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A BERNSTEIN PROPERTY OF MEASURES ON GROUPS AND SYMMETRIC SPACES

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Abstract. In this paper we consider a Bernstein property of probability measures on groups introduced by Neuenschwander. We discuss this property for discrete groups, compact groups, nilpotent groups and some solvable groups. In all these cases we show that a measure having the Bernstein–Neuenschwander property must be concentrated on an Abelian subgroup. We conclude with an application of this result to the Gaussian measures on non-compact symmetric spaces.

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1. Bernstein properties on groups. It is easy to verify that if X and Y are two independent equally distributed real Gaussian random variables, then the random variables

$$X+Y \text{ and } X-Y \text{ are independent.}$$

The famous Bernstein theorem states that the inverse is also true: given two independent real random variables X and Y, the independence of $X+Y$ and $X-Y$ implies that X and Y are Gaussian.

The above Bernstein property of real Gaussian measures may be formulated in the same way on any second countable topological group. We will say that a probability measure μ on such a group G has the *property* (B) if, given two independent random variables X and Y with values in G and having the same probability distribution μ, the random variables

$$X\cdot Y \text{ and } X\cdot Y^{-1} \text{ are independent.} \tag{B}$$

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If G is a locally compact Abelian group, Gaussian measures (in the sens of Parthasarathy) have all the property (B), but the converse is no longer true: there may be other measures than Gaussian with the Bernstein property (see Heyer and Rall [7] and Rukhin [10]). A complete study of these problems was achieved by Feldman [3]. Let us notice that the property (B) has been proposed as a definition of a Gaussian measure on an Abelian metric group, not necessarily locally compact (see Byczkowski [1]).

In the non-commutative case Gaussian measures may not have the property (B): see Byczkowski and Hulanicki [2], where it was showed that Gaussian measures on the Heisenberg group $\boldsymbol{H}_1$ are not stable (the condition (B) trivially implies that if μ is symmetric, then X^2 has the distribution μ^{*4}, which is false for Gaussian measures on the Heisenberg group).

In [8] Neuenschwander proposed, in a non-commutative group case, the study of another Bernstein-like property, which we will call (N). A probability measure μ on a topological second countable group G has the *property* (N) if, given two independent random variables X and Y with values in G and having the same probability distribution μ, the couples of random variables

$$\text{(N)} \qquad (X \cdot Y,\ Y \cdot X) \text{ and } (X \cdot Y^{-1},\ Y^{-1} \cdot X) \text{ are independent.}$$

Remark that on Abelian groups the conditions (B) and (N) are equivalent. The condition (N) was studied in [8] and [9] for simply connected nilpotent Lie groups. In this paper we consider the property (N) on a large class of topological groups, without any use of their eventual Lie structure.

Let us give now some consequences of the property (N) valid on any topological group.

PROPOSITION 1. *Let G be a second countable topological group.*

(1) *Suppose that two identically distributed independent random variables X and Y with values in G verify* (N). *Then*

$$P\{XY = YX\} = 0 \textit{ or } 1.$$

(2) *Let W be an open symmetric neighbourhood of e invariant by all conjugations ($W = xWx^{-1}$ for all $x \in G$) and let $[x, y]$ denote the commutator $[x, y] = xyx^{-1}y^{-1}$ on G. Then, if X and Y satisfy* (N),

$$P\{[X, Y] \in W\} = 0 \textit{ or } 1.$$

(3) *Let G be a topological second countable Hausdorff group and let X and Y be two identically distributed G-valued independent random variables with distribution μ. If $XY = YX$ almost surely, then μ is concentrated on an Abelian subgroup H of G, i.e.* $\operatorname{supp}(\mu) \subset H$.

Proof. (1) The condition (N) implies that the measurable sets

$$\{XY = YX\} \quad \text{and} \quad \{XY^{-1} = Y^{-1}X\}$$

are independent. On the other hand, they are trivially equal. It follows that $P\{XY = YX\} = 0$ or 1.

(2) The set

$$A = \{[X, Y] \in W\} = \{Y^{-1}[X, Y]\, Y \in W\} = \{Y^{-1}XYX^{-1} \in W\}$$

is equal, as $W = W^{-1}$, to the set

$$\{XY^{-1}X^{-1}Y \in W\} = \{[X, Y^{-1}] \in W\} = B.$$

As $[X, Y] = (XY)(YX)^{-1}$ and $[X, Y^{-1}] = (XY^{-1})(Y^{-1}X)^{-1}$, the condition (N) implies that the sets A and B are independent. But $A = B$, and (2) holds.

