y).$$

Proof. The proof of the first part is easy. To prove the second part consider the following two equations:

$$u(x) = \int \varepsilon_r(x-y)\, u(y)\, dy, \quad r < \delta,\ |x-y| > r, \tag{17}$$

and

$$u(x) = \int_{\delta/2}^{\delta} \int \phi(r)\, \varepsilon_r(x-y)\, u(y)\, dr\, dy = \int \varepsilon_r^\phi(x-y)\, u(y)\, dy. \tag{18}$$

If (17) holds, then (18) is an immediate consequence of the Fubini theorem. Conversely, if (18) holds for every ϕ defined in our lemma, then we have

$$u(x) = \int \varepsilon_r(x-y)\, u(y)\, dy$$

for almost every $r \in (\delta/2, \delta)$. Since $\varepsilon_r(z)$ is a continuous function of r for $r < |z|$, we obtain (17). This completes the proof.

THEOREM 5.7. *The set $\mathscr{K}$ is compact and metrizable in the topology of weak convergence of measures. The set $\mathscr{C}$ is a vector lattice (in the sense of the definition from* [1]*).*

Proof. Let $\mu_n \in \mathscr{K}$ for each $n \in N$. Let r be a positive number such that $\overline{B(x_0, r)} \in D$. By the Harnack principle, the functions u_n are uniformly bounded on $\overline{B(x_0, r)}$. Moreover, we have

$$1 = u_n(x_0) = \int_{\bar{D}\setminus B(x_0,r)} \varepsilon_r(x_0 - y)\, d\mu_n(y).$$

Since $\bar{D}$ is compact, $\inf_{y\in D} \varepsilon_r(x_0 - y) > 0$. Therefore, for each n, $\mu_n(\bar{D}\setminus B(x_0, r)) \leqslant c$ for some constant c. This implies that the total masses of the measures μ_n are uniformly bounded. Hence for some sequence n_k we have $\mu_{n_k} \Rightarrow \mu$. By the Harnack principle, the densities u_n are uniformly bounded on every compact set $A \in D$. Therefore, μ must be absolutely continuous on A with respect to the Lebesgue measure and its density u is equal to $\lim_{k\to\infty} u_{n_k}$. Let $x \in A$. Since u_n are α-harmonic on D, by Lemma 5.6, for sufficiently small $\delta > 0$ we have

$$u_n(x) = \int \varepsilon_\delta^\phi(x - y)\, d\mu_n(y).$$

The function $f(x, y) = \varepsilon_\delta^\phi(x-y)$ is uniformly continuous on $K \times \bar{D}$ and $\mu(R^d) < \infty$, so u is continuous on D. Hence we obtain

$$\lim_{k\to\infty} u_{n_k}(x) = u(x) = \int \varepsilon_\delta^\phi(x-y)\, d\mu(y).$$

Clearly, $u(x_0) = 1$. Hence, by Lemma 5.6, μ is α-harmonic on D. Therefore we have proved that every sequence chosen from $\mathscr{K}$ has a point of accumulation which is also an element of $\mathscr{K}$. This shows that $\mathscr{K}$ is compact. Since $\mathscr{K}$ is a bounded set of measures on a compact and metrizable set, the weak convergence topology on $\mathscr{K}$ is a metric one.

Now let us focus on the second part of the theorem. Let $\mu_1, \mu_2 \in \mathscr{H}_0^\alpha(D)$ and let $\mu = \mu_1 \vee \mu_2$. μ is the least measure that dominates both μ_1 and μ_2. Since μ_1 and μ_2 have densities on D, the same property holds for μ and its density is equal to $u = u_1 \vee u_2$. For each $n \in N$, let v_n be an α-harmonic extension of μ onto D_n. Since $\mu \leqslant \mu_1 + \mu_2$, for each $x \in D_n$ we have

$$v_n(x) = E^x \mu(X_{\tau_{D_n}}) \leqslant E^x \mu_1(X_{\tau_{D_n}}) + E^x \mu_2(X_{\tau_{D_n}}) = u_1(x) + u_2(x).$$

