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PARAMETER IDENTIFICATION FOR STOCHASTIC BURGERS' FLOWS VIA PARABOLIC RESCALING

BY

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Dedicated to Professor KAZIMIERZ URBANIK on the Occasion of His 70th Birthday

Abstract. The paper presents a systematic study of classical statistical inference problems (parameter estimation and hypothesis testing) for random fields arising as solutions of the one-dimensional nonlinear diffusion equation with random initial data (the Burgers' turbulence problem). This nonlinear, hydrodynamic-type partial differential equation is an ubiquitous model in physics and engineering. This work can be seen as part of a larger program of developing statistical inference tools for complex stochastic flows governed by nontrivial, physically constrained dynamics.

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CONTENTS

1. INTRODUCTION

1.1. Physical motivation and recent history. A non-linear diffusion equation known as the *Burgers equation* describes various physical phenomena, from non-linear acoustic and kinematic waves to the growth of molecular interfaces, and the formation of large-scale quasi-Voronoi tessellation structures of self--gravitating matter in the late stages of the universe (see, for example, Burgers [11], Chorin [13], Gurbatov et al. [34], Witham [97], Kardar et al. [52], Shandarin and Zeldovich [78], Kofman et al. [54], Weinberg and Gunn [93], Vergassola et al. [89], Molchanov et al. [64], Woyczyński [99]), and other types of irrotational flows. Equations related to the one-dimensional Burgers equation have also emerged in models of financial markets (option pricing) (see Hodges and Carverhill [40]).

Rosenblatt [70], [72] was one of the first to have considered the Burgers equation with random initial data from the rigorous perspective of probability theory and, more recently, numerous researchers studied solutions of the Burgers equation. Bulinski and Molchanov [10], Giraitis et al. [27], Funaki et al. [25] studied solutions of the Burgers equation when the initial condition was either a Gaussian random field or a shot-noise (or Gibbs–Cox) random fields with weak and strong dependence. They obtained Gaussian and non-Gaussian distributions as parabolic scaling limits of distributions of the solution random fields. Leonenko et al. [58]–[60] also obtained Gaussian and non-Gaussian limit distributions in the same context of parabolic scaling in the case when the initial condition was either a Gaussian random field or a chi-square field with long-range dependence. Analogous results under suitable non-Gaussian initial conditions with weak dependence can be found in Surgailis and Woyczynski [84], [86], Hu and Woyczynski [42], Leonenko and Deriev [57], and Deriev and Leonenko [18]. In the Gaussian model with non-integrable oscillating correlations, the limit solution turned out to be non-Gaussian (see Surgailis and Woyczynski [85], [86]). For other results concerning limiting distributions of averaged solutions of Burgers' equation see Rosenblatt [72] and Hu and Woyczynski [44]. Leonenko and Woyczynski [61] obtained results on the rate of convergence to Gaussian fields in parabolic limits with strongly dependent initial data.

Other types of random problems for the Burgers equation have also been considered recently in the mathematical literature. Sinai [81], Albeverio et al. [1], Molchanov et al. [63], Avellaneda and E [3], Hu and Woyczynski [43], Wehr and Xin [92], E et al. [22], and Ryan [73], [74] considered the statistics of shocks in the zero-viscosity limit and related problems of hyperbolic limiting behavior of solutions of Burgers' equation with random data. This type of scaling is of importance in physical applications (see Gurbatov et al. [34], Vergassola et al. [89], among others). Surgailis [83] considered intermediate asymptotics between the smooth parabolic limits and shock-type hyperbolic asymptotics. Recent results on the forced Burgers equation can be found, e.g., in Sinai [80], Holden et al. [41], Bertini et al. [5], Saichev and Woyczynski [75], Molchanov et al. [64] and Kifer [53].

A recent book by Woyczynski [99] contains a fairly complete bibliography of the subject and an exposition of some of the principal results of the theory of Burgers' turbulence.

1.2. Goals of the paper. The present paper provides new statistical inference tools, both *in the space and in the frequency domains*, for the parabolically rescaled one-dimensional Burgers equation with random initial conditions (so--called *Burgers' turbulence problem*). In particular, we discuss estimation of several important physical parameters of the equation itself (such as kinematic viscosity; see Gurbatov et al. [34], Witham [97]) and parameters of the initial data. The statistical tools take advantage of the underlying dynamics governed by the non-linear diffusion Burgers' equation. The parameter identification problems for the multidimensional Burgers turbulence will be addressed in separate papers. Some results in this direction will appear soon in Leonenko and Woyczynski [62].

The problems under consideration belong to a large and recently aggressively studied area of statistics of processes which are characterized by certain singular properties (e.g., vanishing, or unboundedness) of their spectral densities. Statistical problems for discrete processes with singular spectra were studied, among others, by Dzhaparidze and Yaglom [21], Dzhaparidze and Kotz [20], Fox and Taqqu [24], Dahlhaus [16], Giraitis and Surgailis [31] and Robinson

[67]–[69]. Beran's [4] book contains a fairly complete bibliography of the subject. Statistical problems for continuous-time long-memory processes were considered by Viano et al. [90], Comte [14], and Chambers [12].

1.3. Organization of the paper. We begin (Section 2) with a review of results of Bulinski and Molchanov [10], Surgailis and Woyczynski [84], [85], Albeverio et al. [1] and Leonenko et al. [57]–[60] on parabolically rescaled solutions of Burgers' equation with weakly dependent (Theorem 2.1) and strongly dependent (Theorem 2.2) random initial data. They are rephrased here in the form convenient for statistical inference. In particular, the statistical inference for rescaled solutions of the Burgers equation with weakly dependent initial conditions is reduced to the statistical analysis for stationary continuous-parameter Gaussian processes with the covariance function of the form

$$R(x) = c\left(1 - \frac{x^2}{4\mu t}\right)\exp\{-x^2/(8\mu t)\}, \qquad x \in \boldsymbol{R}^1, \tag{1.1}$$

and the spectral density of the form

$$g(\lambda) = q\lambda^2 \exp\{-2\mu t\lambda^2\}, \qquad \lambda \in \boldsymbol{R}^1, \tag{1.2}$$

where $t > 0$ is a fixed parameter (which plays the role of time in the rescaled solution), and c, q, and μ are positive functions of unknown parameters which have to be estimated from the observed data (see (2.5) for an explicit formula for c).

On the other hand, statistical inference for the rescaled solutions of Burgers' equation with strongly dependent initial data can be reduced to an analysis of continuous-parameter stationary Gaussian processes with the spectral density of the form

$$f(\lambda) = p|\lambda|^{1+\alpha} \exp\{-2\mu t\lambda^2\}, \qquad \lambda \in \boldsymbol{R}^1,\ 0 < \alpha < 1, \tag{1.3}$$

where $t > 0$ is fixed, and p and μ are positive functions of unknown parameters. The parameter α, called here the *fractional exponent* (elsewhere it is also called the *self-similarity parameter* or the *Hurst parameter*), is also unknown. This parameter characterizes the decay at infinity of the correlation function of the initial data or, equivalently, the rate of divergence to infinity of its spectral density at the origin (see condition E and (2.18) below). All unknown parameters are to be estimated from observed data. Note that from (1.1)–(1.3) we immediately obtain

$$\int_{-\infty}^{\infty} R(x)\,dx = 0, \qquad g(0) = 0, \qquad f(0) = 0.$$

Reduction of statistical problems for Burgers' turbulence to statistical inference for stationary Gaussian processes (or fields) with spectral densities (1.2) and (1.3) is feasible because of what we call a *Gaussian scenario* which is described in detail in Section 2.

Our continuous-time statistical models with singular spectra appear to be new. Note that Viano et al. [90], Comte [14], and Chambers [12] have proposed long-memory statistical models for continuous stationary processes of ARMA type. However, their spectral densities are different from (1.2) and (1.3). Dzhaparidze and Yaglom [21] did discuss statistical inference for continuous random processes with spectral densities of the form $\lambda^2 e^{-c|\lambda|}$, $\lambda \in \boldsymbol{R}^1$, but their ideas do not seem applicable to our situation either because they can take advantage of the exact likelihood function of the Markovian Ornstein–Uhlenbeck process (with the covariance function $e^{-c|\lambda|}$, $\lambda \in \boldsymbol{R}$, see, for example, Grenander [32], p. 118). Nevertheless, the data transformation technique due to Dzhaparidze and Yaglom [21] and Dzhaparidze and Kotz [20] will be employed in our paper.

Section 3 deals with statistical analysis in the space domain. We use information on parameters contained in the covariance function and the functional limit theorem to estimate the covariance function itself (Theorem 3.1). Fortunately, for the spectral density (1.2) the limiting covariance structure is available via an explicit formula (3.3). This permits us to use the method of moments based on an estimate of the covariance function.

The functional limit theorem contains a statement on convergence of distributions of functionals continuous in the uniform topology (see Theorem 3.1 (iv)). This permits us to develop tests of hypotheses on unknown parameters. Those tests are based on Fernique's type inequalities (see (3.20) or (3.21)).

Section 4 develops three types of discretization (4.1)–(4.3) for the purposes of statistical inference in the frequency domain. As a result we obtain discretized versions of spectral densities (1.2) and (1.3). The spectral densities of these discrete stationary Gaussian processes seem to be completely new and are elegantly expressed via the *elliptic Jacobi theta-functions* (Theorem 4.1) and what we call the *fractional Jacobi functions* (Theorem 4.2). The behavior of these spectral densities at zero depends on the type of discretization. For example, for discretization of the integral type (see (4.1)) we obtain the spectral density (4.8):

$$f_{d1}^{(1)}(\lambda) \sim \mathrm{const} \cdot \lambda^2 \quad \text{as } \lambda \to 0, \tag{1.4}$$

but the discretized versions $f_{di}^{(1)}$, $i = 2, 3$, of (1.2), obtained through discretizations (4.2) and (4.3), have no singularities at zero. The answer to the question about the best discretization scheme depends largely on one's point of view. If one wants to preserve singularities of the original continuous spectral density, then the discretization (4.1) is better. If one wants to smooth them out and, as a result, to obtain simpler statistical tools, then the discretizations (4.2) and (4.3) are more desirable.

Section 4 also presents results in the asymptotic theory of *minimum contrast estimators* (Theorems 4.3 and 4.4) where our conditions are weaker than those required by other authors; see discussion in Subsection 4.2.

Statistical inference on the discretized version of (1.2) in the form (1.4) uses ideas of Dzhaparidze and Kotz [20] who suggested certain transformations of the data before applying standard asymptotic results using either the minimum contrast estimation or Whittle's method of quasi-likelihood. As a result, we obtain consistent, asymptotically normal and asymptotically efficient in Fisher's sense estimates (Theorems 4.5–4.7) for parameters of rescaled solutions with weakly dependent initial conditions. In principle, it would be possible to deal directly with untransformed data but then the asymptotic behavior of the normalized periodogram becomes rather anomalous (see, e.g., Hurvich and Ray [47]).

Finally, we discuss statistical inference for spectral densities obtained from (1.3) through discretizations (4.1)–(4.3). In particular, making again use of Dzhaparidze's data transformation technique, we reduce statistical inference for discretized data (4.1) to statistical inference for stationary Gaussian processes with the property

$$f_{10}(\lambda) \sim \text{const}\cdot|\lambda|^{\alpha-1} \quad \text{as } \lambda \to 0+, \ 0 < \alpha < 1. \tag{1.5}$$

For spectral densities of long-memory processes this is the typical behavior at zero, and several known results are applicable (see Beran [4]). We restrict our attention to results relying on the earlier results of Fox and Taqqu [24], Dahlhaus [16] and Robinson [67]–[69], and on our Theorems 4.3 and 4.4. Consequently, we obtain a semiparametric estimate of the parameter α (Theorem 4.8) and consistent, asymptotically normal estimates of parameters with a more general structure (Theorems 4.9 and 4.10). We plan to continue this line of research using recent important results of Robinson [67]–[69], Anh and Lunney [2], and Hurvich et al. [46], on the semiparametric estimation of the fractal parameter (see, also, Giraitis et al. [28], [29] about the optimality of such an estimator and on the variance-type estimators of the long-memory parameter). Theorem 4.8 describes corresponding statistical properties of Whittle's estimates for parameters of the spectral densities $f_{di}^{(2)}$, $i = 2, 3$.

The proofs are collected in Section 5.

2. PARABOLIC ASYMPTOTICS: THE GAUSSIAN SCENARIO

2.1. The Hopf–Cole solution. We consider the one-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \quad \mu > 0, \tag{2.1}$$

subject to a random initial condition

$$u(0, x) = u_0(x) = \frac{d}{dx}v(x). \tag{2.2}$$

We shall think about the time- and space-dependent random field $u(t, x)$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$, as a velocity field. In this context, the initial datum $v(x)$ is often called the *(random) velocity potential.*

The non-linear parabolic equation (2.1) can be viewed as a simplified version of the Navier–Stokes equation with the non-linear quadratic inertial term uu_x preserved, the pressure term omitted, and the inverse $R = 1/\mu$ of the viscosity parameter μ corresponding to the Reynolds number. Despite its apparent simple form, the Burgers equation (2.1) encompasses some of the important features of the fluid flow, such as steepening of the velocity profiles leading to creation of shocks.

The Burgers equation (2.1) can be linearized by the so-called Hopf–Cole transformation $u(t, x) = -2\mu(\partial/\partial x)\log z(t, x)$ (see, e.g., Witham [97]), which reduces (2.1) to the linear diffusion equation

$$\frac{\partial z}{\partial t} = \mu \frac{\partial^2 z}{\partial x^2}, \qquad t > 0, \ x \in \boldsymbol{R}^1,$$

subject to the initial condition $z(0, x) = \exp\{-v(x)/2\mu\}$, $x \in \boldsymbol{R}^1$. Thus, the solution of equation (2.1) is given by an explicit Hopf–Cole formula

$$u(t, x) = \frac{\int_{-\infty}^{\infty} [(x-y)/t]\, g(t, x-y)\, e^{-v(y)/(2\mu)}\, dy}{\int_{-\infty}^{\infty} g(t, x-y)\, e^{-v(y)/(2\mu)}\, dy} = \frac{I(t, x)}{J(t, x)}, \tag{2.3}$$

where $v(x) = v(0, x)$ is the initial velocity potential (see (2.2)) and

$$g(t, x-y) = (4\pi\mu t)^{-1/2} \exp\{-(x-y)^2/(4\mu t)\}, \qquad x, y \in \boldsymbol{R}^1, \ t > 0,$$

is the Gaussian kernel.

2.2. Parabolic limits for weakly dependent initial conditions. Let now $(\Omega, \mathscr{F}, P)$ be a complete probability space, the initial velocity potential $v(x) = \xi(x) = \xi(\omega, x)$, $\omega \in \Omega$, $x \in \boldsymbol{R}^1$, be a random process, and $u = u(t, x)$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$, be the solution of the random Cauchy problem (2.1) and (2.2). In view of the inner symmetries of Burgers' equation and its connection to the linear diffusion equation via the Hopf–Cole transformation, a study of the limiting behavior of the parabolically rescaled solution random field

$$u = u(t\beta, x\sqrt{\beta}), \qquad \beta > 0,$$

as $\beta \to \infty$ is of obvious interest. If $\mu > 0$ is fixed, under some additional conditions on the random process $\xi(x)$, $x \in \boldsymbol{R}^1$ (to be stated explicitly later in this section), the above rescaled solution obeys asymptotically a "Gaussian scenario" (see, e.g., Bulinski and Molchanov [10], Surgailis and Woyczynski [84], [86], Albeverio et al. [1], Leonenko and Deriev [57], Deriev and Leonenko [18]). Non-Gaussian limits have also been found in some cases when $\xi(x)$, $x \in \boldsymbol{R}^1$, is a stationary random process with long-range dependence (see Surgailis and Woyczynski [85], [86], Albeverio et al. [1], Leonenko et al. [59], Leonenko and Orsingher [58], among others). The latter situation is, however, not considered in the present paper.

In what follows we will need the following assumptions:

A. *The random process* $\xi(x) = \xi(\omega, x)$, $\omega \in \Omega$, $x \in \boldsymbol{R}^1$, *is a real, separable, measurable, stationary, a.s. differentiable Gaussian process with* $E\xi(x) = 0$, $E\xi^2(x) = \sigma^2 > 0$ *and continuous covariance function* $B(x) = B(|x|) = \operatorname{cov}(\xi(0), \xi(x))$, $x \in \boldsymbol{R}^1$.

B. *There exists a real measurable non-random function* $F(u)$, $u \in \boldsymbol{R}^1$, *such that the expectation* $E[\exp\{-F(\xi(0))/2\mu\}]^2 < \infty$, *where* $\xi(x)$, $x \in \boldsymbol{R}^1$, *is a random process satisfying condition* **A.**

Let

$$\phi(u) = (2\pi)^{-1/2} \exp\{-u^2/2\}, \quad u \in \boldsymbol{R}^1,$$

be the density function of a standard $\mathcal{N}(0, 1)$ Gaussian random variable, and $L_2(\boldsymbol{R}^1, \phi(u)\,du)$ be the Hilbert space of functions $f(\cdot)$ such that $E[f(\xi(0))]^2 < \infty$. It is well known (see, e.g., Kwapień and Woyczynski [56]) that Hermite polynomials

$$H_m(u) = (-1)^m \exp\{u^2/2\} \frac{d^m}{du^m} \exp\{-u^2/2\}, \quad u \in \boldsymbol{R}^1, \ m = 0, 1, \ldots$$

form a complete orthogonal system in the Hilbert space $L_2(\boldsymbol{R}^1, \phi(u)\,du)$.

Under assumption **B**, the function $f(u) = \exp\{-F(\sigma u)/2\mu\} \in L_2(\boldsymbol{R}^1, \phi(u)\,du)$, and may be expanded in an $L_2(\boldsymbol{R}^1, \phi(u)\,du)$-convergent series

$$\exp\left\{-F(\sigma u)\frac{1}{2\mu}\right\} = \sum_{k=0}^{\infty} C_k H_k(u)/k!$$

with coefficients

$$C_k = \int_{-\infty}^{\infty} \exp\left\{-F(\sigma u)\frac{1}{2\mu}\right\} \phi(u) H_k(u)\,du.$$

D. *The function* $f(u) = \exp\{-F(\sigma u)/2\mu\}$, $u \in \boldsymbol{R}^1$, *satisfies assumption* **B** *and there exists an integer* $m \geqslant 1$ *such that* $C_1 = \ldots = C_{m-1} = 0$, $C_m \neq 0$. Such an m is called the *Hermitian rank of the function* $f(u)$ (see, for example, Taqqu [88]).

An application of ideas of Breuer and Major [7] (see also Giraitis and Surgailis [30], Ivanov and Leonenko [51]) yields the following theorem which is a version of results proved by Surgailis and Woyczynski [84], Albeverio et al. [1], Leonenko and Deriev [57] (see also Deriev and Leonenko [18]).

