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A GLOBAL APPROACH TO FIRST PASSAGE TIMES

BY

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Abstract. First passage times for discrete-time stochastic processes are studied from a global point of view, in terms of a mapping that takes a numerical sequence to its first passage time function. The continuity properties of this mapping with respect to Skorohod's J_1 and M_1 topologies are examined. One typically has continuity in M_1, but in J_1 only under extra assumptions. The results are applied to random walks and renewal theory.

1. INTRODUCTION

Let $X = \{X_k\}_0^\infty$ be a discrete-time stochastic process with $\sup_k X_k = +\infty$ a.s. The related first passage times are

$$\nu(t) = \min\{k;\ X_k > t\}. \tag{1.1}$$

The results concerning these problems have mainly been focused on asymptotics (Lai and Siegmund [4], [5], Gut [3], and others). The object of this paper is to give a global approach to the subject by mapping the entire process X to the first passage time process ν, and examining various continuity properties of this mapping. Section 3 treats the deterministic setting. Some consequences for stochastic processes are discussed in Section 4.

This paper is based on a section of Larsson-Cohn [6], where further details can be found. The author wishes to thank his supervisor Allan Gut as well as Gerold Alsmeyer, University of Münster, for valuable discussions.

2. PRELIMINARIES

By $\boldsymbol{R}^\infty$ we shall mean $\prod_0^\infty \boldsymbol{R}$, the countable product of the real line equipped with the usual (product) topology. We shall find it convenient to require that $x_0 = 0$ for $x \in \boldsymbol{R}^\infty$. Let $\mathscr{U} \subset \boldsymbol{R}^\infty$ consist of the sequences that are unbounded above. We remark that $\mathscr{U}$, being the intersection of the sets $\{x;\ \sup_k x_k > n\}$, is a G_δ-subset of $\boldsymbol{R}^\infty$, and therefore Polish (separable and metrizable by a complete metric); cf. Cohn [2], Theorem 8.1.4.

For any interval I, $D(I)$ denotes the Skorohod space of càdlàg functions on I. We shall be concerned with two instances of this, namely $D[0, \infty)$ with the J_1 topology and $D(\boldsymbol{R})$ with M_1. Let us call the first space D_1 and the second one D_2. Simply, D will denote either of these two spaces. For easy reference we state the following convergence criteria, cf. Lindvall [7] and Skorohod [8].

PROPOSITION 2.1. *Let f and $\{f_n\}$ be functions in $D[0, \infty)$. Then $f_n \to f\,(J_1)$ iff $r_b f_n \to r_b f$ in $D[0, b]$ for all continuity points b of f, r_b being the restriction to $[0, b]$.*

PROPOSITION 2.2. *Let f and $\{f_n\}$ be monotone functions in $D(I)$. Then $f_n \to f\,(M_1)$ iff $f_n(t) \to f(t)$ for all t that are continuity points of f or end-points of I.*

We define a mapping T by letting it take a (deterministic) sequence of real numbers to its first passage time function ν as in (1.1). Thus T maps $\mathcal{U}$ into D. The corresponding operator in continuous time has been studied in Whitt [9]. Following him, we shall occasionally write x^{-1} instead of $T(x)$.

3. DETERMINISTIC RESULTS

3.1. The J_1 case. We first treat the case $D = D_1 = (D[0, \infty), J_1)$. Before stating the main result, we introduce some terminology. The *ladder epochs* of $x \in \mathcal{U}$ are defined by

$$\tau_0 = 0, \qquad \tau_k = \min\{n > \tau_{k-1};\, x_n > x_{\tau_{k-1}}\},\ k \geqslant 1.$$

The variables x_{τ_k} are the corresponding *ladder heights.*

For integers $0 \leqslant i < j$, let Δ_{ij} consist of those $x \in \mathcal{U}$ that have a ladder epoch equal to i, no further ladder epochs between i and j, and satisfy $x_j = x_i$. Put $\Delta = \bigcup_{i,j} \Delta_{ij}$.