Naturally, $v_n \equiv u \leqslant u_1 + u_2$ on $D\setminus D_n$ and $v_n \equiv \mu$ on D^c. Hence we obtain $|v_n| \leqslant |\mu_1| + |\mu_2|$. The similar arguments as in the first part of the theorem show that for some subsequence n_k we have $v_{n_k} \Rightarrow v \in \mathscr{H}_0^\alpha(D)$ and $v = \lim_{k\to\infty} v_{n_k}$ is the density of v on D (with respect to the Lebesgue measure). Moreover, if U is an open neighbourhood of ∂D, then we obtain

$$\int_{D\cap U} v_n(x)\, dx \leqslant \int_{D\cap U} u_1(x)\, dx + \int_{D\cap U} u_2(x)\, dx,$$

and since u_1, u_2 are integrable, the following statement is true:

$$\forall \varepsilon > 0,\ \exists U \supset \partial D,\ \forall m \geqslant n,\ \int_{D\cap U} v_m(x)\, dx < \varepsilon.$$

But this implies that $\nu \equiv \mu$ on ∂D. Moreover, if $x \in D_n$, then

$$v_n(x) = E^x \mu(X_{\tau_{D_n}}) \geqslant E^x \mu_i(X_{\tau_{D_n}}) = u_i(x), \quad i = 1, 2.$$

Hence $v \geqslant u_i$ on each D_n, so on D. Therefore, ν dominates μ_1 and μ_2 on R^d. Now let $\nu' \in \mathscr{H}_0^\alpha(D)$ be another upper bound for μ_1 and μ_2. We have $\nu' \geqslant \mu$ and $v' \geqslant u_1 \vee u_2$ on D. Then for $x \in D_n$ we get

$$v'(x) = E^x v'(X_{\tau_{D_n}}) \geqslant E^x \mu(X_{\tau_{D_n}}) = v_n(x).$$

Hence $v'(x) \geqslant v(x)$, so $\nu' \geqslant \nu$ on D. Moreover, since $\nu' \geqslant \mu$ and $\nu \equiv \mu$ on ∂D, we see that $\nu' \geqslant \nu$ on ∂D, so $\nu' \geqslant \nu$ on R^d. Therefore, ν is the least upper bound for μ_1, μ_2.

The existence of the greatest lower bound can be proved analogously and the vector part of the definition is immediate. The proof is now completed.

Next we have to identify the extreme points of $\mathscr{K}$.

Definition 5.8. A measure $\mu \in \mathscr{H}^\alpha(D)$ is called *minimal harmonic* on D if for every measure $\nu \in \mathscr{H}^\alpha(D)$ such that $\nu \leqslant \mu$ there exists a number c for which $\nu = c\mu$.

Proposition 5.9. *A measure μ is an extreme point of $\mathscr{K}$ iff μ is minimal harmonic on D.*

Proof. The proof is similar to that for $\alpha = 2$ (cf. [1]).

Now recall the fundamental Choquet theorem:

Theorem 5.10. *If a set A is convex, compact and metrizable, then for every $x \in A$ there exists a probability measure μ supported on the extreme points of A such that x is the barycenter of μ. If, in addition, A is the intersection of some cone C with some hyperplane and C is a vector lattice, then μ is unique.*

Our sets $\mathscr{K}$ and $\mathscr{C}$ satisfy the assumptions of this theorem. Now fix $x \in D$. Define a linear functional L_x as $L_x(\mu) = u(x)$. By the Harnack principle, $|u(x)| \leqslant c$ for each $\mu \in \mathscr{K}$, so L_x is continuous. Therefore

$$u(x) = L_x(\mu) = \int_{\mathscr{K}} L(\nu)\, d\mu_{\mathscr{K}}(\nu) = \int_{\mathscr{K}} v(x)\, d\mu_{\mathscr{K}}(\nu),$$

where $\mu_{\mathscr{K}}$ is a unique probability measure concentrated on the extreme points of $\mathscr{K}$.

