PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 20, Fasc. 1 (2000), pp. 127–140

DISCRETE TIME PERIODICALLY CORRELATED MARKOV PROCESSES

BY

A. R. NEMATOLLAHI AND A. R. SOLTANI (SHIRAZ)

Abstract. We consider a discrete time periodically correlated process $\{X_n\}$ which is also Markov in the wide sense. We provide closed formulas for the covariance function $R(n, m) = EX_n X_m$ and for the spectral density $f = [f_{jk}]$ of such a process. Interestingly, we observe that the covariance function, and also the spectral density, is fully specified only by the values of $\{R(j, j), R(j, j+1), j = 0, 1, \ldots, T-1\}$, where T is the period of the process.

1991 Mathematics Subject Classification: 60J05, 60G12, 60G15, 60G10.

Key words and phrases: periodically correlated processes, Markov processes, covariance characterization, spectral density characterization, second order processes.

1. Introduction. In this work we characterize the covariance function and also the spectral density function of a real-valued periodically correlated Markov (PCM) process $\{X_n, n \in Z\}$, Z stands for the set of integers. Periodically correlated (PC) processes are, in general, nonstationary. This class was first introduced in 1961 by Gladyshev [5]. He studied the structure of the covariance function and provided an interesting spectral representation. Works of Gladyshev initiated a line of research in the theory of second order processes. We cite the works of Hurd [6], Miamee [9], Gardner [4], Makagon et al. [7], Soltani and Shishebor [11]. The book by Gardner [4] is rich in demonstrating various applications of periodically correlated processes in sciences and engineering.

Markov processes are more familiar objects and have been the center of tremendous research activities. We consider the discrete time Markov processes that were studied by Doob [3]. See also the works of Mehr and Mcfadden [8] and Borisov [1] on the structure of the covariance of a Markov process.

This paper is organized as follows. In Section 2 we provide preliminaries on PC processes and Markov processes. Section 3 is devoted to the structure of a PCM process, where a closed formula for the covariance function $R(n, m) = EX_n X_m$ of a PCM process of period T is presented (Theorems 3.1

and 3.2). Interestingly we observed that the covariance can be specified only by the values $\{R(j,j), R(j,j+1), j = 0, 1, \ldots, T-1\}$. We study the spectral density of a PCM process in Section 4 and derive its general form (Theorem 4.1).

2. Preliminaries. Let $\{X_n, n \in Z\}$ be a second order process of centered random variables, i.e., $EX_n = 0$, $E|X_n|^2 < \infty$, $n \in Z$. The process $\{X_n, n \in Z\}$ is said to be *periodically correlated* (PC) if there is a positive integer T for which the covariance function $R(n, m) = EX_n X_m$ satisfies

$$R(n,m) = R(n+T, m+T) \tag{2.1}$$

for all n, m in Z. The smallest T is the *period*. It follows from (2.1) that for each τ the function $R_n(\tau) := R(n+\tau, n)$ is periodic in n with period T, i.e.

$$R_n(\tau) = R_{n+T}(\tau), \qquad n, \tau \in Z. \tag{2.2}$$

Following Doob ([3], p. 90), a real-valued second order process $\{X_n, n \in Z\}$ is Markov in the wide sense if, whenever $t_1 < \ldots < t_n$,

$$\hat{E}\{X_{t_n} | X_{t_1}, \ldots, X_{t_{n-1}}\} = \hat{E}\{X_{t_n} | X_{t_{n-1}}\} \tag{2.3}$$

is satisfied with probability 1, where $\hat{E}$ stands for the linear projection. If the process is Gaussian, then $\hat{E}$ is a version of the conditional expectation. The following facts on covariance of Markov processes, in the wide sense, are essentially due to Doob ([3], p. 233). Let

$$\varrho(n_1, n_2) = \begin{cases} R(n_1, n_2)/R(n_1, n_1) & \text{if } R(n_1, n_1) > 0, \\ 0 & \text{if } R(n_1, n_1) = 0; \end{cases}$$

then $\{X_n, n \in Z\}$ is Markov if and only if ϱ satisfies the functional equation

$$\varrho(n_1, n_2) = \varrho(n_1, n)\varrho(n, n_2), \qquad n_1 \leqslant n \leqslant n_2,$$

which is the same as

$$R(n_1, n)R(n, n_2) = R(n, n)R(n_1, n_2), \qquad n_1 \leqslant n \leqslant n_2. \tag{2.4}$$

It follows from (2.4) that

$$R(n_1, n_2) = G(n_1)H(n_2), \qquad n_1 \leqslant n_2, \tag{2.5}$$

for some functions G and H (see Mehr and Mcfadden [8]). This kind of covariance is known as a *triangular covariance.*

Borisov [1] completed the circle even for continuous time processes, namely, let $R(t_1, t_2)$ be some function defined on $\mathcal{T} \times \mathcal{T}$ and suppose that $R(t_1, t_2) \neq 0$ everywhere on $\mathcal{T} \times \mathcal{T}$, where $\mathcal{T}$ is an interval. Then for $R(t_1, t_2)$ to be the covariance function of a Gaussian Markov process with time space $\mathcal{T}$ it is necessary and sufficient that

$$R(t_1, t_2) = G\big(\min(t_1, t_2)\big) H\big(\max(t_1, t_2)\big), \tag{2.6}$$

where G and H are defined uniquely up to a constant multiple and the ratio G/H is a positive nondecreasing function on $\mathscr{T}$.