THEOREM 2.1. *Let* $u(t, x)$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$, *be a solution of the Cauchy problem* (2.1) *and* (2.2) (see (2.3)) *with a.s. differentiable random initial condition*

$v(x) = F(\xi(x))$, $x \in \boldsymbol{R}^1$, where the random process $\xi(x)$, $x \in \boldsymbol{R}^1$, and the non-random function $F(\cdot)$ satisfy conditions $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$, *and*

$$\int_{-\infty}^{\infty} |B(x)|^m \, dx < \infty, \qquad \int_{-\infty}^{\infty} \left[\sum_{k=m}^{\infty} \frac{C_k^2}{k!\,\sigma^{2k}} B^k(x) \right] dx \neq 0,$$

where $m \geqslant 1$ is the Hermitian rank of the function $f(u) = \exp\{-F(\sigma u)/2\mu\}$, $u \in \boldsymbol{R}^1$. Then the finite-dimensional distributions of the random fields

$$U_\beta(t, x) \equiv \beta^{3/4} u(t\beta, x\sqrt{\beta}), \qquad x \in \boldsymbol{R}^1,\ t > 0, \tag{2.4}$$

converge weakly, as $\beta \to \infty$, to the finite-dimensional distributions of the Gaussian field $U(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, which is stationary in $x \in \boldsymbol{R}^1$, with $EU(t, x) = 0$, and has covariance function of the form

$$\begin{aligned} R(x-y) &= R(x-y, t+s) \equiv EU(t, x)\, U(s, y) \\ &= c \sqrt{\frac{\mu}{\pi} \frac{1}{(t+s)^{3/2}}} \left(1 - \frac{(x-y)^2}{2\mu(t+s)}\right) \exp\left\{ -\frac{(x-y)^2}{4\mu(t+s)} \right\} = \int_{-\infty}^{\infty} e^{i\lambda(x-y)} g(\lambda)\, d\lambda, \end{aligned} \tag{2.5}$$

where

$$g(\lambda) = \frac{2c\mu^2}{\pi} \lambda^2 \exp\{-\lambda^2 \mu(t+s)\}, \qquad \lambda \in \boldsymbol{R}^1, \tag{2.6}$$

and

$$c = c(\mu, B) = \int_{-\infty}^{\infty} G(|u|)\, du, \qquad G(u) = \frac{1}{C_0^2} \sum_{k=1}^{\infty} \frac{C_k^2}{k!\,\sigma^{2k}} B^k(|x|). \tag{2.7}$$

Remark 2.1. The covariance function (2.5) of the Gaussian random field $U(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, which is a limit of the field (2.4), has the spectral density $g(\lambda)$, $\lambda \in \boldsymbol{R}^1$, given by (2.6). It vanishes at zero ($g(0) = 0$). This condition, especially in the context of Burgers' turbulent diffusion, i.e., passive tracer transport in stochastic Burgers' flows, is a consequence of the physical dynamic mass conservation law (see Saichev and Woyczynski [75], p. 1030).

COROLLARY 2.1. *Let, under the assumptions of Theorem 2.1, $F(u) \equiv u$, and $v(x) = \xi(x)$, $x \in \boldsymbol{R}^1$, be a stationary Gaussian process with $E\xi(x) = 0$ and covariance function $B(|x|)$, $x \in \boldsymbol{R}^1$, such that*

$$\int_{-\infty}^{\infty} |B(|x|)|\, dx < \infty, \qquad \int_{-\infty}^{\infty} B(x)\, dx \neq 0.$$

Then the statement of Theorem 2.1 is true with the constant

$$c = c(\mu, B) = \int_{-\infty}^{\infty} \left(\exp\left\{ \frac{B(|x|)}{4\mu^2} \right\} - 1 \right) dx. \tag{2.8}$$

The constant $c\sqrt{\mu/\pi}$ given by (2.5) *and* (2.7) *can be approximated in this case by*

$$\frac{1}{4\pi^{1/2}\mu^{3/2}}\int_{-\infty}^{\infty} B(x)\,dx. \tag{2.9}$$

EXAMPLE 2.1. Let

$$B(x) = B_1(|x|) \equiv \frac{1}{\gamma_1}\exp\left\{-\frac{1}{\gamma_2}x^2\right\}, \qquad x\in \boldsymbol{R}^1,\ \gamma_1>0,\ \gamma_2>0. \tag{2.10}$$

Then the constant $c\sqrt{\mu/\pi}$ in (2.5), where c is given by (2.8), is approximately equal to

$$\frac{\gamma_2^{1/2}}{\gamma_1}\frac{1}{4\mu^{3/2}}. \tag{2.11}$$

EXAMPLE 2.2. Let

$$B(x) = B_2(|x|) \equiv \frac{1}{\gamma_1}\exp\left\{-\frac{1}{\gamma_2}|x|\right\}, \qquad x\in \boldsymbol{R}^1,\ \gamma_1>0,\ \gamma_2>0. \tag{2.12}$$

Then the constant $c\sqrt{\mu/\pi}$ in (2.5), where c is given by (2.8), is approximately equal to

$$\frac{\gamma_2}{\gamma_1}\frac{1}{2\pi^{1/2}\mu^{3/2}}. \tag{2.13}$$

Theorem 2.1 describes limiting distributions of parabolically rescaled solutions of the Burgers equation with random initial condition which is a stationary (possibly, non-Gaussian) random process with weak dependence (the covariance function is integrable or, equivalently, the spectral density is continuous, bounded and bounded away from zero). We note that results similar to Theorem 2.1 have been obtained by Bulinski and Molchanov [10] for the shot noise processes, by Surgailis and Woyczynski [84] for the stationary mixing processes, and by Funaki et al. [25] for Gibbs–Cox random processes. In those papers the limiting Gaussian process has the correlation function of the type (2.5) or, equivalently, the spectral density of the type (2.6), but the structure of the constant depends on the probabilistic structure of the corresponding random initial data.

2.3. Parabolic limits for strongly dependent initial conditions. The present subsection contains a theorem related to the known results on scaling solutions of the Burgers equation with strongly dependent (long-memory) Gaussian initial data (see Giraitis et al. [27], Surgailis and Woyczynski [84]–[86], Albeverio et al. [1], Leonenko et al. [58]–[60]). Those results were obtained by an application of ideas and methods of Dobrushin and Major [19], and Taqqu [88]. We will have need of the following condition:

E. *Condition* A *is satisfied with the covariance function*

$$B(|x|) = E\xi(0)\xi(x) = \frac{L(|x|)}{|x|^{\alpha}}, \qquad 0 < \alpha < 1, \; x \in \boldsymbol{R}^1,$$

where $B(0) = \sigma^2$ *and where* $L(t)$, $t > 0$, *is a function bounded on each finite interval, slowly varying for large values of* t.

THEOREM 2.2. *Let* $u(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, *be a solution of the initial value problem* (2.1) *and* (2.2) *with a.s. differentiable random initial condition* $v(x) = F(\xi(x))$, $x \in \boldsymbol{R}^1$, *where the random process* $\xi(x)$, $x \in \boldsymbol{R}^1$, *and the non-random function* $F(\cdot)$ *satisfy conditions* A, B, E, *and*

$$C_1 = \int_{-\infty}^{\infty} \exp\left\{-F(u\sigma)\frac{1}{2\mu}\right\}\phi(u)\,u\,du \neq 0.$$

Then the finite-dimensional distributions of the fields

$$Y_\beta(t, x) = \frac{\beta^{1/2+\alpha/4}}{L^{1/2}(\sqrt{\beta})} u(t\beta, x\sqrt{\beta}), \qquad t > 0, \; x \in \boldsymbol{R}^1, \; \beta > 0,$$

converge weakly, as $\beta \to \infty$, *to the finite-dimensional distributions of the stationary in* x *Gaussian field* $Y(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, *with* $EY(t, x) = 0$ *and the covariance function of the form*

(2.14) $\quad EY(t, x)\,Y(s, y)$

$$= \frac{C_1^2}{C_0^2}\frac{(2\mu)^{-1-\alpha/2}}{\sqrt{ts}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{w_1 w_2 \phi(w_1)\phi(w_2)\,dw_1\,dw_2}{|(x-y)/\sqrt{2\mu} - (w_1\sqrt{t} - w_2\sqrt{s})|^{\alpha}},$$

$0 < \alpha < 1$, $t, s > 0$, $x, y \in \boldsymbol{R}^1$, *where*

$$C_0 = \int_{-\infty}^{\infty} \exp\left\{-F(u\sigma)\frac{1}{2\mu}\right\}\phi(u)\,du.$$

Remark 2.2. Let $F(u) \equiv u$ and let $v(x) = \xi(x)$, $x \in \boldsymbol{R}^1$, satisfy condition E. Then

$$C_0 = \exp\{\sigma^2/(8\mu^2)\}, \qquad C_1 = -\frac{\sigma}{2\mu}\exp\{\sigma^2/(8\mu^2)\},$$

and

(2.15) $$C_1/C_0 = -\sigma/2\mu.$$

Remark 2.3. If the Gaussian process

$$\xi(x) = \sigma \int_{-\infty}^{\infty} e^{i\lambda x}[b(\lambda)]^{1/2}\,G(d\lambda), \qquad x \in \boldsymbol{R}^1,$$

where $G(\cdot)$ is the complex Gaussian white noise with mean zero and unit variance (that is, its real and imaginary parts have variances equal to 1/2), has the spectral density $b(\lambda)$, $\lambda \in \boldsymbol{R}^1$, which is decreasing for $|\lambda| \geqslant \lambda_0 > 0$, then the limiting Gaussian field $Y(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, in Theorem 2.2 can be represented in the following way:

$$(2.16) \qquad Y(t, x) = \frac{[c_1(\alpha)]^{1/2}}{i} 2\mu \frac{C_1}{C_0} \int_{-\infty}^{\infty} e^{i\lambda x} r(\lambda) G(d\lambda), \qquad t > 0,\ x \in \boldsymbol{R}^1$$

(see Leonenko et al. [60]), and, for $F(u) = u$, we obtain from (2.15)

$$(2.16') \qquad Y_g(t, x) = -\frac{[c_1(\alpha)]^{1/2}}{i} \sigma \int_{-\infty}^{\infty} e^{i\lambda x} r(\lambda) G(d\lambda), \qquad t > 0,\ x \in \boldsymbol{R}^1,$$

where

$$r(\lambda) = \lambda \exp\{-\mu t \lambda^2\} |\lambda|^{(\alpha-1)/2}, \qquad \lambda \in \boldsymbol{R}^1,$$

and

$$(2.17) \qquad c_1(\alpha) = \alpha \Big/ \left[2\Gamma(1+\alpha) \cos\left(\frac{\alpha\pi}{2}\right) \right].$$

Using the Tauberian theorem (see Bingham et al. [6], p. 241), under conditions **A** and **E** we have the following asymptotic representation:

$$(2.18) \qquad b(\lambda) = b(|x|) \sim |\lambda|^{\alpha-1} L\left(\frac{1}{|\lambda|}\right) c_1(\alpha), \qquad \lambda \to 0,\ 0 < \alpha < 1,$$

where the constant $c_1(\alpha)$ is defined by (2.17). Thus $b(\lambda) \uparrow \infty$ as $|\lambda| \to 0$. Using (2.16), we have $r(0) = 0$.

Let $L_2(\Omega)$ be the Hilbert space of random variables with finite second moments. Then the limiting field $Y_g(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, given by (2.16′), is $L_2(\Omega)$-equivalent to the stationary in x Gaussian field

$$(2.19) \qquad Y_g^*(t, x) = [c_1(\alpha)]^{1/2} \sigma \int_{-\infty}^{\infty} \exp\{i\lambda x - \mu t \lambda^2\} |\lambda|^{(1+\alpha)/2} G(d\lambda)$$

with the covariance function

$$EY^*(t, x) Y^*(s, y) = V(x-y, t+s) = EY(t, x) Y(s, y) = \int_{-\infty}^{\infty} e^{i\lambda(x-y)} f(\lambda) d\lambda,$$

where the spectral density

$$f(\lambda) = c_1(\alpha) \sigma^2 \exp\{-\mu\lambda^2(t+s)\} |\lambda|^{1+\alpha}.$$

Remark 2.4. Witham [97] and Gurbatov et al. [34], pp. 14–20, give examples of physical phenomena (in geometric optics and acoustic wave propagation) leading to the Burgers equation containing two parameters:

$$\frac{\partial \varrho}{\partial \tau}+\beta\varrho\frac{\partial \varrho}{\partial z}=\nu\frac{\partial^2 \varrho}{\partial z^2}, \qquad v>0, \tag{2.20}$$

with the initial condition

$$\varrho(0, z)=\varrho_0(z),$$

where $\varrho=\varrho(\tau, z)$, $\tau>0$, $z\in \boldsymbol{R}^1$. The parameters $\beta=\beta(w)>0$, $w\in \boldsymbol{R}^p$, $p\geqslant 1$, and $\nu=\nu(v)>0$, $v\in \boldsymbol{R}^q$, $q\geqslant 1$, themselves are already summary parameters that are functions of several physical constants such as linear sound velocity, dissipation coefficient, adiabatic index, dimensional coefficient, and so on. After transformations

$$u=\varrho, \quad t=\beta\tau, \quad \mu=\nu/\beta \tag{2.21}$$

the equation (2.20) leads to the Burgers equation (2.1) with parameter $\mu=\nu(v)/\beta(w)$. It is obvious that these transformations do not affect parabolic asymptotics. So it is good to keep in mind that the parameter $\mu=\mu(\vartheta)$, $\vartheta\in \boldsymbol{R}^s$, $s\geqslant 1$, which principally represents the viscosity of the medium, may itself depend on a number of other meaningful physical constants $\vartheta=(\vartheta_1, \ldots, \vartheta_s)'\in \boldsymbol{R}^s$.

In the next section we will consider statistical problems of estimation of the unknown parameter ϑ and possibly other parameters (such as constants appearing in the initial conditions) using observations of rescaled solutions of the Burgers equation.

Remark 2.5. Grenander and Rosenblatt [33], pp. 163–173, and Rosenblatt [71], pp. 152–155, considered the problem of measuring turbulence spectra for the three-dimensional Navier–Stokes equation and proposed a non-parametric approach to the problem. However, it is worthwhile to note that the structure of the Reynolds number $R=v\lambda/\nu$ makes it dependent on three other physical parameters: v — the characteristic velocity, λ — the eddy magnitude, and ν — the kinematic viscosity. In the context of the Burgers equation this corresponds to the structure of parameter μ being $\mu=\mu(\vartheta_1, \vartheta_2, \vartheta_3)$ and, in the next section, we will describe statistical inference methods applicable in this situation.

3. INFERENCE IN THE SPACE DOMAIN

3.1. Properties of correlation function estimators. In what follows we will need the following assumption:

F. *The random field $U(t, x)=U_{c,\mu}(t, x)$, $t>0$, $x\in \boldsymbol{R}^1$, is measurable, centered* *and Gaussian, stationary in x, with the covariance function of the form*

$$\begin{aligned}(3.1)\qquad R(x) &= R(x, \eta, \vartheta)\\ &= EU(t, 0)\,U(t, x)\\ &= c(\eta, \vartheta)\sqrt{\mu(\vartheta)}\frac{1}{\pi^{1/2}(2t)^{3/2}}\left(1-\frac{x^2}{4\mu(\vartheta)t}\right)\exp\left\{-\frac{x^2}{8\mu(\vartheta)t}\right\},\end{aligned}$$

$t > 0$, $x \in \boldsymbol{R}^1$, *where*

$$c = c(\eta, \vartheta) = c(\eta_1, \ldots, \eta_r; \vartheta_1, \ldots, \vartheta_k)\colon \Xi \times \Theta \to (0, \infty)$$

and

$$\mu = \mu(\vartheta) = \mu(\vartheta_1, \ldots, \vartheta_k)\colon \Theta \to (0, \infty)$$

are two measurable functions with respect to Borel σ-algebras $\mathscr{B}(\Xi) \times \mathscr{B}(\Theta)$ *and* $\mathscr{B}(\Theta)$, *respectively, and* $\Xi \subset \boldsymbol{R}^r$, $\Theta \subset \boldsymbol{R}^k$, $r \geqslant 1$, $k \geqslant 1$, *are compact sets with non--empty interiors which are assumed to contain the true values* η_0 *and* ϑ_0 *of parameters* η *and* ϑ, *respectively.*

Remark 3.1. Applying results from Cramer and Leadbetter [15], p. 170, we infer that for every fixed $t > 0$ there exists a separable version of the Gaussian field $U(t, x)$, $x \in \boldsymbol{R}^1$, the sample paths thereof are continuous on every compact interval D in $\boldsymbol{R}^1$. So, every such random field U induces a probability measure $Q_{c,\mu}$ on the space $C(D)$ of continuous functions on D with the uniform topology.

In this section our main goal is to estimate the true value of parameters η_0 and ϑ_0 (or some components thereof) from observation of the random field $U(t, x)$, $x \in \Delta \subset \boldsymbol{R}^1$, with a fixed $t > 0$.

Remark 3.2. The Gaussian random fields with covariance functions (3.1) appear as parabolic limits of rescaled solutions of the Burgers equation with weakly dependent initial conditions (see Theorem 2.1). The parameter $c = c(\eta, \vartheta)$ contains all the information about the initial conditions (see (2.6) and (2.7)) and information about parameters depending on ϑ through the viscosity $\mu(\vartheta)$. The parameter $\mu(\vartheta)$ contains information about the viscosity and, possibly, other physical constants (see Remarks 2.3 and 2.4). The structure of parameters $c = c(\eta, \vartheta)$ and $\mu(\vartheta)$ can be chosen in various ways. For example, we may consider $a = c\sqrt{\mu/\pi}2^{-3/2} > 0$ as a single parameter that characterizes the value $R(0)$ which, physically, represents the flow energy. Another possibility would be to consider the factorized structure

$$c(\eta, \vartheta)\sqrt{\mu(\theta)/\pi}2^{-3/2} = b_1(\eta)\,b_2(\vartheta),$$

where $b_1(\eta)$ contains information about the initial data and $b_2(\vartheta)$ contains information on parameters depending on ϑ through viscosity $\mu(\vartheta)$.

For example, for Gaussian initial conditions using the approximate value (2.9), instead of $c\sqrt{\mu/\pi}2^{-3/2}$ with c given by (2.8), we obtain

$$b_1(\eta) = \frac{1}{4\sqrt{\pi}} \int_{-\infty}^{\infty} B(|x|)\,dx, \qquad b_2(\vartheta) = [\mu(\vartheta)]^{-3/2}\,2^{-3/2}.$$

Then, for the correlation function (2.10), we obtain

$$b_1(\eta) = b_1(\gamma_1, \gamma_2) = \sqrt{\gamma_2}/(4\gamma_1\sqrt{\pi}),\ \eta = (\gamma_1, \gamma_2)', \qquad b_2(\vartheta) = [\mu(\vartheta)]^{-3/2}\,2^{-3/2},$$

or, for the correlation function (2.12), we have

$$b_1(\eta) = b_1(\gamma_1, \gamma_2) = \gamma_2/(2\gamma_1),\ \eta = (\gamma_1, \gamma_2)', \qquad b_2(\vartheta) = [\mu(\vartheta)]^{-3/2}\,2^{-3/2}.$$

Unfortunately, such a factorization is not always possible since all the coefficients C_k in the formula (2.7) depend on μ. However, if instead of the initial condition (2.2) we consider an initial condition of the form $u(0, x) = -2\mu dv(x)/dx$, then the coefficients C_k's do not depend on μ.