(3) Let $x, y \in \mathrm{supp}(\mu)$ and let U_n, V_n ($n \in N$) be open neighbourhoods of x and y, respectively, such that $U_{n+1} \subset U_n$, $V_{n+1} \subset V_n$ and $\bigcap_n U_n = \{x\}$, $\bigcap_n V_n = \{y\}$. As $\mu(U_n) > 0$ and $\mu(V_n) > 0$, it follows that for each $n \in N$ there exist $s_n \in U_n$ and $t_n \in V_n$ such that $s_n t_n = t_n s_n$. Letting $n \to \infty$ we see that $xy = yx$, so the support of μ is commutative. H is the subgroup of G generated by $\mathrm{supp}(\mu)$. ∎

2. Property (N) on discrete groups. We consider here the case when G is a discrete group (a countable group equipped with discrete topology). Recall that a subgroup C of G is called *Corwin* if $G = G^{(2)} = \{g^2 \mid g \in G\}$.

THEOREM 2. *Let G be a discrete group. Suppose that two identically distributed independent random variables X and Y with values in G satisfy* (N). *Then the distribution μ of X and Y is concentrated on a finite Abelian subgroup H of G and is equal to a shift by an element of H of a Haar measure of a finite Corwin subgroup of H.*

Proof. X, Y are two independent identically distributed random variables on G, so we see that if $x \in X$ is such that $P\{X = x\} \neq 0$, then

$$P\{XY = YX\} \geqslant P\{X = x,\ Y = x\} > 0.$$

Proposition 1 (1) and (3) implies that $\mathrm{supp}(\mu) \subset H$, an Abelian subgroup of G. In order to characterize μ we use results from Feldman [3], 9.10, for random variables X, Y with values in a locally compact Abelian group. ∎

3. Property (N) on compact groups.

THEOREM 3. *Let G be a compact, second countable Hausdorff topological group. Suppose that two identically distributed independent random variables X and Y with values in G satisfy* (N). *Then the distribution μ of X and Y is concentrated on an Abelian subgroup H of G.*

Proof. If G is compact, then there exists a basis of open neighbourhoods of e each of which is invariant by conjugation and symmetric (see e.g. Heyer [6]). Let W be such a neighbourhood of e. The application

$$\phi\colon (x, y) \to [x, y]$$

is continuous on $G \times G$. Let $x_0 \in \mathrm{supp}(\mu)$. As $\phi(x_0, x_0) = e$, there exists an open neighbourhood U of x_0 such that if $x, y \in U$, then $[x, y] \in W$. It follows that

$$0 < P\{X \in U,\ Y \in U\} \leqslant P\{[X, Y] \in W\}$$

and, by Proposition 1 (2), the set $\{[X, Y] \in W\}$ has probability 1. There exists a decreasing sequence W_n of invariant symmetric neighbourhoods of e such that $\bigcap_n W_n = \{e\}$. This implies that

$$P\{XY = YX\} = 1.$$

The statement follows from Proposition 1 (3). ∎

Recall that a topological group G is called an *SIN-group* if it has a basis of invariant neighbourhoods of e. The only property of compact groups used in the proof of Theorem 3 was the SIN-property, so we have

COROLLARY 4. *Let G be a second countable Hausdorff topological SIN-group. If a probability measure μ on G satisfies* (N), *then* $\mathrm{supp}(\mu) \subset H$, *an Abelian subgroup of G.*

Remark. A precise determination of measures satisfying (N) in the compact (or SIN) case, using the results of Feldman for Abelian groups, depends on properties of Abelian subgroups of G (see Feldman [3], 9.10, and [4]). These results allow us to formulate the following property, easy to verify on the level of the group G.

We denote by $\Gamma(H)$ the set of Gaussian measures in the sense of Parthasarathy on a locally compact Abelian group H and by $I_B(H)$ the set of (shifted) idempotent measures on H, satisfying the condition (B). Recall that an element $x \in G$ is of order 2 if $x \neq e$ and $x^2 = e$.

COROLLARY 5. *Let G be a second countable Hausdorff topological SIN-group. Each measure μ satisfying* (N) *belongs to $I_B(H) * \Gamma(H)$ for an Abelian subgroup H if and only if no connected Abelian subgroup of G contains more than one element of order* 2.

4. Property (N) on nilpotent groups. We formulate first a lemma valid in a general case.

LEMMA 6. *Let G be a topological second countable Hausdorff group and let X and Y be two identically distributed G-valued independent random variables with distribution μ. Suppose that there exists $x_0 \in G$ such that*

$$XY = x_0 YX \ \text{almost surely.}$$

Then $x_0 = e$.