It should be noted that the Borisov result on Gaussian Markov processes can be easily derived in the discrete case for second order Markov processes, in the wide sense, by using Theorem 8.1 of Doob [3], p. 233.

By PCM we mean that a real second order process which is periodically correlated is also Markov in the wide sense.

3. Characterization of the covariance function. In this section we characterize the covariance function of a PCM process $X = \{X_n, n \in Z\}$ of period T and also the covariance function of the T-dimensional stationary process which is customarily associated to it, i.e.

$$Y_n = (X_{nT}, X_{nT+1}, \ldots, X_{nT+T-1}), \qquad n \in Z.$$

Note that under the Markov property it follows from (2.6) that $R_n(\tau)$, given by (2.2), satisfies

$$R_n(\tau) := R(n+\tau, n) = \begin{cases} G(n)H(n+\tau) & \text{if } \tau \geqslant 0, \\ G(n+\tau)H(n) & \text{if } \tau \leqslant 0. \end{cases} \tag{3.1}$$

Assume that $\tau = 0$ in (3.1) to obtain $G(n) = R_n(0)/H(n)$. Therefore (3.1) reveals that

$$R_n(\tau) = \frac{H(n+\tau)}{H(n)} R_n(0), \qquad \tau \geqslant 0, n \in Z. \tag{3.2}$$

Now it follows from (3.2) that for $n > 0$

$$H(n+1) = \frac{R_n(1)}{R_n(0)} H(n) = \ldots = \frac{R_n(1)}{R_n(0)} \frac{R_{n-1}(1)}{R_{n-1}(0)} \cdots \frac{R_0(1)}{R_0(0)} H(0),$$

giving

$$H(n) = H(0) \prod_{j=0}^{n-1} g(j), \qquad n > 0; \tag{3.3}$$

where $g(j) = R_j(1)/R_j(0)$. Hence

$$H(kT+n) = H(0) \prod_{j=0}^{kT+n-1} g(j), \qquad n = 0, 1, \ldots, T-1,$$

$$= H(0) [\prod_{j=0}^{T-1} g(j)]^k \prod_{j=0}^{n-1} g(j).$$

Consequently,

$$H(kT+n) = H(0) [\tilde{g}(T-1)]^k \tilde{g}(n-1)$$

for all $k = 0, 1, 2, \ldots,\ n = 0, 1, \ldots, T-1$, where

$$\tilde{g}(l) = \prod_{j=0}^{l} g(j), \quad \tilde{g}(-1) = 1, \quad g(j) = \frac{R_j(1)}{R_j(0)}. \tag{3.4}$$

On the other hand, let $\tau = 1$ in (3.2); then

$$H(-n) = \frac{R_{-n}(0)}{R_{-n}(1)} H(-n+1) = \frac{R_{-n}(0)\,R_{-n+1}(0)}{R_{-n}(1)\,R_{-n+1}(1)} \cdots \frac{R_{-1}(0)}{R_{-1}(1)} H(0).$$

Therefore

$$H(-n) = H(0) \prod_{j=-1}^{-n} [g(j)]^{-1} = H(0) \prod_{j=1}^{n} [g(-j)]^{-1} = H(0) \prod_{j=1}^{n} [g(T-j)]^{-1}.$$

Hence

$$\begin{aligned} H(-kT+n) &= H(0) \prod_{j=1}^{kT-n} [g(T-j)]^{-1}, \qquad n = 0, 1, \ldots, T-1, \\ &= H(0) \prod_{j=1}^{(k-1)T+T-n} [g(T-j)]^{-1} \\ &= H(0) \Big[\prod_{j=1}^{T} g(T-j)\Big]^{-(k-1)} \prod_{j=1}^{T-n} [g(T-j)]^{-1} \\ &= H(0)\,[\tilde{h}(T)]^{k-1}\,\tilde{h}(T-n), \end{aligned}$$

where $\tilde{h}(l) = \prod_{j=1}^{l} [g(T-j)]^{-1}$. Also note that

$$\tilde{h}(T) = \prod_{j=1}^{T} [g(T-j)]^{-1} = \prod_{j=0}^{T-1} [g(j)]^{-1} = [\tilde{g}(T-1)]^{-1},$$

and

$$\begin{aligned} \tilde{h}(T-n) &= \prod_{j=1}^{T-n} [g(T-j)]^{-1} = \frac{\prod_{j=1}^{T} [g(T-j)]^{-1}}{\prod_{j=0}^{n-1} [g(j)]^{-1}} \\ &= \frac{[\tilde{h}(T)]}{[\tilde{g}(n-1)]^{-1}} = [\tilde{g}(T-1)]^{-1}\,\tilde{g}(n-1). \end{aligned}$$

Consequently,

$$\begin{aligned} H(-kT+n) &= H(0)\,[\tilde{g}(T-1)]^{-k+1}\,[\tilde{g}(T-1)]^{-1}\,\tilde{g}(n-1) \\ &= H(0)\,[\tilde{g}(T-1)]^{-k}\,\tilde{g}(n-1). \end{aligned}$$

So we have proved the following lemma:

LEMMA 3.1. *The function H in* (3.1) *is given by*

$$H(kT+n) = H(0)[\tilde{g}(T-1)]^k \tilde{g}(n-1)$$

for all $k \in Z$, $n = 0, 1, \ldots, T-1$, where $\tilde{g}$ is given by (3.4). ∎

Now we are in a position to characterize the covariance function of a real-valued PCM process. The following theorem provides the details.