To estimate the covariance function $R(x)$ at the fixed point $x \in [0, A]$, where $A > 0$ is a constant, from observation $U(t, x)$, $x \in [0, T+A]$, with $t > 0$ being fixed, we shall use the statistic

$$\hat{R}_T(x) = \frac{1}{T}\int_0^T U(t, b)\,U(t, b+x)\,db. \tag{3.2}$$

Combining results of Ivanov and Leonenko [50] (see, also, Ivanov and Leonenko [51], Chapter IV) and Buldigin [9] we obtain the following theorem:

THEOREM 3.1. *Let $U(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, be a stationary in x Gaussian field satisfying condition* F. *Then, as $T \to \infty$,*

(i) $\hat{R}_T(x) \to R(x)$ *a.s.*;

(ii) $\sup_{x\in[0,A]} |\hat{R}_T(x) - R(x)| \to 0$ *a.s.*;

(iii) *finite-dimensional distributions of the centered process*

$$X_T(x) = \sqrt{T}\,(\hat{R}_T(x) - R(x)), \qquad x \in [0, A],$$

converge weakly to finite-dimensional distributions of a centered Gaussian process $Z(x)$ with the covariance function

$$\varrho(x, y) = \varrho(x, y; \eta, \vartheta) = EZ(x)Z(y) = 4\pi \int_{-\infty}^{\infty} \cos(x\lambda)\cos(y\lambda)\,g^2(\lambda)\,d\lambda \tag{3.3}$$
$$= \frac{4c^2\mu^4}{\sqrt{\pi\mu t}}\left[\exp\left\{-\frac{(x-y)^2}{16\mu t}\right\}\tau(x-y) + \exp\left\{-\frac{(x+y)^2}{16\mu t}\right\}\tau(x+y)\right],$$

$x, y \in [0, A]$, *where $\tau(a) = (8\mu t)^{-4}[a^4 - 48a^2\mu t + 24(\mu t)^2]$, and the spectral density of the field $U(t, x)$ is given by* (2.6), *that is*

$$g(\lambda) = g(\lambda; \eta, \vartheta) = \frac{2c\mu^2}{\pi}\lambda^2 \exp\{-2\mu t\lambda^2\}, \qquad \lambda \in \mathbf{R}^1. \tag{3.4}$$

(iv) *The probability measures $\mathcal{P}_T$ converge weakly to the probability measure $\mathcal{P}$ in the space $C([0, A])$ of continuous functions on $[0, A]$ with the uniform topology, where $\mathcal{P}_T$ are measures induced by the processes $X_T(x)$, $x \in [0, A]$, and $\mathcal{P}$ is a measure induced by $Z(x)$, $x \in [0, A]$, on the space $C([0, A])$.*

Remark 3.3. With covariance function $\varrho(x, y)$ given by the first line of (3.3) statements (i)–(iv) of the above theorem are true for general stationary Gaussian processes, and, with some additional covariance structure and additional conditions (see Ivanov and Leonenko [51], Chapter IV), also for non--Gaussian mixing processes. For a Gaussian process those conditions simplify. For example, if the spectral density $g(\lambda)$, $\lambda \in \boldsymbol{R}^1$, of a Gaussian process (not necessarily of the form (3.4)) satisfies the condition $\int_{-\infty}^{\infty} g^2(\lambda)\,d\lambda < \infty$, then statement (iii) remains true (see Buldigin [9]); and if there exists a $\beta \in (0, 1)$ such that $\int_{-\infty}^{\infty} |\lambda|^{1+\beta} g^2(\lambda)\,d\lambda < \infty$, then statement (iv) remains true (see Ivanov and Leonenko [50], combined with Buldigin [9]).

3.2. Method of moments. In this subsection we will apply the method of moments to estimate parameters η and ϑ.

Let us consider the following system of non-linear equations with respect to η or ϑ:

$$\frac{1}{(2t)^{3/2}}\sqrt{\frac{\mu(\vartheta)}{\pi}}\,c(\eta, \vartheta)\left(1-\frac{\bar{x}_j^2}{4\mu(\vartheta)\,t}\right)\exp\left\{-\frac{\bar{x}_j^2}{8\mu(\vartheta)\,t}\right\} \equiv R(\bar{x}_j), \tag{3.5}$$

$j = 1, \ldots, r+k$, where $\bar{x}_j$ are fixed points from an interval $[0, A] \subset [0, H]$ of monotonicity of the function $R(x)$. If we solve these equations, then we obtain expressions for

$$\eta_i = Q_i(\zeta),\ i = 1, \ldots, r, \qquad \vartheta_i = Q_i(\zeta),\ i = r+1, \ldots, r+k, \tag{3.6}$$

where $\zeta = (\zeta_1, \ldots, \zeta_{r+k})' = \big(R(\bar{x}_1), \ldots, R(\bar{x}_{r+k})\big)'$. The basic idea of the method of moments is to put vector $\zeta_T = \big(\hat{R}_T(\bar{x}_1), \ldots, \hat{R}_T(\bar{x}_{r+k})\big)'$ instead of vector ζ in expressions (3.5) or (3.6), where $\hat{R}_T(x)$ is defined by (3.2), to obtain the estimates

$$\check{\eta}_T = (\check{\eta}_1, \ldots, \check{\eta}_r)' = \big(Q_1(\zeta_T), \ldots, Q_r(\zeta_T)\big)',$$

$$\check{\vartheta}_T = (\check{\vartheta}_1, \ldots, \check{\vartheta}_k)' = \big(Q_{r+1}(\zeta_T), \ldots, Q_{r+k}(\zeta_T)\big)'.$$

Remark 3.4. Recall that if B is a symmetric $n \times n$ positive-definite matrix, then there exists a symmetric positive-definite matrix, denoted by $B^{1/2}$ and called the *symmetric square root* of B, such that $B = B^{1/2}B^{1/2}$.

For a function $Q = Q(\zeta)$, $\zeta = (\zeta_1, \ldots, \zeta_{r+k})'$, denote by

$$\nabla Q = \nabla Q_\zeta = \left(\frac{\partial}{\partial \zeta_1}Q, \ldots, \frac{\partial}{\partial \zeta_{r+k}}Q\right)'$$

its gradient at the point $(\zeta_1, \ldots, \zeta_{r+k}) = \big(R(\bar{x}_1), \ldots, R(\bar{x}_{r+k})\big)'$.

If the functions Q_j, $j = 1, \ldots, r+k$, are continuous, then it follows from Theorem 3.1 that, as $T \to \infty$,

$$\breve{\eta}_T \to \eta_0, \quad \breve{\vartheta}_T \to \vartheta_0 \text{ a.s.,}$$

and

$$\sqrt{T}\left((\breve{\eta}_T, \breve{\vartheta}_T)' - (\eta_0, \vartheta_0)'\right) \xrightarrow{D} \mathcal{N}_{r+k}(0, S), \tag{3.7}$$

where

$$S = S(\eta_1, \vartheta) = (\nabla Q)' \Sigma_1 (\nabla Q), \quad \Sigma_1 \equiv \left(\varrho(\bar{x}_i, \bar{x}_j)\right)_{1 \leqslant i,j \leqslant r+k}, \tag{3.8}$$

and $\varrho(x_i, x_j) = \varrho(x_i, x_j; \eta, \vartheta)$ are defined by (3.3).

If $S > 0$ and $S^{1/2}$ is a symmetric square root of S (see Remark 3.5), and $S^{1/2} = S^{1/2}(\eta, \vartheta)$ is continuous in η_0 and ϑ_0, then, as $T \to \infty$,

$$S^{-1/2}(\breve{\eta}_T, \breve{\vartheta}_T)\sqrt{T}\left((\breve{\eta}_T, \breve{\vartheta}_T)' - (\eta_0, \vartheta_0)\right)' \xrightarrow{D} \mathcal{N}_{r+k}(0, I_{r+k}), \tag{3.9}$$

where I_n stands for the identity matrix of order $n \geqslant 1$. The last statement allows us to construct asymptotic confidence intervals for unknown parameters $(\eta, \vartheta)'$ or for some of their components. This method permits us, first, to use an approximate solutions (3.6) of the non-linear equations (3.5), and then to apply (3.7) and (3.9).

For the sake of simplicity put $t = 1$ and consider the problem of estimation of two parameters only:

$$a = c\sqrt{\mu/\pi}\, 2^{-3/2} \quad \text{and} \quad \mu(\vartheta) = \mu > 0.$$

Parameter a characterizes the value $R(0)$ (energy) in the initial data, and parameter $\mu > 0$ represents the viscosity of the medium (see Remark 2.3). One of the monotonicity intervals of the function $R(x)$ is, in our case, $[0, \sqrt{12\mu}]$. So, for example, for $\mu \geqslant 1/12$, put $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. Then the system of non--linear equations (3.6) becomes

$$R(0) = a, \quad R(1) = ae^{-1/(8\mu)}\left(1 - \frac{1}{4\mu}\right) = ae^{-b/2}(1-b), \quad b = 1/(4\mu) > 0.$$

Thus the methods of moments estimates for parameters a and $b = 1/(4\mu)$ have the form

$$\breve{a}_T = \hat{R}_T(0), \quad \breve{b}_T = u_0(T),$$

where $u_0(T)$ is a unique solution of the non-linear equation

$$\exp\{-u/2\} = s_T/(1-u), \quad 0 \leqslant u < 1, \tag{3.10}$$

with $s_T = \hat{R}_T(1)/\hat{R}_T(0)$.

In practice, one has to use numerical methods (such as the Newton–Raphson method) to solve equation (3.10) in the interval $u \in [0, 1)$. As the first step one can also approximate in the above interval $e^{u/2}$ by $1-u/2$ to obtain

$$u_0(T) \approx u_0^*(T) = \frac{3}{2} - \sqrt{\frac{1}{4} + 2\frac{\hat{R}_T(1)}{\hat{R}_T(0)}}. \tag{3.11}$$

This gives the following statistics as estimators of parameters a and μ:

$$\check{a}_T = \hat{R}_T(0), \quad \check{\mu}_T = \frac{1}{4\mu_0(T)} \quad \text{or} \quad \check{\mu}_T^* = \left[6 - 4\sqrt{\frac{1}{4} + 2\frac{\hat{R}_T(1)}{\hat{R}_T(0)}}\right]^{-1}, \tag{3.12}$$

where $\hat{R}(x)$ is defined by (3.2), and $u_0(T)$ is a unique solution of the non-linear equation (3.10).

The strong consistency of all these estimates (as $T \to \infty$) follows from Theorem 3.1. A construction of asymptotic confidence intervals can be also based on Theorem 3.1 since, as $T \to \infty$,

$$\sqrt{T}(\check{a}_T - \check{\mu}_T)/\sigma_1(\check{a}_T, \check{\mu}_T) \xrightarrow{D} \mathcal{N}(0, 1), \tag{3.13}$$

where

$$\sigma_1^2(a, \mu) = 3a^2\sqrt{\pi\mu}/8 \tag{3.14}$$

is a continuous function of parameters $a > 0$ and $\mu > 0$.

Thus the asymptotic ($T \to \infty$) symmetric ε-confidence interval for the unknown parameter $a > 0$ has the form.

$$(\check{a}_T - u_{1-\varepsilon}\sigma_1(\check{a}_T, \check{\mu}_T)/\sqrt{T},\ \check{a}_T + u_{1-\varepsilon}\sigma_1(\check{a}_T, \check{\mu}_T)/\sqrt{T}),$$

where $\sigma_1^2(a, \mu)$ is given by (3.14) and $u_{1-\varepsilon}$ is a root of the equation

$$\Phi_0(u_{1-\varepsilon}) = 1 - \varepsilon, \qquad \Phi_0(u) = \sqrt{\frac{2}{\pi}} \int_0^u \exp\left\{-\frac{t^2}{2}\right\} dt. \tag{3.15}$$

Construction of an asymptotic confidence interval for the unknown parameter $\mu > 0$ can be based on Theorem 3.1, which, for $T \to \infty$, gives

$$\sqrt{T}(\hat{R}_T(0) - R(0), \hat{R}_T(1) - R(1)) \xrightarrow{D} \mathcal{N}(0, \Sigma_2), \tag{3.16}$$

where

$$\Sigma_2 = 32\sqrt{\frac{\pi}{\mu}}\, a^2 \mu^3 \begin{pmatrix} 2\tau(0) & e^{-1/(16\mu)}[\tau(-1)+\tau(1)] \\ e^{-1/(16\mu)}[\tau(-1)+\tau(1)] & \tau(0) + e^{-1/(4\mu)}\tau(2) \end{pmatrix}, \tag{3.17}$$

where the function $\tau(a)$ depends on μ and is defined by (3.3).

Thus from (3.8) and (3.17), and remembering that $Q(\zeta_1, \zeta_2) = 3/4 - \sqrt{1/4 + 2\zeta_1/\zeta_2}$, we obtain

$$\sqrt{T}(u_0^*(T) - b_0^*)/\sigma_2(\check{a}_T, \check{\mu}_T^*) \xrightarrow{D} \mathcal{N}(0, 1), \tag{3.18}$$

where

$$b_0^* = \frac{3}{4} - \sqrt{\frac{1}{4} + 2\frac{R(1)}{R(0)}}$$

is an approximation to $b_0 \equiv 1/(4\mu)$, and

$$\sigma_2^2(a, \mu) = (\nabla Q)' \Sigma_2 (\nabla Q), \tag{3.19}$$

where ∇Q is taken at the point $(\zeta_1, \zeta_2)' = (R(0), R(1))'$.

If we assume that the function $\sigma_2^2(a, \mu)$ given by (3.19) is continuous at $\check{a}_T$ and $\check{\mu}_T^*$, then we may construct a symmetric asymptotic confidence interval for the unknown parameter μ of the form

$$\left(\frac{1}{4}\left[u_0^*(T) + \frac{u_{1-\varepsilon}}{\sqrt{T}}\sigma_2(\check{a}_T, \check{\mu}_T^*)\right]^{-1}, \frac{1}{4}\left[u_0^*(T) - \frac{u_{1-\varepsilon}}{\sqrt{T}}\sigma_2(\check{a}_T, \check{u}_T^*)\right]^{-1}\right),$$

where $\sigma_2^2(a, \mu)$ is defined by (3.19), $\varepsilon > 0$ is the significance level and $u_{1-\varepsilon}$ is a root of equation (3.15). We need the approximation $\check{\mu}_T^*$ instead of $\check{\mu}_T$ (see (3.13)).

Remark 3.5. Let us consider the special case when the viscosity parameter $\mu > 0$ is known but the parameter $c(\eta, \vartheta) = c(\eta)$, $\eta = (\eta_1, \ldots, \eta_r)' \in \Xi \subset \boldsymbol{R}^r$, is unknown. For the sake of simplicity put $t = 1$. Parameter η contains information on the random initial condition (see (3.1) and Remark 3.2).

We may observe from (3.1) with $t = 1$ that $R(x) = 0$ for $x = 2\sqrt{\mu}$. Let us suppose that $s = [T/2\sqrt{\mu}] - 1 \to \infty$ as $T \to \infty$. Then observations of the Gaussian process $U(1, x)$ at the points $x = 2i\sqrt{\mu}$, $i = 0, 1, \ldots, s-1$, are independent, and statistical inference for parameter η can be based on the exact likelihood function of independent Gaussian observations $Y_i = U(t, 2i\sqrt{\mu})$, $i = 0, 1, \ldots, s-1$, the logarithm thereof has the form

$$\log L(\eta, Y_1, \ldots, Y_{s-1}) = -\frac{1}{2}\sum_{i=0}^{s-1}\left[\frac{Y_i^2}{a(\eta)} + \log a(\eta) + \log(2\pi)\right],$$

where $a(\eta) = c(\eta)\sqrt{\mu/\pi}2^{-3/2}$. In this case the standard maximum likelihood parameter estimation theory for exponential families is applicable. The maximum likelihood estimators are strongly consistent, best asymptotically normal, and asymptotically efficient in the Bahadur sense (see, e.g., Zacks [100], p. 239).

3.3. Testing hypotheses. In this subsection we only establish the asymptotic normality and the large deviation estimate which form the foundation of standard hypothesis testing procedures. We will not dwell here on the latter.

Under assumption F let us consider asymptotic tests for the validity of the hypothesis H_0: $(\eta, \vartheta)' = (\eta_0, \vartheta_0)'$ against its alternative H_1: $(\eta, \vartheta)' \neq (\eta_0, \vartheta_0)'$, where η_0 and ϑ_0 are some fixed points of the sets Ξ and Θ, respectively.

Under hypothesis H_0 we infer from Theorem 3.1, as $T \to \infty$, that

$$\zeta_T \equiv \sqrt{T}\left(\hat{R}_T(x) - R(x;\eta_0,\vartheta_0)\right)/\sqrt{\varrho(x,x;\eta_0,\vartheta_0)} \xrightarrow{D} \mathcal{N}(0,1),$$

where $R(x;\eta,\vartheta_0)$ is defined by (3.1) and $\varrho(x,x;\eta_0,\vartheta_0)$ is defined by the explicit formula (3.3). Thus, for large enough T, we decide in favor of hypothesis H_0 and reject H_1 with significance level $\varepsilon > 0$ whenever, for every $x \in [0, A]$, the quantity $\zeta_T < u_{1-\varepsilon}$, where $u_{1-\varepsilon}$ is the root of equation (3.15).

For construction of the critical regions we will use a modification of Fernique's [23] inequality due to Buldigin [8], and some results from the book of Ivanov and Leonenko [51], where confidence intervals for the correlation function based on Fernique's type inequalities have been developed. In particular, we will apply Theorems 4.6.3 and 4.6.4 of the above-mentioned book for a Gaussian random process with the covariance function (3.1) and the spectral density (3.4).

Using notation of the preceding subsections, we set

$$F_T(\tau) = P\{\sup_{x\in[0,A]} |X_T(x)| > \tau\}, \qquad F(\tau) = P\{\sup_{x\in[0,A]} |Z(x)| > \tau\},$$

where $X_T(x)$ and $Z(x)$ are defined in Theorem 3.1. The statement (iv) of Theorem 4.1 implies that, uniformly in $\tau > 0$,

$$|F_T(\tau) - F(\tau)| \to 0, \qquad T \to \infty.$$

If we estimate the function $F(\tau)$ by the Fernique inequality, then the above relation may serve as a basis for construction of asymptotic functional confidence intervals for the correlation function $R(x)$ and for hypothesis testing.

From Theorem 4.6.3 of Ivanov and Leonenko [51] we infer that if hypothesis H_0 is true, then, for any $\tau > \sqrt{5}K$,

$$\limsup_{T\to\infty} \frac{F_T(\tau)}{G_1(\tau)} \leqslant 1, \tag{3.20}$$

where

$$F(\tau) \leqslant G_1(\tau) = \frac{5K}{2\tau} \exp\left\{-\frac{\tau^2}{2K^2} + 2\right\},$$

and, in the case of the spectral density (3.4),

$$K = K(\eta_0,\vartheta_0) = \frac{2c(\eta_0,\vartheta_0)[\mu(\vartheta_0)]^{3/4}}{\pi^{1/4} e^{1/2} t^{5/4} 2^{3/4}} + \frac{A(2+\sqrt{2})c(\eta_0,\vartheta_0)[\mu_0(\vartheta_0)]^{1/4} 3^{1/2} 2^{1/4}}{8e^{3/2}\pi^{1/4} t^{7/2}}.$$

So, for large enough T, and for all $a \in [u, A]$ and $\tau > \sqrt{5}K$, if the inequality (3.20) is satisfied, then we decide in favor of the hypothesis H_0 and reject the alternative H_1.