Proof. Take $x \in \mathrm{supp}(\mu)$ and U_n a decreasing sequence of open neighbourhoods of x such that $\bigcap_n U_n = \{x\}$. For each n there exist $y_n, z_n \in U_n$ such that $y_n z_n = x_0 z_n y_n$. Otherwise, we would have

$$P\{XY \neq x_0 YX\} \geqslant P\{X \in U_n,\ Y \in U_n\} > 0.$$

As $y_n \to x$ and $z_n \to x$ when $n \to \infty$, it follows that $x^2 = x_0 x^2$ and $x_0 = e$. ∎

We start the study of the nilpotent case by an algebraic property of all nilpotent groups of step 2, that is groups having the property

$$[x, y]z = z[x, y] \tag{1}$$

for all elements x, y, z.

LEMMA 7. *If G is a nilpotent group of step 2, then*

$$[x, y^{-1}] = [y, x].$$

Proof. The equality to be proved is equivalent to

$$y^{-1}[x, y^{-1}] = y^{-1}[y, x],$$

which is true, by applying (1) to the left-hand side. ∎

PROPOSITION 8. *Let G be a second countable Hausdorff nilpotent topological group of step 2. Suppose that two identically distributed independent random variables X and Y with values in G satisfy* (N). *Then the distribution μ of X and Y is concentrated on an Abelian subgroup H of G.*

Proof. On any group we have $[y, x] = [x, y]^{-1}$. Using Lemma 7 we infer that for any $x, y \in G$

$$[x, y]^{-1} = [x, y^{-1}]. \tag{2}$$

By the condition (N), the random variables $[X, Y]$ and $[X, Y^{-1}]$ are independent, and so are $[X, Y]^{-1}$ and $[X, Y^{-1}]$. This together with (2) implies that the random variable $[X, Y]$ is independent of itself. Hence there exists $x_0 \in G$ such that $[X, Y] = x_0$ almost surely. By Lemma 6 we get $x_0 = e$. An application of Proposition 1 (3) completes the proof. ∎

THEOREM 9. *Let G be any second countable Hausdorff nilpotent topological group. Suppose that two identically distributed independent random variables X and Y with values in G satisfy* (N). *Then the distribution μ of X and Y is concentrated on an Abelian subgroup H of G.*

Proof. We prove by induction that for any $n \geqslant 1$ the following holds:

(H_n) *If μ is a probability measure on G, nilpotent of step n, satisfying* (N), *then μ is concentrated on an Abelian subgroup of G.*

This hypothesis is trivially true for $n = 1$. Proposition 8 states that it is true for $n = 2$.

Suppose that G is a nilpotent group of order n and that two identically distributed independent random variables X and Y with values in G satisfy (N). Let Z be the center of G, and $\pi\colon G \to G/Z$ the canonical projection. The group

G/Z is nilpotent of step $n-1$. The identically distributed independent random variables $\pi(X)$ and $\pi(Y)$ with values in G/Z satisfy (N), so by (H_{n-1}) they take their values in an Abelian subgroup A of G/Z.

Let $B=\pi^{-1}(A)$. If $x, y\in B$, then $\pi([x, y])=[\pi(x), \pi(y)]=eZ\in G/Z$, so $[x, y]\in \mathrm{Ker}(\pi)=Z$. This implies that B is nilpotent of order 2 (in other words, the fact that $B/B\cap Z\cong A$ is Abelian is equivalent to B nilpotent of step 2).

As X and Y take values in B, by Proposition 8 we deduce (H_n). ■

Remarks. 1. In [9] it was stated, with an uncomplete proof, that Theorem 9 is true if G is a simply connected nilpotent Lie group. We prove this result in a general nilpotent case, without using the Campbell–Hausdorff formula.

2. The Remark after Corollary 4 and Corollary 5 apply in the case when G is a second countable Hausdorff nilpotent topological group. If $G\cong \boldsymbol{R}^d$ is a simply connected nilpotent Lie group, a measure μ satisfying (N) must be a standard Gaussian measure on an Abelian subgroup $H\cong \boldsymbol{R}^k$.

5. Property (N) on some solvable groups. It is natural to ask whether the statement

μ has the property (N) $\Rightarrow \mu$ is concentrated on an Abelian subgroup,

that we proved for all discrete groups, compact or SIN-groups and nilpotent groups (for which this question may be asked), remains true for other classes of groups. In this section we show that the answer is "yes" also for some solvable groups.