THEOREM 3.1. *Suppose $X = \{X_n, n \in Z\}$ is a real PCM process with covariance function $R_n(\tau)$ given by* (2.2), *for which $R_n(\tau) \neq 0$, $n, \tau \in Z$. Then*

$$R_n(mT+v) = [\tilde{g}(T-1)]^m \tilde{g}(v+n-1)[\tilde{g}(n-1)]^{-1} R_n(0), \tag{3.5}$$

$$R_n(-mT+v) = R_{n+v}((m-1)T+T-v), \tag{3.6}$$

for $m = 0, 1, \ldots$ and $n, v = 0, 1 \ldots, T-1$, where $\tilde{g}(l) = \prod_{j=0}^{l} R_j(1)/R_j(0)$ and $\tilde{g}(-1) = 1$.

Proof. Let $\tau = mT+v \geqslant 0$. Then it follows from (3.2) and Lemma 3.1 that

$$R_n(mT+v) = \frac{H(mT+v+n)}{H(n)} R_n(0) = \frac{H(0)[\tilde{g}(T-1)]^m \tilde{g}(v+n-1)}{H(0)\tilde{g}(n-1)} R_n(0)$$
$$= [\tilde{g}(T-1)]^m \tilde{g}(v+n-1)[\tilde{g}(n-1)]^{-1} R_n(0),$$

for $m = 0, 1, \ldots$ and $n, v = 0, 1, \ldots, T-1$. Also note that

$$R_n(-mT+v) = EX_{n-mT+v} X_n = EX_{n+v} X_{n+mT}$$
$$= R_{n+v}(mT-v) = R_{n+v}((m-1)T+T-v).$$

The proof is complete. ∎

Remark 3.1. Interestingly it follows from Theorem 3.1 that, for each $n = 0, 1, \ldots, T-1$, $R_n(\tau)$ is fully specified by the values of $\{R_j(0), R_j(1), j = 0, 1, \ldots, T-1\}$.

It is natural to investigate conditions on a given set of $2T$ elements $\{R_j(0), R_j(1), j = 0, 1, \ldots, T-1\}$ to insure that the function $R_n(\tau)$, given by (3.5) and (3.6), is a covariance function of a PCM process. The following theorem provides the details.

THEOREM 3.2. *Suppose $\{R_j(0), R_j(1), j = 0, 1, \ldots, T-1\}$ is a set of $2T$ nonzero real numbers that satisfy the following condition*:

$$R_j^2(1) \leqslant R_j(0) R_{j+1}(0), \tag{3.7}$$

for $j = 1, 2, \ldots, T-1$, where $R_T(0) = R_0(0)$. Then the function $R_n(\tau)$, which is formed from $\{R_j(0), R_j(1), j = 0, 1, \ldots, T-1\}$ through (3.5) *and* (3.6), *is a covariance function of a PCM process.*

Proof. Clearly, $R_n(\tau)$ given by (3.5) and (3.6) is periodic in n. For $R_n(\tau)$ to be the covariance function of a Markov process it has to satisfy the Borisov conditions [1], cited in Section 2. Note that

$$\begin{aligned}R_n(\tau) &= R_n(mT+v) = R(n, mT+v+n)\\ &= [\tilde{g}(T-1)]^m \tilde{g}(v+n-1)[\tilde{g}(n-1)]^{-1} R_n(0)\\ &= H(0)[\tilde{g}(T-1)]^m \tilde{g}(v+n-1)[H(0)\tilde{g}(n-1)]^{-1} R_n(0)\\ &= H(0)[\prod_{j=0}^{mT+v+n-1} g(j)][H(n)]^{-1} R_n(0) = H(mT+v+n)G(n).\end{aligned}$$

Therefore

$$R_n(\tau) = R_n(n+\tau, n) = H(\tau+n)G(n),$$

where

$$H(n) = H(0)[\prod_{j=0}^{n-1} g(j)], \qquad G(n) = [H(n)]^{-1} R_n(0).$$

Note that $G(t)/H(t) = R_t(0)/H^2(t)$ is positive. Finally, we show that, under (3.7), $d = G/H$ is nondecreasing on Z. Let $t = mT+n \geqslant 0$ and $0 \leqslant n+1 < T-1$. Then

$$\begin{aligned}\frac{d(t+1)}{d(t)} &= \frac{H^2(t)}{H^2(t+1)} \frac{R_{t+1}(0)}{R_t(0)} = \frac{H^2(mT+n)}{H^2(mT+n+1)} \frac{R_{mT+n+1}(0)}{R_{mT+n}(0)}\\ &= \left[\frac{\tilde{g}(n-1)}{\tilde{g}(n)}\right]^2 \frac{R_{n+1}(0)}{R_n(0)} = \left[\frac{R_n(0)}{R_n(1)}\right]^2 \frac{R_{n+1}(0)}{R_n(0)}\\ &\geqslant \frac{R_n(0)}{R_{n+1}(0)} \frac{R_{n+1}(0)}{R_n(0)} = 1,\end{aligned}$$