Another variant could be to decide in favor of H_0 and against H_1 if, for large enough T, and $\tau > \sqrt{5}K$, the inequality

$$\sqrt{T}\,(\hat{R}_T(x) - R(x, \eta_0, \vartheta_0)) < u_{1-\varepsilon}$$

is satisfied for every $x \in [0, A]$, where $\varepsilon > 0$ is the significance level and $u_{1-\varepsilon}$ is the root of the equation: $1 - G_1(u_{1-\varepsilon}) = 1 - \varepsilon$. In this case, for large enough T,

$$P\{\sup_{a \in [0, A]} \sqrt{T}\,(\hat{R}_T(x) - R(x, \eta_0, \vartheta_0)) < u_{1-\varepsilon}\}$$

will be asymptotically greater than $1 - \varepsilon$.

We may also apply the Rice formula-based Theorem 4.6.4 (Ivanov and Leonenko [51]) to the covariance function (3.1) and the spectral density (3.4). Thus we see under hypothesis H_0 that, for each $\tau > 0$,

$$\limsup_{T \to \infty} \frac{F_T(\tau)}{G_2(\tau)} \leqslant 1, \tag{3.21}$$

where, for the spectral density (3.4),

$$G_2(\tau) = \left(2 + \frac{A}{2\pi}\sqrt{\frac{1}{t\mu(\vartheta_0)}}\right)\exp\{-t^2/2B^2\} + \left(\frac{1}{\sqrt{2}} + \frac{1}{2\pi\sqrt{2}}\sqrt{\frac{1}{t\mu(\vartheta_0)}}\right)\exp\{-t^2/4B^2\},$$

where

$$B^2 = B^2(\eta_0, \theta_0) = \frac{2c^2(\eta_0, \vartheta_0)\,[\mu(\vartheta_0)]^{3/2}}{e\,(2\pi)^{1/2}\,t^{5/2}}.$$

Thus, if T is large enough, and for all $x \in [0, A]$ and $\tau > 0$ the inequality (3.21) is satisfied, then we decide in favor of the hypothesis H_0 and against the alternative hypothesis H_1.

Or, for the significance level $\varepsilon > 0$, we decide in favor of the hypothesis H_0 and against the alternative H_1 if, for large enough T, the inequality

$$\sqrt{T}\,(\hat{R}_T(x) - R(x, \eta_0, \vartheta_0)) < u'_{1-\varepsilon}$$

is satisfied for every $x \in [0, A]$ with $u'_{1-\varepsilon}$ being the root of the non-linear equation: $1 - G_2(u'_{1-\varepsilon}) = 1 - \varepsilon$. In this case, for large enough T,

$$P\{\sup_{x \in [0, A]} \sqrt{T}\,(\hat{R}_T(x) - R(x; \eta_0, \vartheta_0)) < u'_{1-\varepsilon}\}$$

will be asymptotically greater than $1 - \varepsilon$.

Such types of procedures are also useful for construction of goodness-of-fit tests.

4. INFERENCE IN THE FREQUENCY DOMAIN

4.1. The discretization problem. The discretization problem for continuous-time random processes was considered, from different viewpoints, by many authors, see, e.g., Sinai [79], Grenander [32], Grenander and Rosenblatt [33], Rosenblatt [71], Stein [82].

Consider a continuous-parameter random field $\zeta(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, which is stationary in x and has an absolutely continuous spectrum. Following Grenander [32], pp. 249–250, we will consider three main types of discretization in $x \in \boldsymbol{R}^1$, with a fixed $t > 0$:

I. *Locally averaged sampling.* For a fixed step-size $h > 0$, define

$$\zeta_{d1}(t, x) \equiv \frac{1}{h}\int_{(x-1/2)h}^{(x+1/2)h} \zeta(t, y)\,dy, \qquad x \in \boldsymbol{Z}^1 \equiv \{0, \pm 1, \ldots\}. \tag{4.1}$$

II. *Instantaneous sampling.* For a fixed step-size $h > 0$, define

$$\zeta_{d2}(t, x) \equiv \zeta(t, hx), \qquad x \in \boldsymbol{Z}^1. \tag{4.2}$$

III. *Randomized sampling.* For a fixed step-size $h > 0$, $\Delta > 0$, and a sequence of independent, identically $\mathcal{N}(0, \Delta^2)$-distributed random variables v_x, $x \in \boldsymbol{Z}^1$, which are also independent of the random field $\zeta(t, x)$, define

$$\zeta_{d3}(t, x) \equiv \zeta(t, hx + v_x), \qquad x \in \boldsymbol{Z}^1. \tag{4.3}$$

Let now $U(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, be a Gaussian field which is stationary in x and satisfies condition $\mathbb{F}$. Then (see (3.4)) its spectral density has the form

$$g(\lambda) = g(\lambda; \eta, \vartheta) = q\lambda^2 \exp\{-2\mu t\lambda^2\}, \qquad \lambda \in \boldsymbol{R}^1, \tag{4.4}$$

where

$$q = q(w) = \frac{2}{\pi} c(\eta, \vartheta)\,[\mu(\vartheta)]^2, \tag{4.5}$$

with $w = (\eta, \vartheta)' \in W \subset \boldsymbol{R}^l$, $l = r + k \geqslant 1$, and where W is a compact set with non-empty interior containing the true value w_0 of the parameter w (see condition $\mathbb{F}$).

In calculation of the spectral densities of the three discretizations introduced above we will also have need for the *elliptic Jacobi theta-function* $\theta(x, s)$. Recall that

$$\theta(x, s) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \exp\{inx - n^2 s\}, \qquad s > 0,\ x \in \boldsymbol{R}^1 \tag{4.6}$$

(see, for example, Widder [96] and Mumford [65]).

THEOREM 4.1. *Assume that the Gaussian field $U(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, is stationary in x and satisfies condition* $\mathbb{F}$. *Then:*

(i) *Spectral density of the stationary in x Gaussian field $U_{d1}(t, x)$, $t > 0$, $x \in \mathbf{Z}^1$, has the form*

$$
\begin{aligned}
(4.7)\quad f_{d1}^{(1)}(\lambda) &= f_{d1}^{(1)}(\lambda, w) \\
&= \frac{4q}{h^3}\sin^2\frac{\lambda}{2}\sum_{k=-\infty}^{\infty} \exp\left\{-\frac{2\mu t}{h^2}(\lambda+2k\pi)^2\right\} \\
&= \frac{4q}{h^2}\sin^2\frac{\lambda}{2}\sqrt{\frac{\pi}{2\mu t}}\,\theta(\lambda, h^2/(8\mu t)) \\
&= \frac{4q}{h^2}\sin^2\frac{\lambda}{2}\sqrt{\frac{\pi}{2\mu t}}\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k\lambda)\exp\{-k^2h^2/(8\mu t)\}\right] \\
&= \frac{4q}{h^2}\sin^2\frac{\lambda}{2}\sqrt{\frac{\pi}{2\mu t}}\left[\frac{1}{2\pi}\prod_{k=1}^{\infty}(1-\exp\{-k^2h^2/(4\mu t)\})\right] \\
&\quad \times \prod_{k=1}^{\infty}(1-2\exp\{-h^2(2k-1)/(8\mu t)\}\cos\lambda \\
&\quad +\exp\{h^2(4k-2)/(8\mu t)\}),
\end{aligned}
$$

$-\pi < \lambda \leqslant \pi$, *and, as* $\lambda \to 0$,

$$
(4.8)\quad f_{d1}^{(1)}(\lambda) = \lambda^2\left[\frac{q}{h^2}\sqrt{\frac{\pi}{2\mu t}}\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\exp\{-k^2h^2/(8\mu t)\}\right)\right](1+o(1)).
$$

(ii) *Spectral density of the stationary in x Gaussian field $U_{d2}(t, x)$, $t > 0$, $x \in \mathbf{Z}^1$, has the form*

$$
\begin{aligned}
(4.9)\quad f_{d2}^{(1)}(\lambda) &= f_{d2}^{(1)}(\lambda, w) \\
&= \frac{q}{h^3}\sum_{k=-\infty}^{\infty}(\lambda+2k\pi)^2\exp\left\{-\frac{2\mu t}{h^2}(\lambda+2k\pi)^2\right\} \\
&= \frac{qh^2}{32\mu^2t^2}\sqrt{\frac{\pi}{2\mu t}}\left[\frac{8\mu t}{h^2}\theta(\lambda, h^2/(8\mu t))+2\frac{\partial^2}{\partial\lambda^2}\theta(\lambda, h^2/(8\mu t))\right] \\
&= \frac{qh^2}{32\mu^2t^2}\sqrt{\frac{\pi}{2\mu t}}\left[\frac{8\mu t}{h^2}\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\exp\{-k^2h^2/(8\mu t)\}\cos(k\lambda)\right)\right. \\
&\quad \left. -\frac{2}{\pi}\sum_{k=1}^{\infty}k^2\exp\{-k^2h^2/(8\mu t)\}\cos(k\lambda)\right],
\end{aligned}
$$

$-\pi < \lambda \leqslant \pi$, *and*

$$
(4.10)\qquad \lim_{\lambda\to 0} f_{d2}^{(1)}(\lambda) > 0.
$$

(iii) *Spectral density of the stationary in x Gaussian field $U_{d3}(t, x)$, $t > 0$, $x \in \mathbf{Z}^1$, has the form*

$$
\begin{aligned}
(4.11) \quad f_{d3}^{(1)}(\lambda) &= f_{d3}^{(1)}(\lambda, w) \\
&= \frac{\kappa}{2\pi} + \frac{q}{h^3} \sum_{k=-\infty}^{\infty} (\lambda + 2k\pi)^2 \exp\{-(\lambda+2k\pi)^2(\Delta^2 + (2\mu t)/h^2)\} \\
&= \frac{\kappa}{2\pi} + \frac{qh^2\pi^{1/2}}{8[h^2\Delta^2 + 2\mu t]^{5/2}} \left[\frac{4(h^2\Delta^2 + 2\mu t)}{h^2} \theta\left(\lambda, \frac{h^2}{4(h^2\Delta^2 + 2\mu t)}\right) \right. \\
&\quad \left. + 2\frac{\partial^2}{\partial \lambda^2} \theta(\lambda, h^2/4(h^2\Delta^2 + 2\mu t)) \right],
\end{aligned}
$$

$-\pi < \lambda \leqslant \pi$, *where*

$$\kappa = q\sqrt{2\pi}\left[\left(\frac{1}{4\mu t}\right)^{3/2} - \left(\frac{1}{4\mu t + 2\Delta^2 h^2}\right)^{3/2}\right],$$

and

$$(4.12) \qquad \lim_{\lambda \to 0} f_{d3}^{(1)}(\lambda) > 0.$$

Remark 4.1. The parameter dependence structure for fields with spectral densities (4.7), (4.9) and (4.11) is given by (4.5). Discretization procedures (4.2) and (4.3) produce more regular densities $f_{d2}^{(1)}$ and $f_{d3}^{(1)}$ than the discretization procedure (4.1). However, the advantage of the latter is that it preserves the singularity type of the original spectral density of the continuous-time process (see (2.6) and (3.4)).

Let us now consider the discretization procedures I–III for rescaled solutions of the Burgers equation with strongly dependent initial conditions (see Theorem 2.2). In view of (2.16) and (2.19) we will introduce the following condition:

G. *The random field $Y(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, is Gaussian, stationary in x, has an absolutely continuous spectrum, and spectral density of the form*

$$(4.13) \qquad f(\lambda) = f(\lambda, w) = p|\lambda|^{1+\alpha} \exp\{-2\mu t \lambda^2\}, \qquad 0 < \alpha < 1, \ \lambda \in \mathbf{R}^1,$$

with

$$(4.14) \qquad p = c_1(\alpha)\sigma^2, \qquad \mu = \mu(\vartheta),$$

where the constant $c_1(\alpha)$ is given by (2.17), $\mu = \mu(\vartheta)$, $\vartheta \in \Theta \subset \mathbf{R}^k$, $k \geqslant 1$, and $w = (\alpha, \beta, \sigma^2, \vartheta)' \in W \subset \mathbf{R}^{k+3}$, where $\alpha \in (0, 1)$, $\beta = c_1(\alpha) \in (0, \infty)$, $\sigma > 0$. The set W is assumed to be compact with the true value w_0 of the parameter in its interior.

Remark 4.2. Condition G corresponds to the case when $F(u) = u$ in Theorem 2.2, so that we are dealing with Gaussian initial conditions with strong dependence. The structure of the spectral density is chosen according to (2.16) and (2.19). In principle, it is possible to consider a more general situation when the parameter p in (4.13) has a more complex structure than (4.14). For example, according to (2.16), we can put

$$p = 2c_1(\alpha)\mu(\vartheta)d, \qquad d = C_1/C_0,$$

where C_0 and C_1 are defined in Theorem 2.2.

The next result describes the shape of the spectral density for the field Y for the above three methods of discrete sampling (I–III). The random field Y appears as a rescaled solution of the Burgers equation with strongly dependent initial conditions (see Theorem 2.2).

THEOREM 4.2. *Suppose that $Y(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, is a Gaussian random field, stationary in x, and satisfying condition* G. *Then*:

(i) *The spectral density of the discretization $Y_{d1}(t, x)$, $t > 0$, $x \in \boldsymbol{Z}^1$, is of the form*

$$\begin{aligned}(4.15)\quad f_{d1}^{(2)}(\lambda) &= f_{d1}^{(2)}(\lambda, w)\\ &= \frac{p}{h^{2+\alpha}} 4\sin^2(\lambda/2) \sum_{k=-\infty}^{\infty} |\lambda + 2k\pi|^{\alpha-1} \exp\{-2\mu t(\lambda+2k\pi)^2/h^2\}\\ &= \frac{p}{h^{2+\alpha}} 4\sin^2(\lambda/2) \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\lambda} a_m,\end{aligned}$$

$-\pi < \lambda \leqslant \pi$, *where*

$$a_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ims} |s|^{\alpha-1} \exp\{(-2\mu t s^2)/h^2\}\, ds$$

and, as $\lambda \to 0+$,

$$(4.16)\qquad f_{d1}^{(2)}(\lambda) = \lambda^{1+\alpha} \frac{p}{h^{2+\alpha}} (1 + o(1)).$$

(ii) *The spectral density of the discretization $Y_{d2}(t, x)$, $t > 0$, $x \in \boldsymbol{Z}^1$, is of the form*

$$\begin{aligned}(4.17)\quad f_{d2}^{(2)}(\lambda) &= f_{d2}^{(2)}(\lambda, w)\\ &= \frac{p}{h^{2+\alpha}} \sum_{k=-\infty}^{\infty} |\lambda + 2k\pi|^{1+\alpha} \exp\{-2\mu t(\lambda+2k\pi)^2/h^2\}\\ &= \frac{p}{h^{2+\alpha}} \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\lambda} b_m,\end{aligned}$$

$-\pi < \lambda < \pi$, *where*

$$b_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ims} |s|^{1+\alpha} \exp\{(-2\mu t s^2)/h^2\}\, ds$$

and, as $\lambda \to 0+$,

$$\lim_{\lambda \to 0} f_{d2}^{(2)}(\lambda) > 0. \tag{4.18}$$

(iii) *The spectral density of the discretization* $Y_{d3}(t, x)$, $t > 0$, $x \in \mathbf{Z}^1$, *is of the form*

$$\begin{aligned} f_{d3}^{(2)}(\lambda) &= f_{d3}^{(2)}(\lambda, w) \\ &= \frac{p\Gamma((\alpha+2)/2)}{2\pi}\left[\frac{1}{(2\mu t)^{1+\alpha/2}} - \frac{1}{(2\mu t + \Delta^2 h^2)^{1+\alpha/2}}\right] \\ &\quad + \frac{p}{h^{2+\alpha}} \sum_{k=-\infty}^{\infty} |\lambda + 2k\pi|^{1+\alpha} \exp\{-(\lambda+2k\pi)^2(\Delta^2 + (2\mu t)/h^2)\}, \end{aligned} \tag{4.19}$$

$-\pi < \lambda \leqslant \pi$, *and*

$$\lim_{\lambda \to 0} f_{d3}^{(2)}(\lambda) > 0. \tag{4.20}$$

Remark 4.3. The parameter dependence structure of the spectral densities (4.15), (4.17) and (4.19) is given in condition G. Again, the discretization procedure (4.1) gives the spectral density $f_{d1}^{(2)}$ with the same singularity as the spectral density (4.13) of the original continuous-time process, whereas the discretization procedures (4.2) and (4.3) produce spectral densities without singularities (see (4.18) and (4.20)).

Remark 4.4. The function appearing in expressions (4.15), (4.17) and (4.19) contains functions which can be called the *fractional Jacobi theta-functions*; in the general theory of theta-functions the function $\theta(x, s)$ introduced before is usually called the *theta-function of the third kind.*

Remark 4.5. Now consider the case when $t > 0$ is fixed and the discretization step $h > 0$ is also fixed, but the observations of $U(t, x)$, $x \in [0, T]$, are made with $T \to \infty$. This is the so-called increasing domain asymptotics problem (see, for example, Stein [82]). The fixed domain asymptotics problems (see, again, Stein [82]), with T being constant, but $h \to 0$, or T being constant, but statistical inference being done for the spectral density of the fields $h^3 U(h^2 t, hx)$, or $h^{1+\alpha/2} Y(h^2 t, hx)$, as $h \to 0$, where $U(t, x)$ is the parabolically rescaled solution of the Burgers equation with weakly dependent random data, and $Y(t, x)$ is the parabolically rescaled solution of the Burgers equation with strongly dependent data, also remain open.

4.2. Minimum contrast estimators. The *quasilikelihood*, or *minimum contrast*, method of spectral density parameter fitting for discrete-time stationary process was first proposed by Whittle [94], [95], and was later developed by Walker [91], Ibragimov [48], Hannan [37], Rice [66], Guyon [35], [36], Dzhaparidze and Yaglom [21], Dzhaparidze and Kotz [20], Fox and Taqqu [24], Dahlhaus [16], Heyde and Gay [38], [39], Giraitis and Surgailis [31], among others. Also, a few years ago, Tanigushi [87] proposed a wide class of asymptotically efficient estimators which are essentially different from the maximum likelihood estimators. For recent developments, see Dahlhaus and Wefelmeyer [17]. A discretization procedure-based estimation for continuous--time parametric models was considered by Comte [14], Chambers [12], and others.

In this subsection we present general results on consistency and asymptotic normality of the minimum contrast estimators under the Gaussian Whittle contrast. By and large, we follow the approach suggested for random fields by Guyon [36], but our conditions for consistency and asymptotic normality are weaker than those of Guyon and, in particular, are fulfilled in the case of an unbounded spectral density. Our conditions for consistency are also weaker than those of Fox and Taqqu [24], although they did prove the strong consistency of parametric estimators for spectral densities which are unbounded at the origin. In the proof of asymptotic normality we use the results of Heyde and Gay [38], [39].