We now turn back to α-harmonic functions. Recall that each function $u \in \mathscr{H}_0^\alpha(D)$ is a density of some measure $\mu \in \mathscr{H}_0^\alpha(D)$ such that $\mu|_{\partial D} \equiv 0$. We may assume that $u(x_0) = 1$. So if B is a subset of D such that $\bar{B} \in D$ and $x \in B$, then by the Fubini theorem

$$u(x) = E^x \mu(X_{\tau_B}) = \int_{\mathscr{K}} E^x v(X_{\tau_B})\, d\mu_{\mathscr{K}}(\nu).$$

But, from Definition 5.4 we obtain $E^x v(X_{\tau_B}) \leqslant v(x)$, so we get

$$u(x) \leqslant \int_{\mathscr{K}} v(x)\mu_{\mathscr{K}}(v) = u(x).$$

This implies that $\mu_{\mathscr{K}}$ is concentrated on these measures v for which $E^x v(X_{\tau_B}) = v(x)$. Hence we have proved that $\mu_{\mathscr{K}}$ is concentrated on the α-harmonic functions $v \in \mathscr{H}_0^\alpha(D)$ which are minimal harmonic. But we also have the following

PROPOSITION 5.11. *If a function $f \in \mathscr{H}_0^\alpha(D)$ is minimal harmonic on D, then $f \equiv M(\cdot, y)$ for some $y \in \partial_M D$.*

Proof. Since Theorem 5.2 holds, the proof is similar to that for $\alpha = 2$ (see [1]).

We denote by $\partial_m D$ the points $y \in \partial_M D$ for which the kernels $M(\cdot, y)$ are minimal harmonic. The set $\partial_m D$ is called the *minimal Martin boundary*. Thus we have proved the formula

$$u(x) = \int_{\mathscr{K}_1} M(x, y)\, d\mu_{\mathscr{K}}(M(x, y)), \qquad \text{where } \mathscr{K}_1 = \{M(\cdot, y)\colon\ y \in \partial_m D\}.$$

Since $M(\cdot, y_1) \neq M(\cdot, y_2)$ for $y_1 \neq y_2$, we may assume that $\mu_{\mathscr{K}}$ is a measure concentrated on $\partial_m D$. Therefore, we have proved the following

THEOREM 5.12 (the Martin representation for α-harmonic functions). *A function u is an element of $\mathscr{H}_0^\alpha(D)$ iff there exists a unique measure μ concentrated on $\partial_m D$ such that for every $x \in D$*

$$u(x) = \int_{\partial_m D} M(x, y)\, d\mu(y).$$

We will end this paper showing an example of a set D for which $\partial_m D \neq \partial_M D$. The idea comes from K. Bogdan (private communication).

Let points x_n, $n \geqslant 1$, be elements of a ray, which has its origin at x_1. Moreover, let $\operatorname{dist}(x_n, x_1) < \operatorname{dist}(x_{n+1}, x_1)$ for each n, and let $d_n = \operatorname{dist}(B_n, B_{n+1}) > 0$, where $B_n = B(x_n, r_n)$, and $\sum_n (d_n + 2r_n) < \infty$. Then if we set $D = \bigcup_n B_n$, we obtain an open bounded subset of $\boldsymbol{R}^d$.

Put $x_0 = x_1$ and embed D into D^*. Since D^* is compact, we can find some subsequence of $\{x_n\}$ which converges to some $y \in \partial_M D$ as $n \to \infty$. From now on we will denote by x_n' the elements of this subsequence. We will show that $y \notin \partial_m D$. Since $M(\cdot, x_n')$ is α-harmonic on $D \setminus \{x_n'\}$, for every $x \in D$ and sufficiently small $\delta > 0$ we have

$$M(x, x_n') = \int \varepsilon_\delta(x - z)\, M(z, x_n')\, dz.$$

Hence it is enough to show that there exists a positive ε such that for each open neighbourhood U of y there is a positive integer n such that $x_n' \in U$ and