and for the case $n = T-1$

$$\begin{aligned}\frac{d(t+1)}{d(t)} &= \frac{H^2(mT+T-1)}{H^2((m+1)T)} \frac{R_{(m+1)T}(0)}{R_{mT+T-1}(0)} = \left[\frac{\tilde{g}(T-2)}{\tilde{g}(T-1)}\right]^2 \frac{R_T(0)}{R_{T-1}(0)}\\ &= \left[\frac{R_{T-1}(0)}{R_{T-1}(1)}\right]^2 \frac{R_T(0)}{R_{T-1}(0)} \geqslant \frac{R_{T-1}(0)}{R_T(0)} \frac{R_T(0)}{R_{T-1}(0)} = 1.\end{aligned}$$

Therefore G/H is nondecreasing on nonnegative integers, and the same argument provides that it is also nondecreasing on the negative integers. ∎

Remark 3.2. Note that it follows from the Cauchy–Schwarz inequality that $|\tilde{g}(T-1)| \leqslant 1$. Indeed,

$$|\tilde{g}(T-1)| = \left|\prod_{j=0}^{T-1} g(j)\right| = \left|\prod_{j=0}^{T-1} \frac{R_j(1)}{R_j(0)}\right| = \frac{\prod_{j=0}^{T-1} |EX_{j+1}X_j|}{\prod_{j=0}^{T-1} EX_j^2}$$

$$\leqslant \frac{\prod_{j=0}^{T-1} [EX_{j+1}^2 EX_j^2]^{1/2}}{\prod_{j=0}^{T-1} EX_j^2} = \frac{[EX_1^2 EX_0^2 EX_2^2 EX_1^2 \ldots EX_T^2 EX_{T-1}^2]^{1/2}}{EX_0^2 EX_1^2 \ldots EX_{T-1}^2}$$
$$= \frac{[EX_0^2 EX_T^2]^{1/2}}{EX_0^2} = 1,$$

which can also be deduced from the monotonicity of G/H. Therefore $\sum_\tau |R_n(\tau)| < \infty$, $n = 0, 1, \ldots, T-1$. ∎

As observed first by Gladyshev [5], corresponding to every PC process X_n, $n \in Z$, the T-dimensional random sequence $Y_n = (Y_n^0, Y_n^1, \ldots, Y_n^{T-1})$, where $Y_n^k = X_{nT+k}$, $n \in Z$, $k = 0, 1, \ldots, T-1$, is stationary in the wide sense. In order to know more about $\{X_n\}$ it would be useful to study $\{Y_n\}$; see Gladyshev [5], Miamee [9], Makagon et al. [7].

THEOREM 3.3. *Let $X = \{X_n, n \in Z\}$ be a PCM process with the covariance function $R_n(\tau)$ and let $\{Y_n, n \in Z\}$ be its associated T-dimensional stationary process with covariance function $Q(\tau)$. Then*

$$Q(\tau) = CR[\tilde{g}(T-1)]^\tau, \quad \tau \geqslant 0,$$

and $Q(\tau) = Q^*(-\tau)$, where the matrices C and R are given by

$$C = \begin{bmatrix} C_{00} & C_{01} & \ldots & C_{0,T-1} \\ C_{10} & C_{11} & \ldots & C_{1,T-1} \\ \ldots & \ldots & \ldots & \ldots \\ C_{T-1,0} & C_{T-1,1} & \ldots & C_{T-1,T-1} \end{bmatrix},$$

$C_{jk} = \tilde{g}(j-1)[\tilde{g}(k-1)]^{-1}$, and

$$R = \begin{bmatrix} R_0(0) & 0 & \ldots & 0 \\ 0 & R_1(0) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & R_{T-1}(0) \end{bmatrix}.$$

Proof. Let $Q(\tau) = [Q_{jk}(\tau)]_{j,k=0,1,\ldots,T-1}$ be the covariance matrix of Y_n. Then for $j, k = 0, 1, \ldots, T-1$

$$Q_{jk}(\tau) = EY_{n+\tau}^j Y_n^k = EX_{(n+\tau)T+j} X_{nT+k} = EX_{\tau T+j} X_k$$
$$= EX_{\tau T+j-k+k} X_k = R_k(\tau T+j-k).$$

Hence

$$Q(\tau) = \begin{bmatrix} R_0(\tau T) & R_1(\tau T-1) & \ldots & R_{T-1}(\tau T-(T-1)) \\ R_0(\tau T+1) & R_1(\tau T) & \ldots & R_{T-1}(\tau T-(T-2)) \\ \ldots & \ldots & \ldots & \ldots \\ R_0(\tau T+T-1) & R_1(\tau T+T-2) & \ldots & R_{T-1}(\tau T) \end{bmatrix}.$$