We begin by introducing the following general assumption:

H. *Process* $\zeta(x) = \zeta(\omega, x)$, $\omega \in \Omega$, $x \in \boldsymbol{Z}^1$, *is a real stationary, centered random process with covariance function* $R(x)$, $x \in \boldsymbol{Z}^1$, *and spectral density* $f_d(\lambda) = f_d(\lambda, w)$, $w \in W$, $\lambda \in (-\pi, \pi]$, *where* W *is a compact set and the true value of parameter* w_0 *is in the interior of* $W \subset \boldsymbol{R}^r$.

A statistical model with spectral density $f_d(\lambda, w)$ is called *identifiable* if the following condition is satisfied:

I. *If* $w \neq w'$, *then* $f_d(\lambda, w)$ *differs from* $f_d(\lambda, w')$ *on a set of positive Lebesgue measure.*

Now, consider a parametric statistical model of distributions P_w, $w \in W$, and put $P_0 = P_{w_0}$. Let $\zeta(x)$, $x \in \{1, 2, \ldots, T\}$, be an observation from a random process satisfying condition H, and let

$$R_T(x) = R_T(x, \zeta) = T^{-1} \sum_{y=1}^{T-|x|} \zeta(y)\zeta(y+x) \tag{4.21}$$

be the sample covariance function, and

$$I_T(\lambda) = I_T(\lambda, \xi) = \frac{1}{2\pi} \sum_{|x| \leqslant T-1} R_T(x) e^{-i\lambda x} = \frac{1}{2\pi T} \left| \sum_{x=1}^{T} \zeta(x) e^{-i\lambda x} \right|^2 \tag{4.22}$$

be the periodogram.

A *contrast function* for w_0 is a deterministic mapping $K(w_0, \cdot)\colon W \to [0, \infty)$, which has a unique minimum at $w = w_0$. Given $K(w_0, \cdot)$ let us consider *contrast* *processes* $S_T(w)$, $T \in Z^1$, adapted to $\zeta(x)$, $x \in \{1, \ldots, T\}$, defined for all $w \in W$, and such that

$$\liminf_{T\to\infty} [S_T(w) - S_T(w_0)] \geqslant K(w_0, w)$$

in probability P_0. The *minimum contrast estimator* w_T minimizes the values of S_T, i.e.,

$$w_T = \arg\min_{w\in W} S_T(w).$$

We shall consider a Gaussian Whittle contrast, defined by the contrast process

$$S_T(w) = S_T\big(w, I_T(\lambda), f_d(\lambda, w)\big) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\log f_d(\lambda, w) + \frac{I_T(\lambda)}{f_d(\lambda, w)}\right] d\lambda, \qquad w \in W, \tag{4.23}$$

the contrast function

$$K(w_0, w) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\frac{f_d(\lambda, w_0)}{f_d(\lambda, w)} - 1 - \log\frac{f_d(\lambda, w_0)}{f_d(\lambda, w)}\right] d\lambda,$$

and the associated minimum contrast estimator

$$w_T = w_T\big(I_T(\lambda), f_d(\lambda, w)\big) = \arg\min_{w\in W} S_T(w), \tag{4.24}$$

where $S_T(w)$ is defined by (4.23). The functional $S_T(w)$ will also be called *Whittle's functional.* We will use the same notation w_T for the general contrast process and for the Gaussian Whittle contrast process.

THEOREM 4.3. *Assume that the conditions* H *and* I *are satisfied, the function* $f_d^{-1}(\lambda, w)$ *is continuous on* $(-\pi, \pi] \times W$, *and the sample covariance* $R_T(x) \to R(x)$ *in* P_0*-probability, as* $T \to \infty$. *Then, in* P_0*-probability,*

$$S_T(w) \to S(w) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\log f_d(\lambda, w) + \frac{f_d(\lambda, w_0)}{f_d(\lambda, w)}\right] d\lambda, \tag{4.25}$$

and the minimum contrast estimator $w_T \to w_0$ *as* $T \to \infty$.

Assumptions in the above theorem are weaker than those in the comparable result of Guyon [36], p. 145. In particular, he assumes the condition $0 < m \leqslant f_d(\lambda) \leqslant M < \infty$, and the existence and continuity of the second-order derivatives. Fox and Taqqu [24] and Dahlhaus [16] also employ conditions on derivatives of the spectral density but they prove strong consistency of the minimum contrast estimates for long-memory random processes. The assumption of continuity of f_d^{-1} in Theorem 4.3 is satisfied for spectral densities which are unbounded at zero (see, Fox and Taqqu [24], Beran [4]), and so is the sample covariance convergence condition; for Gaussian processes, the latter is a consequence of the ergodic theorem.

In the remainder of this subsection we present results on asymptotic normality of the minimum contrast estimators. The following condition will be utilized:

K. *The process*

$$\zeta(x) = \sum_{y \geqslant 0} a_y \varepsilon_{x-y}, \qquad a_0 = 1, \ \sum_{y \geqslant 0} a_s^2 < \infty,$$

where ε_y, $y \in \boldsymbol{Z}^1$, *are i.i.d. random variables (Gaussian, if the process* ζ *is Gaussian) with zero mean, and variance given by Kolmogorov's formula* $\operatorname{var} \varepsilon_y = \exp\{(1/2\pi)\int_{-\pi}^{\pi} \log f_d(\lambda)\, d\lambda\}$.

Condition **K** is rather weak and is satisfied if

$$\int_{-\pi}^{\pi} \log f_d(\lambda)\, d\lambda > -\infty; \tag{4.26}$$

the latter condition, in turn, plays a fundamental role in the linear prediction theory of stationary processes. Since $\log f_d(\lambda) \leqslant f_d(\lambda)$ and $\int_{-\pi}^{\pi} f_d(\lambda)\, d\lambda = \operatorname{var} \zeta(0) < \infty$, it follows that $\int_{-\pi}^{\pi} \log f_d(\lambda)\, d\lambda$ cannot diverge to $+\infty$, and therefore must be either finite or diverge to $-\infty$. This integral diverges to $-\infty$ if, e.g., $f_g(\lambda)$ vanishes on an interval of positive Lebesgue measure.

We will also need some regularity assumptions on the spectral density.

L. *The spectral density* $f_d(\lambda, w)$, $\lambda \in (-\pi, \pi]$, $w \in W$, *and the vector-valued function*

$$A(\lambda, w) = -\nabla_w f_d^{-1}(\lambda, w), \qquad \lambda \in (-\pi, \pi], \ w \in W,$$

satisfy the following conditions **L1–L7**.

L1. *The parametric integral* $\int_{-\pi}^{\pi} \log f_d(\lambda, w)\, d\lambda$ *is twice differentiable with respect to parameter* w.

L2. *The function* $f_d^{-1}(\lambda, w)$, $\lambda \in (-\pi, \pi]$, $w \in W$, *is twice differentiable with respect to* w *and the derivatives* $\partial f_d^{-1}(\lambda, w)/\partial w_j$ *and* $\partial^2 f_d^{-1}(\lambda, w)/\partial w_j \partial w_k$ *are continuous for all* $w \in W$.

L3. *The vector function* $A(\lambda, w)$ *is symmetric about* $\lambda = 0$, *for* $\lambda \in (-\pi, \pi]$, $w \in W_0 \supset W$, *where* W_0 *is an open set.*

L4. *For all* $w \in W_0$, *the function* $A(\lambda, w) \in L_1((-\pi, \pi])$.

L5. *For all* $w \in W_0$, $f_d(\lambda, w) A(\lambda, w) \in L_1((-\pi, \pi]) \cap L_2((-\pi, \pi])$.

L6. *There exists a* κ, $|\kappa| < 1$, *such that* $|\lambda|^{\kappa} f_d(\lambda, w)$ *is bounded, and* $|\lambda|^{-\kappa} A(\lambda, w) \in L_2((-\pi, \pi])$ *for all* $w \in W_0$.

L7. *The* $r \times r$*-matrix function* $\nabla_w f_d(\lambda, w) A'(\lambda, w) \in L_1((-\pi, \pi])$ *for all* $w \in W$, *where* $W \subset W_0$.

THEOREM 4.4. *Let* $\zeta(x)$, $x \in \mathbf{Z}^1$, *be a Gaussian process satisfying assumptions of Theorem* 4.3 *and conditions* $\mathbb{K}$ *and* $\mathbb{L}$. *If the matrix*

$$\Sigma = \Sigma(f_d(\lambda, w_0)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \nabla_w \log f_d(\lambda, w_0) [\nabla_w \log f_d(\lambda, w_0)]' \, d\lambda \tag{4.27}$$

is nonsingular, then the minimum contrast estimators (4.24) *are asymptotically normal, i.e., as* $T \to \infty$,

$$\sqrt{T}(w_T - w_0) \xrightarrow{D} \mathcal{N}(0, 2\Sigma^{-1}).$$

Remark 4.6. Under some additional conditions on the spectral density, the minimum contrast estmators are asymptotically efficient in the Fisher sense. Those conditions can be found, e.g., in Dzhaparidze and Kotz [20], Dahlhaus [16] and Guyon [36]. In principle, it is possible to prove Theorem 4.4 for the general (not necessarily Gaussian) linear processes using the central limit theorem for quadratic forms of random fields (see Heyde and Gay [39]). In such a case the limiting covariance matrix would contain some supplementary terms, see Giraitis and Surgailis [31]. Again, the above theorem has weaker assumptions than the comparable results of Fox and Taqqu [24], Dahlhaus [16] and Guyon [36], p. 145. This was achieved by having weaker assumptions in the consistency result (Theorem 4.3) and by applying a stronger central limit theorem due to Heyde and Gay [38], [39].

4.3. Estimation of parameters of the rescaled solutions of the Burgers equation with weakly dependent initial conditions. In this subsection we will apply results of Section 4.2 to obtain estimates of parameters of random processes arising in Burgers' turbulence. In our approach we will rely on the approach proposed by Dzhaparidze and Kotz [20] and consider the problem of estimation of parameters of the spectral density (4.5) based on the observations of $U(t, x)$, $x \in [0, T]$, with a fixed $t > 0$, and $T \to \infty$. The locally averaged sampling (4.1) reduces the problem to the parameter estimation problem for the spectral density $d_{d1}^{(1)}(\lambda, w)$, $-\pi < \lambda < \pi$, $w \in W$, given by (4.7), based on the sample $U_{d1}(t, x)$, $x \in \{1, 2, \ldots, T\}$, $T \to \infty$ (see Remark 4.1). We consider a parametric model of distributions P_w and put $P_0 = P_{w_0}$ (see condition $\mathbb{F}$).

Note that the spectral density $f_{d1}^{(1)}$ given by (4.7) admits factorization of the form

$$f_{d1}^{(1)}(\lambda, w) = |1 - e^{-i\lambda}|^2 f_0(\lambda, w), \qquad -\pi < \lambda \leqslant \pi, \; w \in W, \tag{4.28}$$

where

$$f_0(\lambda, w) = \frac{q}{h^2} \sqrt{\frac{\pi}{2\mu t}} \, \theta(\lambda, h^2/(8\mu t)), \tag{4.29}$$

is positive, integrable, bounded and bounded away from zero on $\lambda \in (-\pi, \pi]$ (see (4.7)).

To estimate the time series parameters in the case of a spectral density with fixed zeros (see (4.28)) Dzhaparidze and Kotz [20] proposed to begin with the following transformation of the data:

$$\tilde{V}(z) = V(z) - \frac{1}{T}\sum_{z=1}^{T} V(z), \quad z = 1, 2, \ldots, T, \tag{4.30}$$

where

$$V(z) = U_{d1}(t, 1) + \ldots + U_{d1}(t, z),$$

and then to consider the Whittle functional

$$S_T^{(1)}(w) \doteq S_T\big(w, I_T(\lambda, \tilde{V}), f_0(\lambda, w)\big), \quad w \in W, \tag{4.31}$$

defined by (4.25), and based on the periodogram $I_T(\lambda, \tilde{V})$ of the process $\tilde{V}(z)$, $z = 1, 2, \ldots, T$ (see (4.30)), with the spectral density $f_0(\lambda, w)$ from (4.29).

Let

$$\hat{w}_{T1} = \arg\min_{w \in W} S_T^{(1)}(w)$$

be the minimum contrast estimator of the unknown parameter w obtained from minimization of the function (4.31). In view of Dzhaparidze and Kotz's result ([20], pp. 104–110) and Theorem 4.3, we obtain the following result.

THEOREM 4.5. (i) *Suppose that condition* $\mathbb{F}$ *is satisfied and functions* $c = c(\eta, \vartheta)$ *and* $\mu(\vartheta)$ *are such that the spectral density* $f_0(\lambda, w)$, $-\pi < \lambda \leqslant \pi$, $w \in W$, *satisfies the identifiability condition* $\mathbb{I}$. *Then* $\hat{w}_{T1} \xrightarrow{P} w_0$ *as* $T \to \infty$.

(ii) *Suppose, additionally, that functions* $c = c(\eta, \vartheta)$ *and* $\mu(\vartheta)$ *are such that the function* $f_0(\lambda, w)$, $-\pi < \lambda \leqslant \pi$, $w \in W$, *defined by* (4.29) *is twice differentiable in* $w \in W$, *the second derivatives are continuous on* $(-\pi, \pi]$ *and the matrix* $\Sigma_1 = \Sigma\big(f_0(\lambda, w_0)\big)$ *defined by* (4.27) *is non-singular. Then the estimate* $\hat{w}_{T1}$ *is asymptotically normal and efficient in the Fisher sense, as* $T \to \infty$, *that is*

$$\sqrt{T}(\hat{w}_{T1} - w_0) \xrightarrow{D} \mathcal{N}_l(0, 2\Sigma_1^{-1}).$$

Remark 4.7. If functions $c(\eta, \vartheta)$ and $\mu(\vartheta)$ are twice continuously differentiable with respect to both arguments, and all derivatives of the first and second order are bounded and bounded away from zero when parameters belong to the corresponding compact sets (see condition $\mathbb{F}$), then all the assumptions of Theorem 4.5 (ii) are satisfied. Moreover, for the model with two parameters: $w = (a, \mu)'$ and $a = c\sqrt{\mu/\pi}2^{-3/2}$ characterizing $R(0)$ (we put $t = 1$) and $\mu(\vartheta) = \mu > 0$, condition $\mathbb{I}$ is also satisfied. This fact follows from the uniqueness of the Fourier series (4.9) for functions that are integrable in the square.

Another simple parametrization $w = (q, \mu)'$, where $q > 0$ is a parameter to be estimated (see (4.6)), is also identifiable. Other assumptions of Theorem 4.5 are also satisfied for these two parametrizations.

Dzhaparidze and Kotz's [20] arguments can be simplified if we consider periodograms in the Fourier frequencies

$$\lambda_j = 2\pi j/T, \quad j = 1, 2, \ldots, \tag{4.32}$$

and define

$$\begin{aligned} S_T^*(w) &= S_T^*(w, I_T(\lambda_j, Z), f_0(\lambda, w)) \\ &= \frac{1}{2T}\sum_{j=1}^{T}\left\{\log f_0(\lambda_j, w) + \frac{I_T(\lambda_j, Z)}{f_0(\lambda_j, Z)}\right\}, \end{aligned} \tag{4.32a}$$

where $Z(x)$, $x \in \boldsymbol{Z}^1$, is a stationary Gaussian process with zero mean and the spectral density $f_0(\lambda)$, $-\pi < \lambda \leqslant \pi$, defined by (4.29). Then, by the spectral decomposition theorem,

$$Z(x) = \int_{-\pi}^{\pi} e^{i\lambda x}\sqrt{f_0(\lambda)}\, G(d\lambda), \quad x \in \boldsymbol{Z}^1, \tag{4.33}$$

where $G(d\lambda)$ is a complex Gaussian white noise on $\big((-\pi, \pi], \mathscr{B}((-\pi, \pi])\big)$. From (4.33), we obtain

$$Z(x) - Z(x-1) = \int_{-\pi}^{\pi} e^{i\lambda x}(1 - e^{-i\lambda})\sqrt{f_0(\lambda)}\, G(d\lambda). \tag{4.33a}$$

Then, from (4.21) we infer that the random field

$$U_{d1}^*(t, x) = Z(x) - Z(x-1), \quad x \in \boldsymbol{Z}^1,$$

with fixed $t > 0$, has the covariance function

$$EU_{d1}^*(t, x)\, U_{d1}^*(t, y) = \int_{-\pi}^{\pi} e^{i\lambda(x-y)}|1 - e^{-i\lambda}|^2 f_0(\lambda)\, d\lambda. \tag{4.34}$$

Hence $U_{d1}^*(t, x)$ is $L_2(\Omega)$-equivalent to $U_{d1}(t, x)$, $x \in \boldsymbol{Z}^1$. We shall use the same notation for both processes.

For (4.34) we obtain

$$Z(x) = U_{d1}(t, 1) + \ldots + U_{d1}(t, x) + Z(0), \quad x \in \{1, \ldots, T\}, \tag{4.35}$$

where the unobservable random variable $Z(0)$ is independent of x. It then follows that the finite Fourier transforms at the Fourier frequencies λ_j, defined by (4.32), has the property

$$J_T(\lambda_j, Z) = \frac{1}{\sqrt{2\pi T}} \sum_{x=1}^{T} Z^*(x) \exp\{-ix\lambda_j\} + \frac{1}{\sqrt{2\pi T}} Z(0) \sum_{x=1}^{T} \exp\{-ix\lambda_j\} = J_T(\lambda_j, Z^*), \tag{4.36}$$

because

$$\sum_{x=1}^{T} \exp\{-ix\lambda_j\} = e^{i\lambda_j} \frac{1-\exp\{iT\lambda_j\}}{1-\exp\{i\lambda_j\}} = 0, \quad j = 1, 2, \ldots,$$

where the transformation $Z^*(x)$ of the observed discrete data $U_{d1}(t, x)$, $x \in \{1, 2, \ldots, T\}$, is defined as follows:

$$Z^*(x) = U_{d1}(t, 1) + \ldots + U_{d1}(t, x), \quad x \in \{1, 2, \ldots, T\}. \tag{4.37}$$

It follows from (4.36) that the periodograms (see (4.29)) of the data (4.35) with one unobservable variable coincide with the periodogram of the data (4.37) at the non-zero Fourier frequences:

$$I_T(\lambda_j, Z) = I_T(\lambda_j, Z^*).$$

The above fact reduces the problem of estimation of parameters of the spectral density $f_d^{(1)}(\lambda, w)$, given by (4.7) or (4.28), to statistical inference for the process $Z(x)$, $x \in \boldsymbol{Z}^1$, defined by (4.33), which has the spectral density $f_0(\lambda, w)$ given by (4.29). Statistical inference at the non-zero Fourier frequencies can be done in terms of the periodogram $I_T(\lambda_j, Z^*)$ of the transformed data $Z^*(x)$ given by (4.37).

For

$$\tilde{w}_{T1} = \arg\min_{w \in W} S_T^*(w, I_T(\lambda_j, Z^*), f_0(\lambda_j, w))$$

we have the following result:

THEOREM 4.6. *Under the assumptions of Theorem* 4.3 *the estimate* $\tilde{w}_{T1}$ *is consistent and asymptotically normal* $\mathcal{N}_l(0, 2\Sigma_1^{-1})$, *and asymptotically efficient in the Fisher sense, as* $T \to \infty$.