Let us also say that a non-decreasing, positive integer-valued function f in $D[0, b]$ has the *configuration* $\kappa = \{n_1, n_2, \ldots, n_p\}$, with $n_1 < n_2 < \ldots < n_p$, if it assumes precisely the values in κ on $[0, b]$. This is denoted by $\mathrm{conf}_b(f) = \kappa$.

Finally, the set of continuity points of T is denoted by C_T, its complement being C_T^c.

THEOREM 3.1. $C_T = \mathcal{U} \setminus \Delta$.

Proof. Assume first that $x \in \mathcal{U} \setminus \Delta$, and put $T(x) = \nu$. Let $y^{(n)}$ tend to zero in $\boldsymbol{R}^\infty$, $x + y^{(n)} \in \mathcal{U}$. We must show that

$$\nu^{(n)} := T(x + y^{(n)}) \to T(x) = \nu.$$

Now, ν has a jump at b iff x has a non-zero ladder height equal to b. Hence, by Proposition 2.1 it suffices to show that $\nu^{(n)} \to \nu$ in $D[0, b]$ whenever b is not a ladder height of x. Fix such a number b and take $\varepsilon > 0$. Let d_b be the following metric for J_1 on $D[0, b]$:

$$d_b(f, g) = \inf_{\lambda \in \Lambda} (\|f \circ \lambda - g\| \vee \|\lambda - e\|),$$

where Λ is the time deformation group on $[0, b]$ with identity e, and $\|\cdot\|$ is the supremum norm. Furthermore, let $\operatorname{conf}_b(v) = \kappa = \{n_1, n_2, \ldots, n_p\}$, and put $n_0 = 0$. This means that x has ladder epochs $n_0, \ldots, n_p$, with $x_{n_{p-1}} < b < x_{n_p}$. Moreover, since $x \notin \Delta$, we have $x_j \neq x_{n_r}$ for j between n_r and n_{r+1}, $0 \leqslant r < p$.

Therefore, there exists δ, $0 < \delta < \varepsilon$, such that if $|y_k^{(n)}| < \delta$ for $k \leqslant n_p$, then $v^{(n)}$ also has the configuration κ. Since $y^{(n)} \to 0$, this will indeed be the case for large n. The differences between the times for the corresponding jumps then cannot exceed δ, and so $d_b(v^{(n)}, v) < \varepsilon$ for large n, as was to be proven.

For the converse, take $x \in \Delta_{ij}$. Let $y^{(n)}$ have its j-th component equal to $1/n$ and the others zero, so that $y^{(n)} \to 0$. If $b > x_j$, then, using the notation as above, we obtain $j \in \operatorname{conf}_b(v^{(n)})$, but $j \notin \operatorname{conf}_b(v)$, whence $d_b(v^{(n)}, v) \geqslant 1$. ∎

Remark 3.1. Note that although Δ is dense, it is of the first category in $\mathscr{U}$ (and in $\boldsymbol{R}^\infty$), since Δ_{ij} is contained in the closed and nowhere dense set $\{x;\, x_i = x_j\}$. Hence T is continuous at "most of" $\mathscr{U}$ in a topological sense.

Remark 3.2. Although not continuous, T is still Borel measurable. This follows from the fact that the Borel σ-algebra of J_1 is generated by the finite-dimensional sets and that $x \mapsto x^{-1}(t)$ is measurable for each t; cf. Lindvall [7].

3.2. The M_1 case. Let us now consider the case $D = D_2 = (D(\boldsymbol{R}), M_1)$. Note that $v(t) = 0$ for $t < 0$ by the convention $x_0 = 0$.

THEOREM 3.2. *The first passage time mapping is continuous.*

Proof. Pick x in $\mathscr{U}$ and let $y^{(n)} \to 0$. Using the notation from the proof of Theorem 3.1, we must show that $v^{(n)}(t) \to v(t)$ for continuity points t of v. Thus, we need only consider the case when $t > 0$ is not a ladder epoch of x. Put $v(t) = p$.

This means that $x_k < t$ for $k < p$ and that $x_p > t$. Clearly, the same thing holds for $x + y^{(n)}$ provided that $y_k^{(n)}$ is small enough for $k \leqslant p$, i.e. that n is large enough. But then also $v^{(n)}(t) = p$, and we are done. ∎

Remark 3.3. The continuity would be ruined if we replaced $D(\boldsymbol{R})$ by $D[0, \infty)$, since the convergence at the end-point $t = 0$ would then fail on Δ_{0j}.