Also by the Markov property of $\{X_n\}$ it follows from Theorem 3.1 that for $\tau \geqslant 0$, $0 \leqslant j-k \leqslant T-1$

$$\begin{aligned} R_k(\tau T+j-k) &= [\tilde{g}(T-1)]^{\tau}\tilde{g}(j-k+k-1)[\tilde{g}(k-1)]^{-1}R_k(0) \\ &= [\tilde{g}(T-1)]^{\tau}\tilde{g}(j-1)[\tilde{g}(k-1)]^{-1}R_k(0), \end{aligned}$$

and by using (3.6) we obtain

$$\begin{aligned} R_k(-\tau T+j-k) &= R_{k+j-k}\big((\tau-1)T+T-j+k\big) = R_j\big((\tau-1)T+T-j+k\big) \\ &= [\tilde{g}(T-1)]^{\tau-1}\tilde{g}(T+k-1)[\tilde{g}(j-1)]^{-1}R_j(0) \\ &= [\tilde{g}(T-1)]^{\tau}\tilde{g}(k-1)[\tilde{g}(j-1)]^{-1}R_j(0). \end{aligned}$$

Consequently, for $j \geqslant k$ we have

$$Q_{jk}(\tau) = R_k(\tau T+j-k) = \begin{cases} [\tilde{g}(T-1)]^{\tau}C_{jk}R_k(0) & \text{if } \tau \geqslant 0, \\ [\tilde{g}(T-1)]^{-\tau}C_{kj}R_j(0) & \text{if } \tau < 0, \end{cases} \tag{3.8}$$

where $C_{jk} = \tilde{g}(j-1)[\tilde{g}(k-1)]^{-1}$. Also, note that since $Q(\tau) = Q^*(-\tau)$, where $*$ stands for the transpose, we have

$$R_k(\tau T+j-k) = R_j(-\tau T+k-j),$$

and therefore, for $\tau \geqslant 0$,

$$Q(\tau) = \begin{pmatrix} C_{00}R_0(0) & C_{01}R_1(0) & \dots & C_{0,T-1}R_{T-1}(0) \\ C_{10}R_0(0) & C_{11}R_1(0) & \dots & C_{1,T-1}R_{T-1}(0) \\ \dots & \dots & \dots & \dots \\ C_{T-1,0}R_0(0) & C_{T-1,1}R_1(0) & \dots & C_{T-1,T-1}R_{T-1}(0) \end{pmatrix} [\tilde{g}(T-1)]^{\tau}.$$

Note that since $Q(\tau) = Q^*(-\tau)$, we have characterized Q for all τ. The proof is now complete. ∎

Remark 3.3. It follows from Theorem 3.3 that for each $k = 0, 1, \dots, T-1$ the process $Y_n^k = X_{nT+k}$, $n \in Z$, is an AR(1) process with the covariance function

$$\gamma_k(\tau) = [\tilde{g}(T-1)]^{|\tau|}R_k(0), \quad k = 0, 1, \dots, T-1, \ \tau \in Z.$$

4. A characterization for the spectral density. The spectral density of a PC process was introduced by Gladyshev [5]; if it exists, it is a Hermitian non-negative definite $T \times T$ matrix of functions

$$f(\lambda) = [f_{jk}(\lambda)]_{j,k=0,1,\dots,T-1},$$

for which

$$R_n(\tau) = \sum_{k=0}^{T-1} B_k(\tau)\exp\left(\frac{2\pi ikn}{T}\right),$$

$$B_k(\tau) = \int_0^{2\pi} e^{i\tau\lambda}f_k(\lambda)\,d(\lambda), \tag{4.1}$$

where $f_k(\lambda)$ and $f_{jk}(\lambda)$, $j, k = 0, 1, \ldots, T-1$, are related through

$$f_{jk}(\lambda) = \frac{1}{T} f_{k-j}\big((\lambda - 2\pi j)/T\big), \quad j, k = 0, 1, \ldots, T-1, \ 0 \leqslant \lambda < 2\pi.$$

In this section we characterize f of a PCM process.

Let us denote the spectral distribution matrix of the process Y_n by

$$H(\lambda) = [H_{jk}(\lambda)]_{j,k=0,1,\ldots,T-1},$$

where

$$H_{jk}(\lambda) = \frac{1}{2\pi} Q_{jk}(0)\lambda + \lim_{T\to\infty} \frac{1}{2\pi} \sum_{0<|\tau|<T} Q_{jk}(\tau) \frac{e^{-i\lambda\tau}}{-i\tau}.$$

For more details see Brockwell and Davis [2] and Rozanov [10]. It was proved in Remark 3.2 that $\sum_\tau |Q_{jk}(\tau)| < \infty$, therefore each H_{jk} has a uniformly continuous density $h_{jk}(\lambda) = H_{jk}(d\lambda)/d\lambda$. Moreover,

$$h_{jk}(\lambda) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} Q_{jk}(\tau) e^{-i\lambda\tau}, \quad 0 \leqslant \lambda < 2\pi.$$

LEMMA 4.1. *The spectral density matrix $\boldsymbol{h}(\lambda) = [h_{jk}(\lambda)]_{j,k=0,1,\ldots,T-1}$ of the T-variate stationary process Y_n is specified by*

$$h_{jk}(\lambda) = \frac{1}{2\pi |1 - \tilde{g}(T-1) e^{i\lambda}|^2} \{\tilde{g}(T-1)\big(C_{kj} R_j(0) - C_{jk} R_k(0)\big) e^{i\lambda} + C_{jk} R_k(0) - C_{kj} R_j(0) \tilde{g}^2(T-1)\}, \tag{4.2}$$

for $j \geqslant k$, $0 \leqslant \lambda < 2\pi$. For $j \leqslant k$, $h_{jk}(\lambda) = \overline{h_{kj}(\lambda)}$.