The simplest solvable 2-step group is the so-called "$ax+b$" group. It is the group $S=\{(a, b) \mid a\in\boldsymbol{R}^+, b\in\boldsymbol{R}\}$ with a multiplication $(a_1, b_1)(a_2, b_2)$ given by composition of two affine maps a_1x+b_1 and a_2x+b_2:

$$(a_1, b_1)(a_2, b_2)=(a_1a_2, a_1b_2+b_1), \qquad (a, b)^{-1}=(a^{-1}, -a^{-1}b).$$

Proposition 10. *If S is the "$ax+b$" group and μ is a probability measure on S having the property* (N), *then μ is concentrated on an Abelian subgroup of S.*

Proof. It is easy to verify that if $x=(a_1, b_1)$ and $y=(a_2, b_2)$, then

$$[x, y]=(1, a_1b_2-a_2b_1+b_1-b_2), \tag{3}$$

$$[x, y^{-1}]^{-1}=\big(1, a_2^{-1}(a_1b_2-a_2b_1+b_1-b_2)\big). \tag{4}$$

Let $X=(A_1, B_1)$ and $Y=(A_2, B_2)$ be two independent identically distributed random variables with the same law μ. Let us write

$$U=A_1B_2-A_2B_1+B_1-B_2.$$

We have

$$[Y^{-1}, X]=[X, Y^{-1}]^{-1}=(1, A_2^{-1}U)=(1, W),$$

where we put $W=A_2^{-1}U$. The random variable $[X^{-1}, Y]$ has the same law as

$[Y^{-1}, X]$ since the laws of the couples (X, Y) and (Y, X) are identical. On the other hand, by (4)

$$[X^{-1}, Y] = [Y, X^{-1}]^{-1} = \big(1, A_1^{-1}\cdot(-U)\big) = (1, T),$$

where we put $T = -A_1^{-1}U$. It follows that $W \stackrel{d}{=} T$ (equality in law). By the property (N) the random variables U and W are independent. (N) implies also that the couples (XY, YX) and $(YX^{-1}, X^{-1}Y)$ are independent, so $[X, Y]$ and $[X^{-1}, Y] = (X^{-1}Y)(YX^{-1})^{-1}$ are independent, and U and T are also independent.

The independence of U and W, the independence of U and T, and the equality of distributions of W and T imply that

$$(U, W) \stackrel{d}{=} (U, T) \tag{5}$$

as random variables on $\boldsymbol{R}^2$. By Proposition 1 (1) we know that $P\{[X, Y] = (1, 0)\} = 0$ or 1. Suppose that this probability is 0. By (3) this means that $U \neq 0$ almost surely. Then (5) implies that

$$\frac{W}{U} \stackrel{d}{=} \frac{T}{U} \quad \text{and} \quad A_1^{-1} \stackrel{d}{=} -A_2^{-1}.$$

This is impossible because $A_1 > 0$ and $A_2 > 0$. Hence $P\{XY = YX\} = 1$ and the application of Proposition 1 (3) completes the proof. ∎

Remarks. It is evident that Proposition 10 is also true, with the same proof, for more general 2-step solvable groups $S = \boldsymbol{R}^+ \rtimes \boldsymbol{R}^n$. Moreover, the action of $\boldsymbol{R}^+$ on $\boldsymbol{R}^n$ may be non-homogeneous, i.e. if $a > 0$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$, then

$$a\cdot \boldsymbol{x} = (a^{d_1}x_1, \ldots, a^{d_n}x_n) \quad \text{for } d_1, \ldots, d_n \text{ fixed.}$$

Let us notice that a slight modification of the proof of Proposition 10 allows us to see that it remains true for the group $\boldsymbol{R}^* \rtimes \boldsymbol{R}^n$. As A_1 and A_2 are independent random variables with values in the Abelian group $\boldsymbol{R}^* \cong \boldsymbol{R} + \boldsymbol{Z}(2)$ such that $A_1 A_2$ and $A_1 A_2^{-1}$ are independent, they must have Gaussian distributions and, in particular, they are both concentrated on $\boldsymbol{R}^+$ or on $\boldsymbol{R}^-$. Hence the equality $A_1^{-1} \stackrel{d}{=} -A_2^{-1}$ is also impossible.

6. Gaussian measures on symmetric spaces and property (N). Let G be a semisimple Lie group with finite centre and K its maximal compact subgroup. In this section we show how to deduce from the results of Section 3 the answer to the question whether Gaussian measures on a Riemannian symmetric space of non-compact type G/K have the property (N). Recall that these measures are defined as the measures in the heat semigroup $\{\gamma_t\}_{t>0}$ generated by the Laplace–Beltrami operator on G/K.