Remark 4.8. The limiting covariance matrix Σ_1 in Theorems 4.3 and 4.4 can be consistently estimated by

$$\hat{\Sigma}_1(w_0) = \frac{1}{2T} \sum_{s=1}^{T} \nabla_w \log f_0(\lambda_s, w) [\nabla_w \log f_0(\lambda_s, w)]', \tag{4.38}$$

which is computationally simpler than (4.27).

Usually, it is not possible to solve the non-linear estimating equations $\nabla_w S_T^*(w) = 0$ exactly (they are usually non-linear). Instead, one could determine the estimates by the Newton–Raphson iteration. The resulting estimates have the same asymptotic properties as $\hat{w}_{T1}$ and $\tilde{w}_{T1}$.

Remark 4.9. Using the standard arguments (see Hannan [37], Rice [66]) one can prove that Theorems 4.3 and 4.4 remain true for estimates $w_T^* = \arg\min_{w\in W} S_T^*(w)$, where $S_T^*(w)$ is defined by (4.32a) with $f_0(\lambda, w)$ replaced by $f_d(\lambda, w)$ (which fulfills assumptions of the above-mentioned theorems).

Now let us turn to the parameter estimation problem for the spectral density (4.5), based on the observations $U = U(t, x)$, $x \in [0, T]$, with fixed $t > 0$ and using discretizations (4.2) and (4.3). These discretizations reduce the problem to parameter estimation of spectral densities $f_{di}^{(1)}(\lambda, w)$, $-\pi < \lambda \leqslant \pi$, $w \in W$, $i = 2, 3$, given by (4.9) and (4.11), respectively.

Let $U_{di} = \big(U_{di}(t, 1), \ldots, U_{di}(t, T)\big)'$, $i = 2, 3$, be data discretized through sampling procedures (4.2) and (4.3), respectively. Consider the corresponding Whittle functionals

$$S_T^{(i)}(w) = S_T^{(i)}\big(w, I_T(\lambda, U_{di}), f_{di}^{(1)}(\lambda, w)\big), \qquad i = 2, 3,$$

and

$$\hat{w}_{Ti} = \arg\min_{w\in W} S_T^{(i)}(w), \qquad i = 2, 3.$$

THEOREM 4.7. (i) *Let $i = 2$ or 3, and assume that conditions* F *and* I (*with f_d replaced by $f_{di}^{(1)}$*) *are satisfied. Then, as $T \to \infty$, $\hat{w}_{Ti} \xrightarrow{P_0} w_0$.*

(ii) *Suppose, additionally, that functions $c(\eta, \vartheta)$ and $\mu(\vartheta)$ are such that the functions $f_{di}^{(1)}(\lambda, w)$, $-\pi < \lambda \leqslant \pi$, $w \in W$, defined by* (4.9) *and* (4.11), *respectively, are twice differentiable in $w \in W$, their second derivatives are continuous in λ on $(-\pi, \pi]$, and the matrix $\Sigma_i = \Sigma_2\big(f_{di}^{(1)}(\lambda, w_0)\big)$ defined by* (4.27) *is non-singular. Then the estimator $\hat{w}_{Ti}$ is asymptotically normal and efficient in the Fisher sense, as $T \to \infty$, that is*

$$\sqrt{T}\,(\hat{w}_{Ti} - w_0) \xrightarrow{D} \mathcal{N}_l(0, 2\Sigma_i^{-1}).$$

Remark 4.10. If functions $c(\eta, \vartheta)$ and $\mu(\vartheta)$ are twice continuously differentiable and the derivatives are bounded and bounded away from zero, then all the assumptions of Theorem 4.7 are satisfied. For a model with only two parameters $w = (q, \mu)' \in W \subset (0, \infty)^2$, the assumptions in Theorem 4.7 (i)–(ii) are always satisfied. The condition I is also satisfied (see Remark 4.7).

4.4. Estimation of parameters of the rescaled solutions of the Burgers equation with strongly dependent initial conditions. We will now consider the problem of parameter estimation for the spectral density $f(\lambda, w)$, $\lambda \in \boldsymbol{R}^1$, $w \in W$, defined by (4.13), based on the observation $Y(t, x)$, $x \in [0, T]$, with fixed $t > 0$. We consider a parametric model with probabilities P_w and put $P_0 = P_{w_0}$.

After the locally averaged discretization (4.1) the problem is reduced to the problem of estimation of parameters of the spectral density

$$f_{1d}^{(2)}(\lambda) = f_{1d}^{(2)}(\lambda, w), \qquad -\pi < \lambda \leqslant \pi, \; w \in W,$$

given by (4.15) and based on the sample $Y_{d1}(t, x)$, $x \in \{1, \ldots, T\}$. This spectral density is singular at zero (see (4.16)). Normalized periodograms of processes with such singularities can have some anomalous properties (see Hurvich and Ray [47]). In particular, the expectation of the normalized periodogram tends to infinity in this case (*ibidem*, Theorem 2).

For the sake of simplicity, we select the discretization step in (4.15), (4.17) and (4.19) to be $h = 1$, but accommodating other step sizes can be routinely accomplished by replacing λ by λ/h.

The spectral density $f_{d1}^{(2)}$, given by (4.15), can be written in the form

$$f_{d1}^{(2)}(\lambda, w) = |1-e^{-i\lambda}|^2 f_{10}(\lambda, w), \tag{4.39}$$

where

$$\begin{aligned} f_{10}(\lambda) &= f_{10}(\lambda, w) \\ &= p\,|\lambda|^{\alpha-1}\big[\exp\{-2\mu t\lambda^2\} \\ &\quad + |\lambda|^{1-\alpha} \sum_{k\neq 0} |\lambda+2k\pi|^{\alpha-1} \exp\{-2\mu t(\lambda+2k\pi)^2\}\big], \end{aligned} \tag{4.40}$$

$-\pi < \lambda \leqslant \pi$, $w \in W$. In analogy to (4.33)–(4.37) we can introduce the transformed data

$$\tilde{Y}_1(x) = Y_{d1}(t, 1) + \ldots + Y_{d1}(t, x), \qquad x \in \{1, 2, \ldots, T\}, \tag{4.41}$$

and

$$Y_1(x) = \tilde{Y}_1(x) + Y(0), \qquad x \in \{1, 2, \ldots, T\}, \tag{4.42}$$

where $Y_1(x)$, $x \in \boldsymbol{Z}^1$, is a stationary process with spectral density f_{10} given by (4.40).

As in the previous section (see (4.33)–(4.37)) we see that the periodograms of these two processes coincide at the non-zero Fourier frequencies (4.32):

$$I_T(\lambda_j, \tilde{Y}_1) = I_T(\lambda_j, Y_1). \tag{4.43}$$

Thus the parameter $w \in W$ estimation problem for the spectral density $f_{d1}^{(2)}$ is reduced to the estimation problem for the parameter $w \in W$ for the spectral density f_{10} (see (4.40)), based on the periodogram at the non-zero Fourier frequencies. The periodogram $I_T(\lambda_j, \tilde{Y}_1)$ can be obtained in terms of discretized observations (4.41), and $I_T(\lambda_j, Y_1)$ is a periodogram of the stationary Gaussian process $Y_1(x)$, $x \in \boldsymbol{Z}^1$, with the spectral density f_{10} given by (4.40). It follows from (4.40) that, as $\lambda \to 0$,

$$f_{10}(\lambda) \sim p\lambda^{\alpha-1}, \qquad 0 < \alpha < 1. \tag{4.44}$$

This spectral density has a pole at zero. Statistical inference problems for processes of such type (called the *long-memory processes* or *processes with long-range dependence*) were considered by many authors, including Fox and Taqqu [24], Dahlhaus [16], Beran [4], Robinson [67]–[69], and Anh and Lunney [2]. It would be of interest to investigate the parameter estimation problems for the spectral density (4.39) without using transformation (4.41) and compare the two methods from the efficiency viewpoint.

We begin with the special problem of estimation of the parameter $\alpha \in (0, 1)$ alone and employ the semiparametric methods (see Robinson [67]–[69]). Parameter α is called by various authors the *parameter of self-similarity*, the *Hurst parameter*, or the *fractional parameter*.

The first simple method of estimation of parameter α was proposed in a paper by Geweke and Porter–Hudak [26]. It reduced the problem to the regression problem with asymptotically independent errors. Unfortunately, for long-memory processes it is not true that the normalized periodogram's ordinates at the Fourier frequencies may be treated as asymptotically i.i.d. exponential random variables (see Künsch [55], Hurvich and Beltrao [45], Iglói [49], Robinson [68]). In fact, both the limiting distribution and the limiting expectation of the normalized periodogram's ordinates at the Fourier frequencies λ_j depend on j. In view of these anomalies in the behavior of the periodogram at very low frequencies (i.e., frequencies of order $1/T$) it seems natural to contemplate removal from the regression scheme of a block of very low frequencies of the periodogram.

This idea was pursued by Iglói [49] and Robinson [68] who proposed to choose the ordinary least-squares estimates associated with the regression scheme

$$Z_j = b+(\alpha-1)\,d_j+\xi_j, \qquad j = l+1, \ldots, m, \tag{4.45}$$

where $b = \log p-\gamma$, $\gamma \approx 0.5772\ldots$, is the Euler constant([1]),

$$Z_j = \log I_T(\lambda_j, \tilde{Y}_1), \qquad d_j = \log \lambda_j, \qquad \xi_j = \log \frac{I_T(\lambda_j, \tilde{Y}_1)}{p\,|\lambda_j|^{\alpha-1}}+\gamma,$$

and λ_j are Fourier frequencies (4.30).

In what follows we shall have need for the following condition:

M. *The quantities l, $m \to \infty$ in such a way that, as $T \to \infty$,*

$$\frac{l}{m}+\frac{\sqrt{m}\log m}{l}+\frac{\log^2 T}{m}+\frac{m^{3/2}}{T} \to 0.$$

([1]) Recall that the Euler constant γ can be defined via the asymptotic relation $H(N) \sim \log(N+1)+\gamma$, $N \to \infty$, where $H(N) = 1+1/2+1/3+\ldots+1/N$ is the partial sum of the harmonic series; see, e.g., Saichev and Woyczynski [77], p. 97; also $\gamma = \exp\{-\int_{-\infty}^{\infty} e^{-t} \log t\, dt\}$.

Under condition **M** the random variables ξ_j in (4.45) are asymptotically i.i.d. random variables with the *Gumbel distribution* $F(u) = 1 - \exp\{-e^u\}$, $u \in \mathbf{R}^1$. Its expectation is $-\gamma$, and its variance is $\pi^2/6 \approx 1.645\ldots$

The least-squares estimator of parameter α obtained from (4.45) has the form

$$\hat{\alpha}_T = 1 + Z'V(V'V)^{-1},$$

where $Z = (Z_{l+1}, \ldots, Z_m)'$, $V = (d_{l+1}, \ldots, d_m)'$. In the one-dimensional case, utilizing Robinson's [68] result we obtain the following theorem:

THEOREM 4.8. *Under conditions* **G** *and* **M**,

$$E(\hat{\alpha}_T - \alpha)^2 = O(1/m) \quad \text{and} \quad 2\sqrt{m}(\hat{\alpha}_T - \alpha) \xrightarrow{D} \mathcal{N}(0, \pi^2/6).$$

We plan to continue work on this type of semiparametric estimates of the fractional parameter α using recent results of Robinson [67]–[69], Anh and Lunney [2], and Hurvich et al. [46]. Note that Giraitis et al. [28] presented an optimal choice of m in such estimates.

Next, consider the more general problem of estimation of the parameter $w = (\alpha, \beta, \sigma^2, \vartheta)' \in W$ (see condition **G**) of the spectral density f_{10} given by (4.40), based on data (4.41), and using (4.43). These issues have been studied before. Fox and Taqqu [24] have developed the theory of estimation of parameters of spectral densities satisfying condition (4.44). Dahlhaus [16] proved the asymptotic efficiency of their procedure and extended their results to more general parametrizations. Dahlhaus [16] also proved the asymptotic normality for the exact maximum likelihood estimates. Our approach will use that of Heyde and Gay [38], [39].

Following notation (4.32a), let

$$S^*_{T1}(w) = S^*_T(w, I_T(\lambda_j, \tilde{Y}_1), f_{01}(\lambda, w)), \qquad w \in W,$$

and

$$w^*_{T_1} = \arg\min_{w \in W} S^*_T(w).$$

THEOREM 4.9. (i) *Suppose that the condition* **G** *is satisfied and that the spectral density* f_{01}, *given by* (4.40), *satisfies the identifiability condition* **I**. *Then* $w^*_{T_1} \to w_0$ *in* P_0*-probability, as* $T \to \infty$.

(ii) *If, additionally, the function* $\mu(\vartheta)$ *is twice continuously differentiable in* $\vartheta \in \Theta$, *all its derivatives of the first and second order are bounded away from zero on* $\vartheta \in \Theta$, *and the matrix* $\Gamma_1 = \Sigma(f_{01}(\lambda, w_0))$ (see (4.27)) *is not singular, then, as* $T \to \infty$,

$$\sqrt{T}(w^*_{T_1} - w_0) \xrightarrow{D} \mathcal{N}(0, 2\Gamma_1^{-1}).$$

Remark 4.11. For the parametrization $w = (\alpha, p, \mu)' \in W \subset \mathbf{R}^3$ (W a compact set) all the assumptions of Theorem 4.9 are satisfied.

Remark 4.12. Theorem 4.9 provides an alternative way of estimating fractional parameter $\alpha \in (0, 1)$ as the only parameter (see Theorem 4.8), if we consider the one-parameter model $w = \alpha \in [\alpha_0, \alpha_1] \subset (0, 1)$ and assume that other parameters are known.

Remark 4.13. In Dahlhaus' [16] paper one can find additional conditions under which our procedure becomes asymptotically efficient in the Fisher sense.

Finally, consider the problem of estimation of the parameter $w \in W$ for the spectral densities $f_{di}^{(2)}$, $i = 2, 3$, given by (4.17) and (4.18), respectively. These densities have no singularities at zeto or at infinity (see (4.19) and (4.20)).

Let $Y_{di}(t, x)$, $x \in \{1, \ldots, T\}$, $i = 2, 3$, be discretized data obtained from the observation of $Y(t, x)$, $x \in [0, T]$, using discretizations (4.2) and (4.3). Consider the minimum contrast estimates (see (4.22) and (4.24))

$$\tilde{w}_{T_i} = w_T\big(I_T(\lambda, Y_{di}), f_{di}^{(2)}(\lambda, w)\big),$$

where $w = (\alpha, \beta, \sigma^2, \vartheta) \in \boldsymbol{R}^{k+3}$ (see condition $\mathbf{G}$).

THEOREM 4.10. (i) *Let $i = 2$ or 3. Assume that the condition $\mathbf{G}$ is satisfied and that the spectral density $f_{di}^{(2)}$ satisfies the identifiability condition $\mathbf{I}$. Then $\tilde{w}_{T_i} \to w_0$ in P_0-probability, as $T \to \infty$.*

(ii) *Suppose, additionally, that the function $\mu(\vartheta)$ is such that the function $f_{di}^{(2)}$ is twice differentiable in $\vartheta \in \Theta$, its second derivatives are continuous on $(-\pi, \pi]$, and the matrix $\Gamma_i = \Sigma\big(f_{di}^{(2)}(\lambda, w_0)\big)$ defined by (4.27) is non-singular. Then the estimator $\tilde{w}_{T_i}$ is asymptotically normal and efficient in the Fisher sense, i.e., as $T \to \infty$,*

$$\sqrt{T}\,(\tilde{w}_{T_i} - w_0) \xrightarrow{D} \mathcal{N}_{k+3}(0, 2\Gamma_i^{-1}).$$

Remark 4.14. For the parametrization $w = (\alpha, \beta, \sigma^2, \mu)'$, all the assumptions of Theorem 4.10 are satisfied.

5. PROOFS

Proof of Theorem 2.1. We shall only indicate the main steps of the proof which is an adaptation of ideas of Albeverio et al. [1] for dimension $n = 1$. Following their Theorem 3, from (2.3) we obtain

$$(5.1)\qquad \operatorname{cov}\big(J(t_1, x_1), J(t_2, x_2)\big)$$
$$= \exp\{B(0)/(4\mu^2)\} \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} g(t_2, u)\, g\big(t_1, u + (x_1 - x_2 - z)\big)\, du\Big]$$
$$\times \big(\exp\{B(z)/(4\mu^2)\} - 1\big)\, dz$$

$$= \exp\{B(0)/(4\mu^2)\} \int_{-\infty}^{\infty} [4\pi\mu(t_1+t_2)]^{-1/2} \exp\left\{-\frac{|x_1-x_2-z|^2}{2\mu(t_1+t_2)}\right\}$$
$$\times \left(\exp\{B(z)/(4\mu^2)\}-1\right) dz,$$

and

$$\lim_{\beta\to\infty} \beta^{1/2} \operatorname{cov}\left(J(\beta, x_1\sqrt{\beta}), J(\beta, x_2\sqrt{\beta})\right)$$
$$= \kappa_1 (8\mu t)^{-1/2} \exp\left\{-\frac{|x_1-x_2|^2}{8\mu}\right\}, \qquad x_1, x_2 \in \boldsymbol{R}^1,$$

where

$$\kappa_1 = \exp\{B(0)/(4\mu^2)\} \int_{-\infty}^{\infty} \left(\exp\{B(z)/(4\mu^2)\}-1\right) dz.$$

Then, using the method of moments with the diagram formalism (see, e.g., Breuer and Major [7], Giraitis and Surgailis [30], Leonenko and Deriev [57]) it is easy to prove that the finite-dimensional distributions of the renormalized field

$$\tilde{J}(\beta, x) = \beta^{1/4}\left(J(\beta, x\sqrt{\beta}) - \exp\{B(0)/(4\mu^2)\}\right)$$

converge weakly, as $\beta \to \infty$, to the centered, homogeneous Gaussian process with the covariance function given by (5.1). By the functional central limit theorem (see Albeverio et al. [1], Theorem 2) we infer that the finite-dimensional distributions of the random fields $\tilde{J}(t\beta, x)$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$ tend, as $\beta \to \infty$, to the distributions of the stationary Gaussian field

$$\sqrt{\kappa_1}\, B(t, x) = \int_{-\infty}^{\infty} g(t, x-y)\, G(dy),$$

with the covariance function $\kappa_1 \operatorname{cov}\left(B(t_1, x_1), B(t_2, x_2)\right) = g(t_1+t_2, x_1-x_2)$, where $G(\cdot)$ is a complex Gaussian white noise.