Proof. For $j-k \geqslant 0$ it follows from (3.8) that

$$\begin{aligned}
h_{jk}(\lambda) &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} R_k(\tau T + j - k) e^{-i\lambda\tau} \\
&= \frac{1}{2\pi} \Big[\sum_{\tau=-\infty}^{-1} R_k(\tau T + j - k) e^{-i\lambda\tau} + R_k(j-k) + \sum_{\tau=1}^{\infty} R_k(\tau T + j - k) e^{-i\lambda\tau} \Big] \\
&= \frac{1}{2\pi} \{C_{jk} R_k(0) + \sum_{\tau=1}^{\infty} [\tilde{g}(T-1)]^\tau C_{kj} R_j(0) e^{i\lambda\tau} \\
&\quad + \sum_{\tau=1}^{\infty} [\tilde{g}(T-1)]^\tau C_{jk} R_k(0) e^{-i\lambda\tau}\} \\
&= \frac{1}{2\pi} \Big\{ C_{jk} R_k(0) + C_{kj} R_j(0) \frac{\tilde{g}(T-1) e^{i\lambda}}{1 - \tilde{g}(T-1) e^{i\lambda}} + C_{jk} R_k(0) \frac{\tilde{g}(T-1) e^{-i\lambda}}{1 - \tilde{g}(T-1) e^{-i\lambda}} \Big\}
\end{aligned}$$

$$= \frac{1}{2\pi|1-\tilde{g}(T-1)e^{i\lambda}|^2}\{C_{jk}R_k(0)(1-\tilde{g}(T-1)e^{i\lambda}-\tilde{g}(T-1)e^{i\lambda}+\tilde{g}^2(T-1))$$
$$+C_{kj}R_j(0)\tilde{g}(T-1)e^{i\lambda}(1-\tilde{g}(T-1)e^{-i\lambda})$$
$$+C_{jk}R_k(0)\tilde{g}(T-1)e^{-i\lambda}(1-\tilde{g}(T-1)e^{i\lambda})\}$$
$$= \frac{1}{2\pi|1-\tilde{g}(T-1)e^{i\lambda}|^2}\{\tilde{g}(T-1)(C_{kj}R_j(0)-C_{jk}R_k(0))e^{i\lambda}$$
$$+C_{jk}R_k(0)-C_{kj}R_j(0)\tilde{g}^2(T-1)\}. \blacksquare$$

Remark 4.1. Lemma 4.1 provides the spectral density of each $Y_n^k = X_{nT+k}$,

$$h_k(\lambda) = \frac{R_k(0)(1-\tilde{g}^2(T-1))}{2\pi|1-\tilde{g}(T-1)e^{i\lambda}|^2}, \quad k = 0, 1, \ldots, T-1, \ 0 \leqslant \lambda < 2\pi,$$

which is the spectral density of an AR(1) process.

THEOREM 4.1. *The spectral density of a PCM process is a Hermitian non-negative definite $T\times T$ matrix of functions*

$$f(\lambda) = [f_{jk}(\lambda)]_{j,k=0,1,\ldots,T-1},$$

where

$$f_{jk}(\lambda) = \frac{1}{T^2 2\pi|1-\tilde{g}(T-1)e^{i\lambda}|^2} \tag{4.3}$$
$$\times\Bigg\{(1-\tilde{g}^2(T-1))\sum_{p=0}^{T-1}\exp\left(\frac{-2\pi ip(k-j)}{T}\right)R_p(0)$$
$$+\sum_{p=0}^{T-1}\sum_{l=p+1}^{T-1}\Bigg[\tilde{g}(T-1)(C_{pl}R_l(0)-C_{lp}R_p(0))$$
$$\times\left(\exp\left(\frac{i(p-l+T)\lambda-2\pi i(pk-jl)}{T}\right)\right.$$
$$\left.+\exp\left(-\frac{i(p-l+T)\lambda+2\pi i(lk-jp)}{T}\right)\right)$$
$$+(C_{lp}R_p(0)-\tilde{g}^2(T-1)C_{pl}R_l(0))$$
$$\times\left(\exp\left(\frac{i(p-l)\lambda-2\pi i(pk-jl)}{T}\right)\right.$$
$$\left.+\exp\left(-\frac{i(p-l)\lambda+2\pi i(lk-jp)}{T}\right)\right)\Bigg]\Bigg\}$$

for any $j, k = 0, 1, \ldots, T-1$, $0 \leqslant \lambda < 2\pi$.