$\int_U M(z, x_n')\,dz \geqslant \varepsilon$. For if this condition holds, then the sequence of measures with the densities (with respect to the Lebesgue measure) $M(\cdot, x_n')$ is convergent to an α-harmonic measure with a positive mass at y and with the density on D equal to $M(\cdot, y)$ (see Theorem 5.7). Hence we will obtain the inequality

$$M(x, y) > \int \varepsilon_\delta(x-z) M(z, y)\,dz.$$

Let n be a natural number for which $B_n \subset U$. We use the following estimation (see [7]):

$$g_D(z, x_n) \geqslant g_{B_n}(z, x_n) \geqslant A_1 \min\left(\frac{1}{|z-x_n|^{d-\alpha}}, \frac{\delta^{\alpha/2}(x_n)\,\delta^{\alpha/2}(z)}{|z-x_n|^d}\right),$$

where $\delta(x) = \operatorname{dist}(x, \partial B_n)$ and A_1 is a constant which depends only on d and α. In our case $\delta(x_n) = r_n$, $\delta(z) = r_n - |z-x_n|$. Therefore,

$$g_D(z, x_n) \geqslant g_{B_n}(z, x_n) \geqslant A_1 \int_A \frac{\delta^{\alpha/2}(x_n)\,\delta^{\alpha/2}(z)}{|z-x_n|^d}\,dz,$$

where

$$A = \left\{z\colon \frac{\delta^{\alpha/2}(x_n)\,\delta^{\alpha/2}(z)}{|z-x_n|^d} \leqslant \frac{1}{|z-x_n|^{d-\alpha}}\right\} = \{z\colon (r_n - |z-x_n|)^{\alpha/2} r_n^{\alpha/2} \leqslant |z-x_n|^\alpha\}.$$

Substituting $|z-x_n| = r_n t$ we see that

$$A = \{z\colon 0 \leqslant (1-t)^{\alpha/2} \leqslant t^\alpha \leqslant 1\} = \{z\colon \varepsilon_0 \leqslant t \leqslant 1\},$$

where $\varepsilon_0 = (\sqrt{5}-1)/2$. Hence

$$\int_U g_D(z, x_n)\,dz \geqslant A_1 \int_{r_n \geqslant |z-x_n| \geqslant \varepsilon_0 r_n} \frac{(r_n - |z-x_n|)^{\alpha/2} r_n^{\alpha/2}\,dz}{|z-x_n|^d}$$

$$= A_2 r_n^{\alpha/2} \int_{\varepsilon_0}^{1} (tr_n)^{d-1} \frac{(r_n - r_n t)^{\alpha/2}}{(tr_n)^d} r_n\,dt = A_2 r_n^\alpha \int_{\varepsilon_0}^{1} \frac{(1-t)^{\alpha/2}}{t}\,dt = C_1 r_n^\alpha,$$

where the constant C_1 does not depend on U.

Now, since $x_0 \in B_1$, for $n \geqslant 2$ we have

$$g_D(x_0, x_n) = E^{x_n} g_D(x_0, X_{\tau_{B_n}}) = \int_{D\setminus B_n} g_D(x_0, z) P^{x_n}(X_{\tau_{B_n}} \in dz)$$

$$\leqslant \int_{D\setminus B_n} u(x_0, z) P^{x_n}(X_{\tau_{B_n}} \in dz) = \sum_{i\neq n} \int_{B_i} \frac{C_{d,\alpha} r_n^\alpha\,dz}{|x_0 - z|^{d-\alpha}(|x_n - z|^2 - r_n^2)^{\alpha/2} |x_n - z|^d}.$$