Remark 3.4. The above result is not very surprising. Indeed, the problems arising from the events Δ_{ij} depend on a special behaviour of the first passage time functions, where one jump is replaced by two successive smaller ones, cf. Larsson-Cohn [6]. Such paths are close in M_1, but not in J_1.

Remark 3.5. Theorems 3.1 and 3.2 can be compared to Theorems 7.1 and 7.2 of Whitt [9] for continuous time. In that case one has continuity (M_1) everywhere and continuity (J_1) for strictly increasing functions. However, the latter condition is not necessary (consider $x(t) = -I_{[0,1)}(t) + tI_{[1,\infty)}(t)$), and so it seems that simple necessary and sufficient conditions are not known.

4. CONSEQUENCES FOR STOCHASTIC PROCESSES

4.1. General results. The results of Section 3 have immediate consequences for discrete-time stochastic processes due to the continuous mapping theorem.

PROPOSITION 4.1. *Let* X *and* $X^{(n)}$ *be discrete-time stochastic processes that are unbounded above a.s.*

(a) *Suppose that* $P(X \in \Delta) = 0$. *If* $X^{(n)}$ *converges almost surely, in probability, or weakly to* X *in* $\boldsymbol{R}^{\infty}$, *then* $(X^{(n)})^{-1}$ *converges in the same way to* X^{-1} *in* D_1.

(b) *If the assumption that* X *does not belong to* Δ *is dropped, then the same holds with* D_1 *replaced by* D_2.

The case of convergence in probability can be given a more abstract formulation. Namely, for any separable metric space S, let $L^0(S)$ be the (metrizable) space of random elements of S on some fixed probability space, endowed with the topology of convergence in probability. The mapping $T: \mathscr{U} \to D$ induces a mapping from $L^0(\mathscr{U})$ into $L^0(D)$, which we call $\mathscr{T}$. If $D = D_2$, then $\mathscr{T}$ is continuous; if $D = D_1$, then $\mathscr{T}$ is continuous at X iff $P(X \in \Delta) = 0$.

Just like in Remark 3.1, the set $C_{\mathscr{T}}^c = \{X;\, P(X \in \Delta) > 0\}$ is dense (as is its complement) in $L^0(\mathscr{U})$. However, we do not know if it is of the first category. The sets $\{X;\, P(X \in \Delta_{ij}) = 0\}$ are also dense in interesting cases, cf. Larsson-Cohn [6].

4.2. Random walks and renewal theory. The conditions in Proposition 4.1 (a) are particularly simple to deal with if X is a random walk with positive drift, i.e. if $X_n = \sum_1^n Y_k$, where $\{Y_k\}$ are i.i.d. with positive mean. Indeed, if $\{X \in \Delta_{0j}\}$ is a null set, then $P(X_j = 0) = 0$ by stationarity. Conversely, if $P(X_j = 0)$ vanishes for all positive j, then so does $P(X_j - X_i) = 0$, and X does not belong to Δ_{ij}. Thus, T is continuous at X iff $P(X_n = 0) = 0$ for all $n \geqslant 1$, which is perhaps most simply characterized in terms of the point masses of Y_1:

PROPOSITION 4.2. *Let* X *be as above. Then* $P(X \in \Delta) = 0$ *if and only if there do not exist point masses of* Y_1 *(distinct or not) that sum up to zero.*

Thus, it suffices to have Y_1 continuous or $Y_1 > 0$ a.s. In the latter case, X is a renewal process and $\{\nu(t) - 1\}$ is the classical renewal counting process. For a concrete example: if $X^{(n)}$ converges weakly in $\boldsymbol{R}^{\infty}$ to an i.i.d. sequence of exponential variables, then $(X^{(n)})^{-1}$ converges weakly in D_1 to a Poisson process starting at 1. Note that weak convergence in $\boldsymbol{R}^{\infty}$ is equivalent to convergence of the finite-dimensional distributions, cf. Billingsley [1].

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