Proof. It was proved by Gladyshev ([5], Theorem 2) that

$$f(\lambda) = \frac{1}{T} U(\lambda) h(\lambda) U^{-1}(\lambda),$$

where $f(\lambda)$ is the spectral density matrix of the $\{X_n\}$, and $U(\lambda) = [U_{jk}(\lambda)]_{j,k=0,1,\ldots,T-1}$ is a unitary matrix depending on λ with elements

$$U_{jk}(\lambda) = T^{-1/2} \exp\left(\frac{2\pi ijk - ik\lambda}{T}\right).$$

Note that $U^{-1}(\lambda) = U^*(\lambda)$; therefore

$$U_{jk}^{-1}(\lambda) = \overline{U_{kj}(\lambda)} = T^{-1/2} \exp\left(\frac{ij\lambda - 2\pi ijk}{T}\right).$$

Consequently,

$$\begin{aligned} f_{jk}(\lambda) &= \frac{1}{T} \sum_{p=0}^{T-1} \sum_{l=0}^{T-1} U_{jl}(\lambda) h_{lp}(\lambda) U_{pk}^{-1}(\lambda) \\ &= \frac{1}{T} \sum_{p=0}^{T-1} \sum_{l=0}^{T-1} T^{-1/2} \exp\left(\frac{2\pi ijl - il\lambda}{T}\right) h_{lp}(\lambda) T^{-1/2} \exp\left(\frac{ip\lambda - 2\pi ipk}{T}\right). \end{aligned}$$

Hence

$$\begin{aligned} f_{jk}(\lambda) = \frac{1}{T^2} \Bigg[&\sum_{p=0}^{T-1} \sum_{l=p}^{T-1} \exp\left(\frac{i(p-l)\lambda - 2\pi i(pk-jl)}{T}\right) h_{lp}(\lambda) \\ &+ \sum_{p=0}^{T-1} \sum_{l=0}^{p-1} \exp\left(\frac{i(p-l)\lambda - 2\pi i(pk-jl)}{T}\right) \overline{h_{pl}(\lambda)} \Bigg], \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} h_{lp}(\lambda) = \frac{1}{2\pi |1 - \tilde{g}(T-1) e^{i\lambda}|^2} \big[&\tilde{g}(T-1) \big(C_{pl} R_l(0) - C_{lp} R_p(0)\big) e^{i\lambda} \\ &+ C_{lp} R_p(0) - C_{pl} R_l(0) \tilde{g}^2(T-1) \big], \qquad l \geqslant p, \end{aligned}$$

with $C_{jk} = \tilde{g}(j-1) [\tilde{g}(k-1)]^{-1}$. By substituting h in (4.4) we obtain

$$\begin{aligned} f_{jk}(\lambda) = \frac{1}{T^2 2\pi |1 - \tilde{g}(T-1) e^{i\lambda}|^2} \Bigg\{ &\sum_{p=0}^{T-1} \sum_{l=p}^{T-1} \exp\left(\frac{i(p-l)\lambda - 2\pi i(pk-jl)}{T}\right) \\ &\times \big[\tilde{g}(T-1) \big(C_{pl} R_l(0) - C_{lp} R_p(0)\big) e^{i\lambda} + C_{lp} R_p(0) - C_{pl} R_l(0) \tilde{g}^2(T-1)\big] \\ &+ \sum_{p=0}^{T-1} \sum_{l=0}^{p-1} \exp\left(\frac{i(p-l)\lambda - 2\pi i(pk-jl)}{T}\right) \big[\tilde{g}(T-1) \big(C_{lp} R_p(0) - C_{pl} R_l(0)\big) e^{-i\lambda} \\ &+ C_{pl} R_l(0) - C_{lp} R_p(0) \tilde{g}^2(T-1)\big] \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T^2 2\pi |1-\tilde{g}(T-1)e^{i\lambda}|^2} \Bigg\{ \sum_{p=0}^{T-1} \exp\left(\frac{-2\pi i p(k-j)}{T}\right) \\
&\quad \times \left[\tilde{g}(T-1)\left(C_{pp}R_p(0)-C_{pp}R_p(0)\right)e^{i\lambda}+C_{pp}R_p(0)-C_{pp}R_p(0)\tilde{g}^2(T-1)\right] \\
&\quad + \sum_{p=0}^{T-1}\sum_{l=p+1}^{T-1} \exp\left(\frac{i(p-l)\lambda-2\pi i(pk-jl)}{T}\right)\left[\tilde{g}(T-1)\left(C_{pl}R_l(0)-C_{lp}R_p(0)\right)e^{i\lambda}\right. \\
&\quad \left. +C_{lp}R_p(0)-C_{pl}R_l(0)\tilde{g}^2(T-1)\right] + \sum_{p=0}^{T-1}\sum_{l=0}^{p-1} \exp\left(\frac{i(p-l)\lambda-2\pi i(pk-jl)}{T}\right) \\
&\quad \times \left[\tilde{g}(T-1)\left(C_{lp}R_p(0)-C_{pl}R_l(0)\right)e^{-i\lambda}+C_{pl}R_l(0)-C_{lp}R_p(0)\tilde{g}^2(T-1)\right]\Bigg\} \\
&= \frac{1}{T^2 2\pi |1-\tilde{g}(T-1)e^{i\lambda}|^2} \Bigg\{ \sum_{p=0}^{T-1} \exp\left(\frac{-2\pi i p(k-j)}{T}\right) R_p(0) \\
&\quad - \sum_{p=0}^{T-1} \exp\left(\frac{-2\pi i p(k-j)}{T}\right) R_p(0)\tilde{g}^2(T-1) \\
&\quad + \sum_{p=0}^{T-1}\sum_{l=p+1}^{T-1} \exp\left(\frac{i(p-l)\lambda-2\pi i(pk-jl)}{T}\right)\left[\tilde{g}(T-1)\left(C_{pl}R_l(0)-C_{lp}R_p(0)\right)e^{i\lambda}\right. \\
&\quad \left. +C_{lp}R_p(0)-C_{pl}R_l(0)\tilde{g}^2(T-1)\right] + \sum_{p=0}^{T-1}\sum_{l=p+1}^{T-1} \exp\left(\frac{-i(p-l)\lambda-2\pi i(lk-jp)}{T}\right) \\
&\quad \times \left[\tilde{g}(T-1)\left(C_{pl}R_l(0)-C_{lp}R_p(0)\right)e^{-i\lambda}+C_{lp}R_p(0)-C_{pl}R_l(0)\tilde{g}^2(T-1)\right]\Bigg\} \\
&= \frac{1}{T^2 2\pi |1-\tilde{g}(T-1)e^{i\lambda}|^2} \Bigg\{ \left(1-\tilde{g}^2(T-1)\right) \sum_{p=0}^{T-1} \exp\left(\frac{-2\pi i p(k-j)}{T}\right) R_p(0) \\
&\quad + \sum_{p=0}^{T-1}\sum_{l=p+1}^{T-1} \Bigg[\tilde{g}(T-1)\left(C_{pl}R_l(0)-C_{lp}R_p(0)\right) \\
&\quad \times \left(\exp\left(\frac{i(p-l+T)\lambda-2\pi i(pk-jl)}{T}\right)+\exp\left(-\frac{i(p-l+T)\lambda+2\pi i(lk-jp)}{T}\right)\right) \\
&\quad + \left(C_{lp}R_p(0)-\tilde{g}^2(T-1)C_{pl}R_l(0)\right)\left(\exp\left(\frac{i(p-l)\lambda-2\pi i(pk-jl)}{T}\right)\right. \\
&\quad \left. +\exp\left(-\frac{i(p-l)\lambda+2\pi i(lk-jp)}{T}\right)\right)\Bigg]\Bigg\}. \qquad \blacksquare
\end{aligned}
$$