For $i = 1$ we see that $|x_n - z| \geqslant d_1 + r_n$ and we have

$$\int_{|x_0 - z| \leqslant r_1} \frac{C_{d,\alpha} r_n^\alpha\,dz}{|x_0 - z|^{d-\alpha}(|x_n - z|^2 - r_n^2)^{\alpha/2} |x_n - z|^d}$$

$$\leqslant \int_{|x_0-z|\leqslant r_1} \frac{C_{d,\alpha} r_n^\alpha \, dz}{|x_0-z|^{d-\alpha}(2r_n d_1+d_1^2)^{\alpha/2}\,|d_1+r_n|^d}$$

$$\leqslant A_3 \int_0^{r_1} \frac{\varrho^{d-1} r_n^\alpha \, d\varrho}{\varrho^{d-\alpha} d_1^{\alpha+d}} = A_3 r_1^\alpha \frac{1}{\alpha}\left(\frac{1}{d_1}\right)^{\alpha+d} r_n^\alpha = A_4 r_n^\alpha,$$

where the constant A_4 does not depend on U.

If we assume that the sequence $(d_i)_{i\geqslant 1}$ is nonincreasing, then for $i \geqslant 2$ we have $|z-x_n| \geqslant d_i+r_n$ and $|z-x_0| \geqslant d_1$, so

$$\int_{|x_i-z|\leqslant r_i} \frac{C_{d,\alpha} r_n^\alpha \, dz}{|x_0-z|^{d-\alpha}(|x_n-z|^2-r_n^2)^{\alpha/2}\,|x_n-z|^d}$$

$$\leqslant \frac{C_{d,\alpha} r_n^\alpha}{d_1^{d-\alpha}(2r_n d_i+d_i^2)^{\alpha/2}\,|d_i+r_n|^d}|B_i| \leqslant A_5 r_n^\alpha \left(\frac{1}{d_1}\right)^{d-\alpha}\left(\frac{1}{d_i}\right)^{\alpha+d} \omega_d r_i^d,$$

and, again, the constant A_5 does not depend on U. Hence

$$g_D(x_0, x_n) \leqslant A_4 r_n^\alpha + A_6 r_n^\alpha \sum_{i=2}^\infty \left(\frac{1}{d_i}\right)^{\alpha+d} r_i^d.$$

If we set $d_i = 2^{-i}$ and $r_i = 4^{-i}$, then we obtain $g_D(x_0, x_n) \leqslant C_2 r_n^\alpha$, where C_2 does not depend on U. Thus we have proved that for every positive integer n such that $B_n \in U$,

$$\int_U M(z, x_n)\, dz = \frac{\int_U g_D(z, x_n)\, dz}{g_D(x_0, x_n)} \geqslant C_1/C_2,$$

which we wanted to show.

Remark. If we put $D' = D \setminus \bigcup_n \{x_n\}$, then the above calculations show that there exists a sequence of α-harmonic functions on D' which is convergent on D' and its limit is not α-harmonic on D'.

REFERENCES

[1] R. F. Bass, *Probabilistic Techniques in Analysis*, Springer, New York 1995.

[2] R. F. Bass and M. Cranston, *Exit times for symmetric stable processes in* $\boldsymbol{R}^d$, Ann. Probab. 11 (1983), pp. 578–588.

[3] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York 1969.

[4] K. Bogdan, *The boundary Harnack principle for fractional Laplacian*, Studia Math. 123 (1997), pp. 43–80.

[5] K. Bogdan, *Representation of* α*-harmonic functions in Lipschitz domains*, Hiroshima Math. J. 29 (1999), pp. 227–243.

[6] M. Brelot, *On Topologies and Boundaries in Potential Theory*, Springer, 1971.

[7] T. Kulczycki, *Exit time and Green function of cone for symmetric stable processes*, Probab. Math. Statist. 19 (1999), pp. 337–374.
[8] H. Kunita and T. Watanabe, *Markov processes and Martin boundaries, Part I*, Illinois J. Math. 8 (1964), pp. 485–526.
[9] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer, Berlin 1969.

Institute of Mathematics
Technical University of Wrocław
50-370 Wrocław, Poland
E-mail: michalik@kac.im.pwr.wroc.pl

Received on 20.12.1999