We first formulate a general result.

PROPOSITION 11. *Let G be a metrizable group. Let $\{\mu_n\}_{n\in N}$ be a sequence of probability measures on G, each of which has the property* (N). *Suppose that*

$$\mu_n \Rightarrow \mu, \quad n \to \infty,$$

where μ is a probability measure on G. Then μ also has the property (N).

Proof. In the case of a locally compact Abelian group this property may be seen, writing a functional equation for the Fourier transform of the distributions μ_n equivalent to (N) and going with $n \to \infty$. This does not work in the non-commutative case.

Let $\mu_n \Rightarrow \mu$. Using results of Skorokhod [11] we may construct two independent sequences of random variables $\{X_n\}$ and $\{Y_n\}$ such that

(i) X_n and Y_n have the distribution μ_n for all $n \in N$;

(ii) $X_n \to X$ and $Y_n \to Y$ almost surely, where X and Y are two independent random variables;

(iii) X and Y have the distribution μ.

As the measures μ_n satisfy the condition (N), it holds for each couple (X_n, Y_n). We write the condition (N) in the form

$$\forall f, g \in C_b(G \times G) \;\; Ef(\xi)g(v) = Ef(\xi)Eg(v) \tag{6}$$

for $\xi = (X_n Y_n, Y_n X_n)$ and $v = (X_n Y_n^{-1}, Y_n^{-1} X_n)$. The property (6) is equivalent to the independence of random variables ξ and v with values in a metric space (see Feller [5]; the proof in the real case may be modified to metric spaces). By the dominated convergence theorem, when $n \to \infty$, we obtain (6) for $\xi = (XY, YX)$ and $v = (XY^{-1}, Y^{-1}X)$. If (N) holds for one pair of independent random variables with distribution μ, it holds true for any other pair of independent random variables with distribution μ. In fact, the condition (N) may be written down in terms of the distribution μ and the joint distribution $\mu \otimes \mu$ only. ∎

Remark. Proposition 11 remains true for the property (B) instead of the property (N).

THEOREM 12. *The Gaussian measures γ_t on a Riemannian symmetric space G/K of non-compact type, with K non-commutative, do not satisfy the condition* (N) *for sufficiently small $t > 0$.*

Proof. We know by Theorem 3 that the Haar measure κ of K does not satisfy (N). On the other hand, if $\{\gamma_t\}_{t>0}$ is the continuous semigroup of K-invariant Gaussian measures on G/K (seen as K-biinvariant measures on G), we have

$$\gamma_t \Rightarrow \kappa, \quad t \to 0+.$$

The theorem follows from Proposition 11. ∎

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REFERENCES

[1] T. Byczkowski, *Gaussian measures on metric Abelian groups*, in: *Probability Measures on Groups*, Proc. 5th Conf. Oberwolfach, 1978, pp. 41–53.

[2] T. Byczkowski and A. Hulanicki, *Gaussian measure of normal subgroups*, Ann. Probab. 11 (1983), pp. 685–691.

[3] G. M. Feldman, *Arithmetic of Probability Distributions and Characterization Problems on Abelian Groups*, Math. Monographs, AMS, Vol. 116 (1993).

[4] G. M. Feldman, *On groups for which the Bernstein characterization of the Gaussian distribution is valid*, Preprint de l'Université d'Angers, 1999.

[5] W. Feller, *An Introduction to Probability Theory*, Vol. 2, Wiley, New York 1966.

[6] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer, 1977.

[7] H. Heyer and Ch. Rall, *Gaussche Wahrscheinlichmasse auf Cornwinschen Gruppen*, Math. Z. 128 (1972), pp. 343–361.

[8] D. Neuenschwander, *Gauss measures in the sense of Bernstein on the Heisenberg group*, Probab. Math. Statist. 14 (1993), pp. 253–256.

[9] D. Neuenschwander, B. Roynette and R. Schott, *Characterization of Gauss measures on nilpotent Lie groups and symmetric spaces*, C. R. Acad. Sci. Paris 324 (1997), pp. 87–92.

[10] A. Rukhin, *Some statistical and probabilistic problems on groups*, Proc. Steklov Inst. Math. 111 (1970), pp. 59–129.

[11] A. V. Skorokhod, *Limit theorems for stochastic processes*, Theory Probab. Appl. 1 (1956), pp. 261–290.

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