Then, asymptotically, distributions of the velocity field (2.3) can be obtained from the formula $u(t, x) = -2\mu\left(\partial\left(J(t, x)\right)/\partial x\right)/J(t, x)$, where, in view of the law of large numbers, the denominator tends in probability to the constant $EJ(t, x) = C_0 = \exp\{B(0)/(8\mu^2)\}$, while in the numerator the derivative commutes with the passage to the limit. Hence, by Slutsky's arguments, the finite-dimensional distributions of the rescaled velocity fields $t^{3/4} u(\beta t, x\sqrt{\beta})$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$, converge weakly, as $\beta \to \infty$, to the finite-dimensional distributions of the Gaussian field

$$U(t, x) = -2\mu\sqrt{\kappa_2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} g(t, x-y)\, G(dy),$$
$$\kappa_2 = \int_{-\infty}^{\infty} \left(\exp\{B(z)/(4\mu^2)\}-1\right) dz.$$

The limiting field $U(t, x)$ has a moving average representation, so that it is stationary in x, zero mean, Gaussian, with the covariance function

$$R(x-y) = \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x} g(t, x-u)\right]\left[\frac{\partial}{\partial y} g(t, y-u)\right] du = \int_{-\infty}^{\infty} e^{i\lambda(x-y)} g(\lambda)\, d\lambda, \tag{5.2}$$

where the spectral density

$$g(\lambda) = (2\pi)^{-1} \gamma_t(\lambda) \overline{\gamma_s(\lambda)}, \qquad \gamma_t(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda v} \frac{\partial}{\partial v} g(t, v)\, dv.$$

Putting $\sigma^2 = 2\pi\mu t$ in the formula

$$\int_{-\infty}^{\infty} \lambda e^{-i\lambda v} \exp\{-|v|^2/(2\sigma^2)\} (2\pi\sigma^2)^{-1/2}\, dv = i\lambda\sigma^2 \exp\{(-\lambda^2\sigma^2)/2\}, \qquad \lambda \in \boldsymbol{R}^1, \sigma^2 > 0,$$

we obtain

$$\gamma_t(\lambda) = i2\mu\sqrt{\kappa_2} \exp\{(-\lambda^2\sigma^2)/2\}\, \lambda,$$

and

$$g(\lambda) = (2\pi)^{-1} (2\mu)^2 \kappa_2 \lambda^2 \exp\{-\lambda^2 \mu (t+s)\}, \qquad \lambda \in \boldsymbol{R}^1. \tag{5.3}$$

Thus we proved (2.6) in the case $F(u) = u$ (see Corollary 2.1).

Then, taking derivatives of both sides of the identity

$$\psi(x) = \int_{-\infty}^{\infty} \exp\{i\lambda x - \lambda^2/(2\kappa^2)\}\, d\lambda = \sqrt{2\pi\kappa^2} \exp\{(-x^2\kappa^2)/2\}, \qquad x \in \boldsymbol{R}^1,$$

we obtain

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \kappa^2 \sqrt{2\pi\kappa^2} (\kappa^2 x^2 - 1) \exp\{(-x^2\kappa^2)/2\}. \tag{5.4}$$

Putting $\kappa^2 = 1/(2\mu(t+s))$ in the above formula we obtain, with the help of (5.2)–(5.4), the formula (2.5) in the case $F(u) = u$. For non-Gaussian processes, the full proofs are presented in Leonenko and Deriev [57] in the case $t = s = 1$, and the general case can be proved in a similar fashion. The paper also contains specific applications of the diagram formula (see Step 5 in the proof of their Theorem 2.1). Related parabolic asymptotics is discussed, in a variety of contexts, in Surgailis and Woyczynski [84], [86], and in Funaki et al. [25].

Proof of Theorem 2.2. Theorem 2.2 can be proved following the main steps of Leonenko et al. [58], [59], where the case $F(u) = u$ and $t = s = 1$ is considered. There is a little difference since in those papers the initial conditions are Gaussian. So, for example, in formulas (3.8) and (3.9) of Leonenko and Orsingher [58] one only has to substitute

$$C_k = \int_{-\infty}^{\infty} \exp\{-F(\sigma u)/(2\mu)\}\, \phi(u)\, H_k(u)\, du, \qquad k = 0, 1, \ldots$$

The expression for the limiting variance (2.14) can be obtained by methods similar to those of (3.17) in Leonenko and Orsingher [58]. Below we just indicate the main steps in the proof.

First, let us consider the Hilbert space $L_2(\Omega)$ Hermite expansion for the local functional

$$\exp\left\{-\frac{F(\xi(y))}{2\mu}\right\} = \sum_{j=0}^{\infty} \frac{C_j}{j!} H_k(\xi(y)),$$

and apply it to the numerator of (2.3), to obtain

$$(5.5)\qquad I(\beta t, x\sqrt{\beta}) = \int_{-\infty}^{\infty} \frac{x\sqrt{\beta}-y}{t\beta} g(t\beta, x\sqrt{\beta}-y)\, e^{-F(\xi(y))/(2\mu)}\, dy$$

$$= C_0 \eta_{0\beta}(t, x) + C_1 \eta_{1\beta}(t, x) + \sum_{j\geqslant 2} \frac{C_j}{j!} \eta_{j\beta}(t, x) + R_\beta,$$

where random variables

$$\eta_{j\beta}(t, x) = \int_{-\beta}^{\beta} \frac{x\sqrt{\beta}-y}{t\beta} g(t\beta, x\sqrt{\beta}) H_j(\xi(y))\, dy, \qquad j = 0, 1, 2, \ldots,$$

$$R_\beta = \int_{|y|\geqslant\beta} \frac{x\sqrt{\beta}-y}{t\beta} g(t\beta, x\sqrt{\beta}) \exp\{-F(\xi(y))/(2\mu)\}\, dy.$$

Observe that $\eta_{1\beta}(t, x)$, $(t, x)\in(0, \infty)\times \boldsymbol{R}^n$, is a Gaussian field because $H_1(u) = u$. Since (see, e.g., Ivanov and Leonenko [51], p. 55)

$$EH_r(\xi(y_1))\, H_m(\xi(y_2)) = m!\,\delta_m^r B^m(|y_1-y_2|), \qquad r, m \geqslant 0,$$

where δ_m^r is the Kronecker symbol, we have

$$E\eta_{m\beta}(t, x)\, \eta_{r\beta}(s, x') = \delta_m^r E\eta_{m\beta}(t, x)\, \eta_{m\beta}(s, x'),$$

where

$$\varrho_{mij}^2(\beta) = E\eta_{m\beta}(t, x)\, \eta_{m\beta}(s, x')$$

$$= m! \int_{-t}^{t} \int_{-t}^{t} \frac{x\sqrt{\beta}-y}{t\beta} \frac{x'\sqrt{\beta}-y'}{t\beta}$$

$$\times g(t\beta, x\sqrt{\beta}-y)\, g(s\beta, x'\sqrt{\beta}-y')\, B^m(|y-y'|)\, dy\, dy'.$$

Changing variables via the transformations

$$w^2/2 = (x\sqrt{\beta}-y)^2/(4\mu t\beta) \quad \text{and} \quad z^2/2 = (x'\sqrt{\beta}-y')^2/(4\mu t\beta),$$

and utilizing the basic properties of the slowly varying function L from condition E (see, e.g., Ivanov and Leonenko [51], p. 56), we have, for $0 < \alpha m < 1$, $m \geqslant 1$, and $\beta \to \infty$,

$$\varrho^2_{mij}(\beta) = \frac{2\mu m!}{\beta\sqrt{ts}} \int_{D_\beta(t)} \int_{D_\beta(s)} wz\phi(w)\phi(z) B^m \tag{5.6}$$

$$\times \left(\sqrt{2\mu\beta} \left| \frac{x-x'}{\sqrt{2\mu}} - (w\sqrt{t} - z\sqrt{s}) \right| \right) dw\, dz$$

$$= (2\mu)^{1-(m\alpha)/2} m! \varrho^2_{mij}(x-x') \frac{L^m(\sqrt{\beta})}{\beta^{1+(m\alpha)/2}} (1+o(1)), \qquad \beta \to \infty,$$

where $D_\beta(t) = [x/\sqrt{2\mu t} - \sqrt{\beta/(2\mu t)},\ x/\sqrt{2\mu t} + \sqrt{\beta/(2\mu t)}]$, and

$$\varrho_{mij}(x-x') = \frac{1}{\sqrt{ts}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{wz\phi_n(w)\phi_n(z)}{|(x-x')/\sqrt{2\mu} - (w\sqrt{t} - z\sqrt{s})|^{m\alpha}} dw\, dz, \qquad 0 < \alpha < 1/m.$$

By the law of large numbers, the denominator of (2.3) converges in probability to a constant, as $\beta \to \infty$,

$$J(t\beta, x\sqrt{\beta}) = \int_{-\infty}^{\infty} g(t\beta, x\sqrt{\beta} - y) e^{-F(\xi(y))/(2\mu)} dy \to C_0, \tag{5.7}$$

and it is easy to prove that $\lim_{\beta\to\infty} C_0 \eta_0(t, x) = 0$. Then, as in Leonenko and Orsingher [58], we can show that, for $0 < \alpha < 1$ and $\beta \to \infty$,

$$\frac{\beta^{1/4+\alpha/2}}{L^{1/2}(\sqrt{\beta})} \left[\sum_{j \geqslant 2} \frac{C_j}{j!} \eta_{j\beta}(t, x) + R_\beta \right] \to 0 \tag{5.8}$$

in probability. From (2.3) and (5.5)–(5.8) we obtain, with the help of Slutsky's argument,

$$\tilde{Y}_\beta(t, x) = \frac{\beta^{1/2+\alpha/4}}{L^{1/2}(\sqrt{\beta})} \frac{C_1}{C_0} \eta_{1\beta}(t, x) \xrightarrow{D} Y(t, x), \qquad Y_\beta(t, x) \xrightarrow{D} Y(t, x),$$

as $\beta \to \infty$, where $Y_\beta(t, x)$ is defined in Theorem 2.2.

Next, (5.6) implies that the covariance function of the field $Y(t, x)$ is as claimed in (2.14). The Gausianness of $Y(t, x)$ follows from the fact that $\eta_{1\beta}(t, x)$ is a Gaussian field defined by the first Hermite polynomial.

The remainder of the proof is a modification of arguments in Leonenko et al. [60]. Let us show that the stationary in x random field $Y(t, x)$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^1$, has the spectral representation (2.16) and the spectral density given by (2.19). Using the self-similarity property $G(d(a\lambda)) \stackrel{D}{=} \sqrt{a} G(d\lambda)$,

$a > 0$, of the Gaussian white noise G, we can write

(5.9)
$$\tilde{Y}_\beta(t, x) \stackrel{D}{=} \frac{C_1}{C_0} \frac{\beta^{1/2+\alpha/4}}{L^{1/2}(\sqrt{\beta})} \times \int_{-\infty}^{\infty} \left[\int_{-\sqrt{\beta}}^{\sqrt{\beta}} \frac{x-y}{t(4\pi\mu t)^{1/2}} \exp\{-(x-y)^2/(4\mu t)+i\lambda t\}\, dy \right] \times \sqrt{g(|\lambda|/\sqrt{\beta})}\, \beta^{-3/4}\, G(d\lambda),$$

$(t, x) \in (0, \infty) \times \boldsymbol{R}^1$. Utilizing the identity

(5.10)
$$\int_{-\infty}^{\infty} \frac{\exp\{i\lambda z-(z-x)^2/(4\mu t)\}}{(4\pi\mu t)^{1/2}}(x-z)\, dz = \frac{2\mu t\lambda}{i} \exp\{it\lambda-\mu t\lambda^2\}, \quad \lambda \in \boldsymbol{R}^1,$$

we have, with the help of (2.12) and (5.9), (5.10),

(5.11)
$$\begin{aligned} E|\tilde{Y}_\beta(t, x)-Y(t, x)|^2 &= \frac{C_1^2\, \beta^{1+\alpha/2}}{C_0^2\, L(\sqrt{\beta})}(2\mu)^2 \\ &\times E\Bigg| \frac{1}{2\mu} \int_{-\infty}^{\infty} \left[\int_{-\sqrt{\beta}}^{\sqrt{\beta}} \frac{x-y}{t(4\pi\mu t)^{1/2}} \exp\{-(x-y)^2/(4\mu t)+i\lambda t\}\, dy \right] \\ &\times \sqrt{g(|\lambda|/\sqrt{\beta})}\, \beta^{-3/4}\, G(d\lambda) \\ &- \frac{[c_1(\alpha)]^{1/2}}{i} \int_{-\infty}^{\infty} \exp\{it\lambda-\mu t\lambda^2\}\, \lambda\, |\lambda|^{(\alpha-1)/2}\, G(d\lambda) \Bigg|^2 \\ &\leqslant \frac{C_1^2\, \beta^{1+\alpha/2}}{C_0^2\, L(\sqrt{\beta})}(2\mu)^2\, E\Bigg| - \int_{-\infty}^{\infty} \frac{\lambda \exp\{it\lambda-\mu t\lambda^2\}}{i\beta^{3/4}} \sqrt{g(|\lambda|/\sqrt{\beta})}\, G(d\lambda) \\ &+ \int_{-\infty}^{\infty} \left[\int_{|y|>\sqrt{\beta}} \frac{(x-y)\exp\{i\lambda x-(x-y)^2/(4\mu t)\}}{2\mu t(4\pi\mu t)^{1/2}\, \beta^{3/4}}\, dy \right] \sqrt{g(|\lambda|/\sqrt{\beta})}\, G(d\lambda) \\ &+ \int_{-\infty}^{\infty} \frac{\lambda \exp\{it\lambda-\mu t\lambda^2\}\, L^{1/2}(\sqrt{\beta})}{i|\lambda|^{(1-\alpha)/2}\, \beta^{1/2+\alpha/4}} \sqrt{c_1(\alpha)}\, G(d\lambda) \Bigg|^2 \\ &\leqslant \frac{C_1^2}{C_0^2}(2\mu)^2 \Bigg[\int_{-\infty}^{\infty} \Bigg| \frac{\lambda \exp\{ix\lambda-\mu t\lambda^2\}\, \beta^{1/2+\alpha/4}}{i\beta^{3/4}\, L^{1/2}(\sqrt{\beta})} \\ &\times \sqrt{g(|\lambda|/\sqrt{\beta})} - [c_1(\alpha)]^{1/2} \frac{\lambda \exp\{ix\lambda-\mu t\lambda^2\}}{|\lambda|^{(1-\alpha)/2}} \Bigg|^2 d\lambda \\ &+ \frac{\beta^{1+\alpha/2}}{L(\sqrt{\beta})} \int_{-\infty}^{\infty} \Bigg| \int_{|y|>\sqrt{\beta}} \frac{(x-y)\exp\{ix\lambda-(x-y)^2/(4\mu t)\}}{2\mu t(4\pi\mu t)^{1/2}\, \beta^{3/4}}\, dy \Bigg|^2 g(|\lambda|/\sqrt{\beta})\, d\lambda \Bigg] \\ &= A_1(\beta)+A_2(\beta), \end{aligned}$$

where $c_1(\alpha)$ is given by (2.17).

The function $\psi(\lambda) = \lambda^2 \exp\{2i\lambda x - 2\mu t\lambda^2\}/|\lambda|^{1-\alpha}$, $\lambda \in \boldsymbol{R}^1$, is absolutely integrable. Using the Tauberian result (2.18), and properties of slowly varying functions, we infer from (5.11) that

$$\lim_{\beta\to\infty} A_1(\beta) = \lim_{\beta\to\infty} \int_{-\infty}^{\infty} \psi(\lambda) Q_\beta(\lambda)\, d\lambda = 0,$$

where, as $\beta \to \infty$,

$$Q(\lambda) \sim \frac{C_1^2}{C_0^2}(2\mu)^2 c_1(\alpha) \left| \frac{L^{1/2}(\sqrt{\beta}/|\lambda|)}{L^{1/2}(\sqrt{\beta})} - 1 \right|^2.$$

For $A_2(\beta)$ we have the following estimate:

$$A_2(\beta) \leqslant \frac{(C_1/C_0)^2 (2\mu)^2}{L(\sqrt{\beta})\, \beta^{(1-\alpha)/2}} \int_{-\infty}^{\infty} g\left(\frac{|\lambda|}{\sqrt{\beta}}\right) \left| \int_{|y|>\sqrt{\beta}} \frac{(x-y)\exp\{-(x-y)^2/(4\mu t)\}}{(2\mu t)(4\pi\mu t)^{1/2}}\, dy \right|^2 d\lambda$$
$$\leqslant \frac{\text{const}}{L(\sqrt{\beta})\, \beta^{(1-\alpha)/2}}.$$

Hence $\lim_{\beta\to\infty} A_2(\beta) = 0$, and $\lim_{\beta\to\infty} E|\tilde{Y}_\beta(t, x) - Y(t, x)|^2 = 0$, where $Y(t, x)$ is defined by (2.16). Finally, applying the Cramer–Wold argument we obtain the assertion of Theorem 2.2, including the statement in Remark 2.3.

Proof of Theorem 3.1. The statement (i) is a consequence of Theorem 4.2.1, and statements (ii) and (iv) are consequences of Theorem 4.2.2 in Ivanov and Leonenko [51], but we must note that the convergence of the finite-dimensional distributions has to be proved by using the mixing condition (see Ivanov and Leonenko [51], p. 34). On the other hand, Buldigin [9] proved by the method of moments, without using mixing conditions, the central limit theorem for correlograms of Gaussian processes with a square-integrable spectral density; and the proof of compactness of probability measures in the uniform topology (see Lemma 4.3.2 of Ivanov and Leonenko [51]) does not use the mixing condition. So, Theorem 3.1 is a combination of the results mentioned above. The proof of the last formula (3.3) is based on the formulas

$$\varrho(x, y) = 2\pi \int_{-\infty}^{\infty} [e^{i\lambda(x-y)} + e^{i\lambda(x+y)}]\, g^2(\lambda)\, d\lambda$$

and

$$\int_{-\infty}^{\infty} e^{i\lambda h} \lambda^4 \frac{\exp\{-\lambda^2/(2\sigma^2)\}}{\sqrt{2\pi\sigma^2}}\, d\lambda = \frac{\partial^4}{\partial h^4} \exp\{(-h^2\sigma^2)/2\}$$
$$= \exp\{(-h^2\sigma^2)/2\}\, \sigma^4 (h^4\sigma^4 - 6h^2\sigma^2 + 3), \qquad h \in \boldsymbol{R}^1,\ \sigma^2 > 0.$$

Proof of Theorem 4.1. Let $t > 0$ be fixed. If $f_c(\lambda)$, $\lambda \in \boldsymbol{R}^1$, is the spectral density of the stationary in x random field $\zeta(t, x)$, $t > 0$, $x \in \boldsymbol{R}^1$, then the spectral densities $f_{di}(\lambda)$, $-\pi < \lambda \leqslant \pi$, $i = 1, 2, 3$, of random fields $\zeta_{di}(t, a)$, $t > 0$, $a \in \boldsymbol{Z}^1$, obtained from $\zeta(t, a) > 0$, $a \in \boldsymbol{R}^1$, through discretizations (4.1)–(4.3), respectively, take the form (see Grenander [32], p. 250)

$$f_{d1}(\lambda) = \frac{4}{h}\sin^2\frac{\lambda}{2}\sum_{k=-\infty}^{\infty}(\lambda+2k\pi)^{-2} f_c\left(\frac{\lambda+2k\pi}{h}\right), \tag{5.12}$$

$$f_{d2}(\lambda) = \frac{1}{h}\sum_{k=-\infty}^{\infty} f_c\left(\frac{\lambda+2k\pi}{h}\right), \tag{5.13}$$

$$f_{d3}(\lambda) = \frac{\kappa}{2\pi}+\frac{1}{h}\sum_{k=-\infty}^{\infty}|\psi(\lambda+2k\pi)|^2 f_c\left(\frac{\lambda+2k\pi}{h}\right), \tag{5.14}$$

where

$$\kappa = \int_{-\infty}^{\infty} f_c(\lambda)\left[1-|\psi(h\lambda)|^2\right]d\lambda, \qquad \psi(\lambda) = E\exp\{i\lambda v_x\},\ \lambda \in \boldsymbol{R}^1.$$

We note (see Mumford [65], p. 12, or Widder [96], p. 86) that the Jacobi theta-function may be defined as follows:

$$\theta(x, s) = \frac{1}{\sqrt{4\pi s}}\sum_{k=-\infty}^{\infty}\exp\left\{\frac{-(x+2k\pi)^2}{4s}\right\}, \tag{5.15}$$

$$(x, s) \neq (2m\pi, 0),\ m = 0, \pm 1, \ldots,$$

and in this case the formula (4.4) is the Fourier series of a 2π-periodic function (5.15) (see Widder [96], p. 90).