In the spectral analysis of PC processes it is usually desirable to obtain the f_k in (4.1). Let $T=2$. Then $f_{00}(\lambda)=\frac{1}{2}f_0(\lambda/2)$ and $f_{11}(\lambda)=\frac{1}{2}f_0\left((\lambda-2\pi)/2\right)$ for $\lambda\in[0, 2\pi)$, and therefore

$$
f_0(\lambda)=\begin{cases} 2f_{00}(2\lambda) & \text{if } 0\leqslant\lambda<\pi, \\ 2f_{11}(2(\lambda-\pi)) & \text{if } \pi\leqslant\lambda<2\pi. \end{cases}
$$

Also note that by doing some algebraic simplifications we obtain

$$f_{00}(\lambda) = \frac{\big(R_0(0)+R_1(0)\big)\big(1-\tilde{g}^2(1)\big)}{8\pi\,|1-\tilde{g}(1)\,e^{i\lambda}|^2}\,[1+2\alpha\cos(\lambda/2)],$$

$$f_{11}(\lambda) = \frac{\big(R_0(0)+R_1(0)\big)\big(1-\tilde{g}^2(1)\big)}{8\pi\,|1-\tilde{g}(1)\,e^{i\lambda}|^2}\,[1-2\alpha\cos(\lambda/2)],$$

where

$$\alpha = \frac{R_0(1)+R_1(1)}{\big(R_0(0)+R_1(0)\big)\big(1+\tilde{g}(1)\big)}.$$

Hence

$$f_0(\lambda) = \frac{\big(R_0(0)+R_1(0)\big)\big(1-\tilde{g}^2(1)\big)}{4\pi\,|1-\tilde{g}(1)\,e^{i2\lambda}|^2}\,[1+2\alpha\cos(\lambda)], \qquad 0 \leqslant \lambda < 2\pi.$$

Since $f_0 \geqslant 0$, we infer from Brockwell and Davis ([2], p. 121) that $|\alpha| \leqslant \frac{1}{2}$, which can also be seen directly by using the Cauchy–Schwarz inequality and doing some manipulations. Also $f_{01}(\lambda) = \frac{1}{2} f_1(\lambda/2)$ and $f_{10}(\lambda) = \frac{1}{2} f_{-1}\big((\lambda-2\pi)/2\big) = \frac{1}{2} f_1\big((\lambda-2\pi)/2\big)$ for $\lambda \in [0, 2\pi)$. Therefore

$$f_1(\lambda) = \begin{cases} 2f_{01}(2\lambda) & \text{if } 0 \leqslant \lambda < \pi, \\ 2f_{10}\big(2(\lambda-\pi)\big) & \text{if } \pi \leqslant \lambda < 2\pi. \end{cases}$$

Hence

$$f_1(\lambda) = \frac{\big(R_0(0)-R_1(0)\big)\big(1-\tilde{g}^2(1)\big)}{4\pi\,|1-\tilde{g}(1)\,e^{i2\lambda}|^2}\,[1+i\,2\beta\sin(\lambda)], \qquad 0 \leqslant \lambda < 2\pi,$$

where

$$\beta = \frac{R_1(1)-R_0(1)}{\big(R_0(0)-R_1(0)\big)\big(1-\tilde{g}(1)\big)}.$$

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Department of Statistics
College of Science
Shiraz University
Shiraz 71454, Iran

Received on 15.3.1999