Now, apply (5.12) to the spectral density $f_c(\lambda) = g(\lambda)$, $\lambda \in \boldsymbol{R}^1$, given by (4.4), to obtain from (4.4), (4.5), (5.12) and (5.15) the formula (4.7). The last representation in (4.7) is based on the product expansion for the theta-function $\theta(x, t)$ given, for example, in Widder [96], p. 92. Thus we have proved (4.7). Using the asymptotic relation $4\sin^2(\lambda/2) \sim \lambda^2$, $\lambda \to 0$, we obtain (4.8).

Next, let $\kappa(x, s) = -2\partial/\partial x\theta(x, s)$. Using Theorem 2.1 of Widder [96], p. 88, we obtain from (5.15)

$$\frac{\partial}{\partial x}\kappa(x, s) = \frac{1}{s}\theta(x, s)-\frac{1}{2s^2\sqrt{4\pi s}}\sum_{k=-\infty}^{\infty}(x+2k\pi)^2\exp\{-(x+2k\pi)^2/(4s)\}$$

and

$$\sum_{k=-\infty}^{\infty}(x+2k\pi)^2\exp\{-(x+2k\pi)^2/(4s)\} \tag{5.16}$$

$$= \left[\frac{1}{s}\theta(x, s)-\frac{\partial}{\partial x}\kappa(x, s)\right]2s^2\sqrt{4\pi s} = \left[\frac{1}{s}\theta(x, s)+2\frac{\partial^2}{\partial x^2}\theta(x, s)\right]2s^2\sqrt{4\pi s}$$

$$= \left[\frac{1}{s}\theta(x, s)+2\frac{\partial}{\partial x}\theta(x, s)\right]2s^2\sqrt{4\pi s},$$

because the theta-function $\theta(x, s)$ is one of the fundamental solutions of the heat equation (see Mumford [65], p. 12).

Put $f_c(\lambda) = g(\lambda)$, $\lambda \in \boldsymbol{R}^1$ (see (4.5)), in formula (5.13). Then, in view of (5.16) with $s = h^2/(8\mu t)$, the Fourier series expansion for the theta-function yields (4.9). The relation (4.10) follows from (4.9) and Theorem 2.1 of Widder [96], p. 88. Formulas (4.11) and (4.12) may be proved by a similar argument, using (5.14) in place of (5.13), and a direct calculation of the constant κ in (5.14). We omit the details.

Proof of Theorem 4.2. The first part of (4.15) is a result of substitution of (4.13) into (5.12). The second part is the Fourier series for the first part because the series in the first part is a 2π-periodic function satisfying the Dirichlet conditions, and the Fourier coefficients of the function

$$d(\lambda, t) = \sum_{k=-\infty}^{\infty} \Psi(\lambda + 2k\pi, t),$$

$$\Psi(\lambda, t) = |\lambda|^{\alpha-1} \exp\left\{-\frac{2\mu t\lambda^2}{h^2}\right\}, \qquad -\pi < \lambda \leqslant \pi, \ t > 0,$$

are given by

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} d(s, t) e^{-ims} ds = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} e^{-ims} \Psi(s + 2k\pi, t) ds$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+2)\pi} e^{-ims} \Psi(s, t) ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(s, t) e^{-ims} ds.$$

Then, as $\lambda \to 0$,

$$f_{d1}^{(2)}(\lambda) = \frac{p}{h^{2+\alpha}} 4\sin^2\frac{\lambda}{2}\left[|\lambda|^{\alpha-1} \exp\left\{-\frac{2\mu t\lambda^2}{h^2}\right\}\right.$$

$$\left. + \sum_{k\neq 0} |\lambda + 2k\pi|^{\alpha-1} \exp\left\{-\frac{2\mu t(\lambda+2k\pi)^2}{h^2}\right\}\right]$$

$$\sim \lambda^{1+\alpha} \frac{p}{h^{2+\alpha}} + \lambda^2 F(\lambda), \qquad \lim_{\lambda\to 0} F(\lambda) > 0.$$

Thus (4.15) and (4.16) are proved. Similarly, if we substitute (4.13) into (5.13) we obtain (4.17), but in this case

$$\lim_{\lambda\to 0} f_{d2}^{(2)}(\lambda) = \lim_{\lambda\to 0} \frac{p}{h^{2+\alpha}}\left[|\lambda|^{\alpha+1} \exp\left\{-\frac{2\mu t\lambda^2}{h^2}\right\}\right.$$

$$\left. + \sum_{k\neq 0} |\lambda + 2k\pi|^{\alpha+1} \exp\left\{-\frac{2\mu t(\lambda+2k\pi)^2}{h^2}\right\}\right] > 0.$$

The formulas (4.19) and (4.20) may be proved by similar arguments, using (4.13), (5.14) and direct calculations which are omitted.

We will precede the proof of Theorem 4.3 by quoting a result of Guyon [36], pp. 119–120, the proof thereof is based on the well-known Wald argument.

THEOREM 5.1. *Assume* $w \mapsto K(w_0, w)$, $w \mapsto S_T(w)$, *are* P_0*-a.e. continuous, and that there exist* $\delta_i \to 0$ *such that* $\lim_{T\to\infty} P_0(H_T(1/i) \geqslant \delta_i) = 0$, *where* $H_T(\varepsilon)$ *is the modulus of continuity of* $S_T(w)$, *i.e.,*

$$H_T(\varepsilon) = \sup\{|S_T(w) - S_T(w')|:\ w, w' \in W, |w - w'| \leqslant \varepsilon\}. \tag{5.17}$$

Then $w_T \to w_0$ *in* P_0*-probability, as* $T \to \infty$.

Proof of Theorem 4.3. The proof is based on ideas of Walker [91], Ibragimov [48] and Hannan [37], and for its completion it is sufficient to verify assumptions of Theorem 5.1.

Let $h(\varepsilon) = \sup\{|f_d^{-1}(\lambda) - f_d^{-1}(\lambda')|:\ |\lambda - \lambda'| \leqslant \varepsilon\}$ be the modulus of continuity of the function $f_d^{-1}(\lambda)$. Defining

$$V_T(w) = \int_{-\pi}^{\pi} I_T(\lambda) f_d^{-1}(\lambda, w)\, d\lambda,$$

we have

$$\sup\{|V_T(w) - V_T(w')|:\ w, w' \in W,\ |w - w'| \leqslant h(\varepsilon)\} \leqslant h(\varepsilon) \int_{-\pi}^{\pi} I_T(\lambda)\, d\lambda = 2\pi h(\varepsilon) R_T(0),$$

where $R_T(x)$ is defined by (4.21), and $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$. This gives (5.17).

Let us now prove that convergence in probability of the sample covariance $R_T(x) \to R(x)$, $T \to \infty$ (see the assumptions of Theorem 4.3), implies that $S_T(w) \to S(w)$ in P_0-probability, as $T \to \infty$, where $S_T(w)$ is defined by (4.23), and $S(w)$ is defined by (4.25). Actually, it is sufficient to show that, for any continuous function $q(\lambda)$,

$$\int_{-\pi}^{\pi} q(\lambda) I_T(\lambda)\, d\lambda \to \int_{-\pi}^{\pi} q(\lambda) f_d(\lambda, w_0)\, d\lambda, \tag{5.18}$$

in P_0-probability, as $T \to \infty$. Let $M(\lambda) = \sum_{|\nu| \leqslant r} e^{i\lambda\nu} s_\nu$ be an arbitrary trigonometric polynomial. Then, as $T \to \infty$, in P_0-probability,

$$\lim_{T\to\infty} \int_{-\pi}^{\pi} M(\lambda) I_T(\lambda)\, d\lambda = \lim \sum s_\nu R_T(\nu) = \sum s_\nu R(\nu) = \int_{-\pi}^{\pi} M(\lambda) f_d(\lambda, w_0)\, d\lambda.$$

Now, let $q(\lambda)$ be an arbitrary continuous function on $(-\pi, \pi]$. For any $\varepsilon > 0$, one can find a trigonometric polynomial $M(\lambda)$ such that $\max_\lambda |q(\lambda) - M(\lambda)| \leqslant \varepsilon$. Furthermore, we have

$$\begin{aligned} &\int_{-\pi}^{\pi} q(\lambda) I_T(\lambda)\, d\lambda - \int_{-\pi}^{\pi} q(\lambda) f_d(\lambda, w_0)\, d\lambda \\ &\qquad \leqslant \Big| \int_{-\pi}^{\pi} M(\lambda)\big(I_T(\lambda) - f_d(\lambda, w_0)\big)\, d\lambda \Big| + \varepsilon \int_{-\pi}^{\pi} \big(I_T(\lambda) - f_d(\lambda, w_0)\big)\, d\lambda. \end{aligned} \tag{5.19}$$

The first term on the right-hand side of (5.19) converges in P_0-probability to zero, the second term is positive, and its expectation does not exceed $2\varepsilon\int_{-\pi}^{\pi} f_d(\lambda, w_0)\,d\lambda$. Thus (5.18) is proved.

From (5.18) it follows that, in P_0-probability, $S_T(w)-S_T(w_0)\to K(w_0, w)$ as $T\to\infty$, with

$$(5.20)\qquad K(w_0, w)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\left(\frac{f_d(\lambda, w_0)}{f_d(\lambda, w)}-1\right)-\log\frac{f_d(\lambda, w_0)}{f_d(\lambda, w)}\right]d\lambda\geqslant 0,$$

because $x-1\geqslant\log x$ for $x>0$ and $x-1=\log x$ only for $x=1$.

Now, (5.20) and the identifiability condition **I** imply that the function $K(w_0, w)$ has a unique minimum at $w=w_0$, and $K(w_0, w)\geqslant 0$. Thus $K(w_0, w)$ is a contrast function for the Gaussian Whittle contrast, which completes the proof of Theorem 4.3.

Proof of Theorem 4.4. It follows from Theorem 1 in Heyde and Gay [38], [39] that under conditions **K** and **L**, as $T\to\infty$,

$$(5.21)\qquad \sqrt{T}\Big[\int_{-\pi}^{\pi} A(\lambda, w)\big(I_T(\lambda)-EI_T(\lambda)\big)\,d\lambda\Big]\xrightarrow{D}\mathcal{N}_r\big(o, 4\pi\Sigma\big(f_d(\lambda, w_0)\big)\big),$$

where $\Sigma\big(f_d(\lambda, w_0)\big)$ is defined by (4.27). By the mean value theorem, under conditions **L1** and **L2** we obtain

$$\nabla S_T(w_T)=\nabla S_T(w_0)+\nabla^2 S_T(w^*)(w_T-w_0),$$

where $|w_T^*-w_0|<|w_T-w_0|$, and $\nabla^2=\nabla\cdot\nabla'=(\partial^2/\partial w_k\,\partial w_j)_{1\leqslant k,j\leqslant r}$. Since w_0 is in the interior of W, Theorem 4.3 implies that, for large T, W_T is also in the interior of W, and

$$(5.22)\qquad \nabla S_T(w_0)=\big[-\nabla^2 S_T(w_T^*)\big](w_T-w_0).$$

Since $\nabla^2 S_T(w)=(2\pi)^{-1}\int_{-\pi}^{\pi}\nabla^2 f_d^{-1}(\lambda, w)\,I_T(\lambda)\,d\lambda$, it follows from Theorem 4.3 and condition **L3** that, in P_0-probability,

$$(5.23)\qquad \nabla^2 S_T(w_T^*)\to 2\pi\int_{-\pi}^{\pi} f_d(\lambda, w_0)\big[\nabla^2 f^{-1}(\lambda, w_0)\big]\,d\lambda=\Sigma\big(f(\lambda, w_0)\big),$$

uniformly in $w\in W$, because the function $\nabla^2 f_d^{-1}(\lambda, w)$ is jointly continuous in $(\lambda, w)'$ in view of condition **L2**. Remembering that, as $T\to\infty$,

$$(5.24)\qquad -\sqrt{T}\big(\nabla S_T(w_0)-E\nabla S_T(w_0)\big)=\sqrt{T}\Big[\int_{-\pi}^{\pi} A(\lambda, w)\big(I_T(\lambda)-EI_T(\lambda)\big)\Big]\,d\lambda$$
$$=\sqrt{T}\Big[\int_{-\pi}^{\pi} A(\lambda, w)\big(I_T(\lambda)-f(\lambda, w_0)\big)\Big]\,d\lambda+o(1),$$

we obtain the statement of Theorem 4.4 from (5.22) and (5.21), (5.23), and (5.24), by Slutsky's argument.

Proof of Theorem 4.5. There exist constants H and K such that $0 < H \leqslant f_0(\lambda, w) \leqslant K < \infty$, $-\pi < \lambda \leqslant \pi$, and, from (4.7), (4.21) and (5.15), we can select

$$K = \frac{q}{h^2}\sqrt{\frac{\pi}{2\mu t}}\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\exp\left\{-\frac{k^2 h^2}{8\mu t}\right\}\right] \quad \text{and} \quad H = \frac{q}{h^3}\exp\left\{-\frac{4\pi^2 \mu t}{h^2}\right\}.$$

So, the statement (i) follows from Theorem 4.3, and (ii) is a consequence of Dzhaparidze and Kotz's [20], Theorem 1, p. 113, and Theorem 2, p. 114.

Proof of Theorem 4.6. By a standard argument (see, e.g., Hannan [37], Rice [66], Dzhaparidze and Kotz [20]) one obtains, as $T \to \infty$,

$$\sqrt{T}\left(S_T^{(1)}(w) - S_T^*\big(w, I_T(\lambda_j, Y^*), f_0(\lambda_j, w)\big)\right) \to 0,$$

in probability, where $S_T^{(1)}$ is defined by (4.31). Thus Theorem 4.3 follows from Theorem 4.1.

Proof of Theorem 4.7. It follows from (4.9), (4.11), and (5.4) that there exist constants H and K such that $0 < H \leqslant f_{di}^{(1)}(\lambda, w) \leqslant K < \infty$, $-\pi < \lambda \leqslant \pi$, $i = 2, 3$. For example, for $i = 2$ we can choose

$$H = qh^{-2}(2\pi)^2 \exp\{-2\mu t(3\pi)^2 h^{-2}\}$$

and

$$K + \frac{qh^2}{32\mu^2 t^2}\sqrt{\frac{\pi}{2\mu t}}\left[\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\exp\{-k^2 h^2/(8\mu t)\}\right)\frac{8\mu t}{h^2} + \frac{2}{\pi}\sum_{k=1}^{\infty} k^2 \exp\{-k^2 h^2/(8\mu t)\}\right].$$

Thus the results of Dzhaparidze and Kotz [20], Theorem 1, p. 113, and Theorem 2, p. 114, are applicable.

Proof of Theorem 4.8. In the univariate case, the conditions given by Robinson [68] are, in addition to condition **M**, reduced to the following ones:

(1) There exist constants $C > 0$, $\nu \in (-1, 1)$, $\eta \in (0, 2]$, such that $\tilde{f}_0(\lambda) = C\lambda^{-\nu} + O(\lambda^{\eta-\nu})$ as $\lambda \to 0$.

(2) There exists a $\nu \in (-1, 1)$ so that $\tilde{f}_0(\lambda)$ differentiable and $(d/d\lambda)\tilde{f}_0(\lambda) = O(\lambda^{-1-\nu/2})$ in a neighborhood $(0, \varepsilon)$ of the origin.

It follows from (4.40) that (1) and (2) are satisfied with $\nu = 1-\alpha \in (0, 1)$ and $\eta = \nu$.

Proof of Theorem 4.9. Both assumptions of Theorem 4.3 are satisfied by the ergodicity and property (4.40). Thus we obtain statement (i).

We also note that (4.26) is fulfilled for the spectral density $f_{10}(\lambda)$ given by (4.40). To prove statement (ii) we need to verify the rest of the conditions in

Theorem 4.4. First, we note that in view of (4.40), as $|\lambda| \to 0$ (see the proof of Theorem 4.2)

$$F_{10}(\lambda) = \varrho(\vartheta)\{\psi_1(\lambda)+\psi_2(\lambda)\}, \tag{5.25}$$

where $\psi_1(\lambda) = O(|\lambda|^{\alpha-1})$, $0 < m \leqslant \psi_2(\lambda) \leqslant M < \infty$. The constant that appear in the "big O" may depend on the parameter w.

All the first and second derivatives of the function $f_{10}^{-1}(\lambda, w)$ with respect to w_j, $j \neq 1$, are, by assumption, bounded and bounded away from zero on the compact set W. So, we need to check the conditions **L1–L7** for the first component of the function $A(\lambda, w) = -\nabla f_{10}^{-1}(\lambda, w) = (A_1, \ldots, A_r)'$, where $A_1(\lambda, w) = -(\partial/\partial\alpha) f_{10}^{-1}(\lambda, w)$. Using (5.25) one can prove that, as $|\lambda| \to 0$,

$$f_{10}^{-1}(\lambda, w) = O(|\lambda|^{1-\alpha}), \qquad f_{10}(\lambda, w) = O(|\lambda|^{\alpha-1}), \tag{5.26}$$

$$(\partial/\partial\alpha) f_{10}^{-1}(\lambda, w) = O(|\lambda|^{1-\alpha}\log|\lambda|), \tag{5.27}$$

$$(\partial^2/\partial\alpha^2) f_{10}^{-1}(\lambda, w) = O(|\lambda|^{1-\alpha}\log^2|\lambda|), \tag{5.28}$$

$$(\partial^2/\partial w_j \partial\alpha) f_{10}^{-1}(\lambda, w) = O(|\lambda|^{1-\alpha}\log|\lambda|). \tag{5.29}$$

Thus the condition **L6** is satisfied, i.e., $|\lambda|^{\kappa} f_{01}(\lambda, w)$ is bounded for some κ, $0 < \kappa < 1$, and $A_1(\lambda, w)|\lambda|^{-\kappa} \in L_2((-\pi, \pi])$. The rest of the conditions **L1–L5** and **L7** are also satisfied as is easy to check using (5.26)–(5.29) and like arguments. We omit the details.

Proof of Theorem 4.10. Part (i) follows directly from Theorem 4.3, while part (ii) is a consequence of a result from Dzhaparidze and Kotz [20], pp. 104–110.

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